Assignment: Uncertainties and data processing

1. Suppose you measure three independent variables as,

$$x = 10 \pm 2, \quad y = 7 \pm 1, \quad \theta = (40 \pm 3)^\circ.$$

   based on which you calculate the following quantity,

$$q = \frac{x+2}{x + y\cos(4\theta)},$$

   What would be your answer for q and uncertainty?

2. To find the acceleration of a cart, a student measures its initial and final velocities, v_i and v_f, and then calculates the difference $(v_f - v_i)$. Her data in two separate trials is,

$$\text{First run:} \quad v_i = 14.11\,\text{cm/s}, \quad v_f = 18.12\,\text{cm/s}.$$
$$\text{Second run:} \quad v_i = 19.21\,\text{cm/s}, \quad v_f = 20.61\,\text{cm/s}.$$

   Each of these has an uncertainty of $\pm 1\%$.

   (a) How would she calculate the absolute uncertainties in all four measurements and quote the final values of v_i and v_f for the two trials.

   (b) Find the change in velocity ($v = v_f - v_i$) and its uncertainty in both trail. Quote your value of v along with its uncertainty? Find the percentage uncertainty in v for the two trials.

3. Suppose you are an experimental physicist and you measure three quantities x, y and z as follows,

$$x = 8.15, \qquad y = 5.1, \qquad z = 4.2$$

   Compute the following expressions and in the true spirit of being faithful to the concept of precision,

   (a) $q_1 = (x^2 y)/z$

   (b) $q_2 = (xy)/\sqrt{z}$

   (c) $q_3 = (x^2 - y^2)/z^2$

8.16	8.14	8.12	8.16	8.18	8.10	8.18	8.18	8.18	8.24
8.16	8.14	8.17	8.18	8.21	8.12	8.12	8.17	8.06	8.10
8.12	8.10	8.14	8.09	8.16	8.16	8.21	8.14	8.16	8.13

TABLE I: Model data of time t (s).

4. **(a)** Calculate the mean and standard deviation for the following 30 measurements of a time t (s),

   (b) After several measurements, we can expect about 68% of the observed values to be within σ_t of $< t >$ (i.e., $< t > \pm \sigma_t$).

   (c) For the measurements of part (a),

   How many would you expect to lie the range ($< t > \pm \sigma_t$)? How many actually do?

   (d) How many would you expect to lie the range ($< t > \pm 2\sigma_t$)? How many actually do?

   (e) What does $2\sigma_t$ correspond to?

5. In an experiment, one measures values of current (I) for different applied voltages (V) across a light bulb, thus calculating the resistance (R) and power dissipation (P) in the bulb. The schematic of the setup is shown in Figure (1). The data collected during the experiment is shown in Table II.

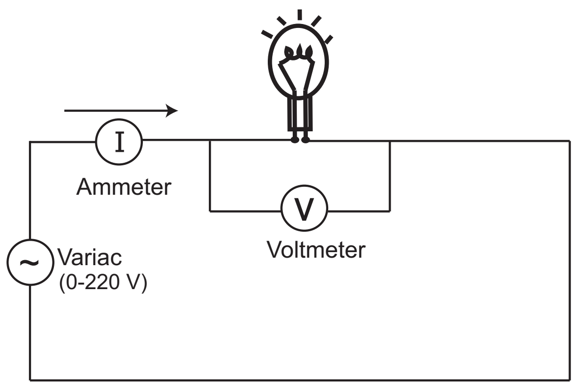

FIG. 1: A schematic diagram of the experiment.

Utilizing the relations for power dissipation ($P = I^2R$), emitted radiation by a black body ($P = A\sigma T^4$) and the power law ($T \propto R^\gamma$), we get,

$$P = \text{cosntant} * R^{4\gamma}. \tag{1}$$

V (V)	I (mA)	R (Ω)	P (W)
33	100.4	328.7	3.31
57	126.3	451.3	7.24
76	146.6	518.4	11.14
96	165.0	581.8	15.84
109	176.3	618.3	19.22
143	210.1	681.1	30.03
164	220.2	745.5	36.08
183	230.3	795.7	42.09
207	250.5	828.2	51.75

TABLE II: Model table for experimental results.

(a) Find uncertainties in the dependent and independent variables by utilizing the fact that the voltage and current readings are taken by using a digital multimeter with a rating of 1%.

(b) Make a table quoting the uncertainties in V, I, R and P.

(c) Linearize Equation (1). Plot a graph between $\log(P)$ versus $\log(R)$. Calculate uncertainties in $\log(P)$ and $\log(R)$, also plot a graph with error bars, both in the dependent and independent variables.

(d) Transfer all the uncertainties in the dependent variable. Plot another graph with error bars in the dependent variable only.

(e) Calculate the values of the slope and intercept using the mathematical relationship for a weighted fit of a straight line.

6. In an experiment to find the acceleration of gravity (g) using a simple pendulum, a student records the following readings for the time period (T) and length of the pendulum (l),

Length, l (cm): 57.3 61.1 73.2 83.7 95.0

Time, T (s): 1.521 1.567 1.718 1.835 1.952

The length of the pendulum (l) is measured by a ruler, which is an analog device, and a time is measured using a digital stopwatch (rating= 0).

The student uses the following relation to calculate (g),

$$g = \frac{4\pi^2 l}{T^2}. \tag{2}$$

(a) Calculate the uncertainty in each reading of l and T using probability distribution function and record them in a table.

(b) The relation for the time period is given as,

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{3}$$

Using Matlab, draw a plot between T and $\sqrt{l}$, also calculate slope of the straight line using least squares curve fitting.

(c) Plot uncertainties in T and $\sqrt{l}$ along with errobars.

PH-100

Exam: Experimental Physics Lab-I

Attempt all questions.

Q.No.1

The volume V of a rectangular block is determined by measuring the length l_x, l_y and l_z of its sides. From the scatter of the measurements a standard error of 0.01% is assigned to each dimension. What is the standard error in V, if,

(a) the scatter is due to errors in setting and reading the measuring instrument,

(b) and if it is due to temperature fluctuations.

Q.No.2

After measuring the speed of sound u several times, a student concludes that the standard deviation σ_u of her measurements is $\sigma_u = 10\,\mathrm{m/s}$. If all uncertainties were truly random, she could get any desired precision by making enough measurements and averaging.

(a) How many measurements are needed to give a final uncertainty of $\pm 3\,\mathrm{m/s}$?

(b) How many for a final uncertainty of only $\pm 0.5\,\mathrm{m/s}$?

Q.No.3

Suppose we wish to measure the spring constant k of a spring by timing the oscillations of a mass m fixed to its end. The time period for such oscillations is,

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Thus by measuring T and m, we can find k as,

$$k = 4\pi^2\,\frac{m}{T^2}.$$

The set of data for careful measurements of T corresponding to each mass m is given in Table (I).

Calculate the best estimate of k by combining each value of m with its corresponding period T. Also find out the uncertainty in the best estimated value of k.

Mass m (kg)	0.513	0.581	0.634	0.691	0.752	0.834	0.901	0.950
Period T (s)	1.24	1.33	1.36	1.44	1.50	1.59	1.65	1.69

TABLE I: Measured values of mass and time period.

Solution key for the exam

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Q.No.1

The standard error in each dimension is 0.01%.

(a) The volume of a rectangular block is,

$$V = l_1 l_2 l_3.$$

The error affects the three sides independently. Hence, the standard error in V can be calculated as,

$$\begin{aligned}(\Delta V)^2 &= \left(\frac{\partial V}{\partial l_1}\right)^2 (\Delta l_1)^2 + \left(\frac{\partial V}{\partial l_2}\right)^2 (\Delta l_2)^2 + \left(\frac{\partial V}{\partial l_3}\right)^2 (\Delta l_3)^2, \\ &= (l_2 l_3)^2 (\Delta l_1)^2 + (l_1 l_3)^2 (\Delta l_2)^2 + (l_1 l_2)^2 (\Delta l_3)^2.\end{aligned}$$

Doing a step we get the result,

$$(\Delta V)^2 = \left(\frac{l_1 l_2 l_3}{l_1}\right)^2 (\Delta l_1)^2 + \left(\frac{l_1 l_2 l_3}{l_2}\right)^2 (\Delta l_2)^2 + \left(\frac{l_1 l_2 l_3}{l_3}\right)^2 (\Delta l_3)^2,$$

implying

$$\begin{aligned}\left(\frac{\Delta V}{V}\right)^2 &= \left(\frac{\Delta l_1}{l_1}\right)^2 + \left(\frac{\Delta l_2}{l_2}\right)^2 + \left(\frac{\Delta l_3}{l_3}\right)^2, \\ \left(\frac{\Delta V}{V}\right) &= \sqrt{(0.01)^2 + (0.01)^2 + (0.01)^2} = 0.017\%\,, \\ &\approx 0.02\,\%.\end{aligned}$$

(b) For temperature variations, all sides are affected equally. Therefore, one can use the formula for volume with equal lengths,

$$V = l^3.$$

The error in volume will be,

$$\left(\frac{\Delta V}{V}\right)^2 = \left(\frac{3\Delta l}{l}\right)^2.$$

$$\left(\frac{\Delta V}{V}\right) = \left(\frac{3\Delta l}{l}\right), \\ = 0.03\ \%.$$

This result shows that the overall uncertainty can increase, if the errors are not independent nor random.

Q.No.2

(a) The standard deviation of the measured data is $\sigma_u = 10\,\mathrm{m/s}$.

The standard error in the mean can be find out using the following relationship,

$$\sigma_{\mathrm{mean}} = \frac{\sigma}{\sqrt{n}}, \tag{1}$$

implying,

$$n = \left(\frac{\sigma_u}{\sigma_{\mathrm{mean}}}\right)^2, \\ = \left(\frac{10}{3}\right)^2, \\ = 11.$$

(b) Let's see how many times we will repeat the experiment to get a final uncertainty of $0.5\,\mathrm{m/s}$.

Using Equation (1), we get,

$$n = \left(\frac{10}{0.5}\right)^2, \\ = 400.$$

Hence we conclude that we need to repeat the measurements 11 and 400 times for minimizing error to $3\,\mathrm{m/s}$ and $0.5\,\mathrm{m/s}$, respectively.

Q.No.3

The spring constant k measured by timing the oscillations of a mass m fixed to its end is given as,

$$k = 4\pi^2 \frac{m}{T^2}.$$

As the measured masses are not different measurements of the same quantity, therefore the process of averaging can't work. On the other hand, we are not sure about the uncertainties in our measurements, so we need to calculate k first by combining each value of m with its corresponding period T as given in Table (1).

Mass m (kg)	Period T (s)	Spring constant k (N/m)
0.513	1.24	13.17
0.581	1.33	12.97
0.634	1.36	13.53
0.691	1.44	13.16
0.752	1.50	13.19
0.834	1.59	13.02
0.901	1.65	13.07
0.950	1.69	13.13

Table 1: Measurement of spring constant k.

The mean of the measured values of spring constant k is,

$$\begin{aligned}\bar{k} &= \frac{\sum_i k_i}{n} \\ &= \frac{13.17 + 12.97 + 13.53 + 13.16 + 13.19 + 13.02 + 13.07 + 13.13}{8} \\ &= 13.16\,\mathrm{N/m}.\end{aligned}$$

Deviation from the mean value is,

$$d_i = k_i - \bar{k},$$

and the deviations are,

d_i (N/m)	$(d_i\ (\mathrm{N/m}))^2$
0.01	1×10^{-4}
-0.19	0.0361
0.37	0.1369
0	0
0.03	9×10^{-4}
-0.14	0.0196
-0.09	8.1×10^{-3}
-0.03	9×10^{-4}

The standard deviation, s, is,

$$\begin{aligned}s &= \sqrt{\frac{\sum_i d_i^2}{n}} = \sqrt{\frac{0.2026}{8}} \\ &= 0.159 \approx 0.16\,\mathrm{N/m}.\end{aligned}$$

The standard error, σ, can be find out using the following relationship,

$$\begin{aligned}\sigma &= \sqrt{\frac{n}{n-1}}\,s \\ &= 0.17\,\mathrm{N/m}.\end{aligned}$$

We can say that he expected standard error is approximately equal to the standard deviation, $\sigma \approx s$.

Now the standard error in the mean is,

$$\begin{aligned} \sigma_m &= \frac{\sigma}{\sqrt{n}} \\ &= 0.06\,\mathrm{N/m}. \end{aligned}$$

Hence the final result can be written as,

$$k = (13.16 \pm 0.06)\,\mathrm{N/m}.$$

Solution Assignment: Uncertainties and data processing

1. Suppose you measure three independent variables as,

$$x = 10 \pm 2, \quad y = 7 \pm 1, \quad \theta = (40 \pm 3)^\circ.$$

based on which you calculate the following quantity,

$$q = \frac{x+2}{x + y\cos(4\theta)},$$

What would be your answer for q and uncertainty?

Solution:

Given values are:

$$x = 10 \pm 2, \quad y = 7 \pm 1, \quad \theta = (40 \pm 3)^\circ.$$

The mathematical expression for q is,

$$\begin{aligned} q &= \frac{x+2}{x + y\cos(4\theta)} = \frac{10+2}{10 + 7(\cos(160^\circ))} \\ &= \frac{12}{10 + 7(-0.939)} = \frac{12}{3.427} \\ &= 3.50 \end{aligned}$$

Using Taylor series relationship for finding uncertainty in q,

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x}\,\Delta x\right)^2 + \left(\frac{\partial q}{\partial y}\,\Delta y\right)^2 + \left(\frac{\partial q}{\partial \theta}\,\Delta \theta\right)^2}. \tag{1}$$

Where,

$$\begin{aligned} \left(\frac{\partial q}{\partial x}\right) &= \frac{[x + y\ \cos(4\theta)](1) - (x+2)(1)}{[x + y\cos(4\theta)]^2} \\ &= \frac{3.427 - 12}{(3.427)^2} \\ &= -0.732 \end{aligned}$$

Likewise differentiating w.r.t y yields,

$$\left(\frac{\partial q}{\partial y}\right) = \frac{-(x+2)\cos(4\theta)}{[x+y\cos(4\theta)]^2}$$

$$= \frac{-(12)(-0.939)}{(3.427)^2}$$

$$= 0.960$$

and w.r.t θ gives,

$$\left(\frac{\partial q}{\partial \theta}\right) = \frac{-(x+2)(-4y\sin 4\theta)}{[x+y\cos(4\theta)]^2}$$

$$= \frac{-(12)(-28\times 0.342)}{(3.427)^2}$$

$$= \frac{114.912}{(3.427)^2}$$

$$= 9.784$$

Substituting the above expressions in Equation (1) results in,

$$\Delta q = \sqrt{(-0.732\times 2)^2 + (0.960\times 1)^2 + (9.784\times \frac{3\times 3.14}{180})^2}$$

$$= \sqrt{2.1433 + 0.9216 + 0.2622}$$

$$= 1.82$$

Finally, the value of q along with its uncertainty can be quoted as,

$$q = (3.5 \pm 1.8).$$

2. To find the acceleration of a cart, a student measures its initial and final velocities, v_i and v_f, and then calculates the difference $(v_f - v_i)$. Her data in two separate trials is,

$$\text{First run:} \quad v_i = 14.11\,\text{cm/s}, \quad v_f = 18.12\,\text{cm/s}.$$
$$\text{Second run:} \quad v_i = 19.21\,\text{cm/s}, \quad v_f = 20.61\,\text{cm/s}.$$

Each of these has an uncertainty of $\pm 1\%$.

(a) How would she calculate the absolute uncertainties in all four measurements and quote the final values of v_i and v_f for the two trials.

(b) Find the change in velocity $(v = v_f - v_i)$ and its uncertainty in both trail. Quote your value of v along with its uncertainty? Find the percentage uncertainty in v for the two trials.

Solution:

(a) Since each quantity has an uncertainty of $\pm 1\%$, the absolute uncertainties of the measurand can be calculated as,

1^{st} run:

$$v_i = 0.01 \times 14.11 = 0.1411\,\text{cm/s}.$$
$$v_f = 0.01 \times 18.12 = 0.1812\,\text{cm/s}.$$

2^{nd} run:

$$v_i = 0.01 \times 19.21 = 0.1921\,\text{cm/s}.$$
$$v_f = 0.01 \times 20.61 = 0.2061\,\text{cm/s}.$$

The final values of the initial and final velocities alongwith uncertainties in both the trials can be quoted as,

1^{st} run:

$$v_i = (14.11 \pm 0.14)\,\text{cm/s}.$$
$$v_f = (18.12 \pm 0.18)\,\text{cm/s}.$$

While quoting the final value of the calculated uncertainty, care must be taken that uncertainty should have only one significant figure, at the most 2 significant figures can be considered. After deciding about the significant figure, the best approximated value can be rounded off so that the decimal places of the best estimated value and uncertainty should be at the same position.

2^{nd} run:

$$\begin{aligned} v_i &= (19.21 \pm 0.19)\,\text{cm/s}. \\ v_f &= (20.61 \pm 0.21)\,\text{cm/s}. \end{aligned}$$

(b) The change in velocity can be find out simply by subtracting final and initial velocities, and given as,

1^{st} run:

$$\begin{aligned} v_f - v_i &= [18.0 - 14.0] = 4\,\text{cm/s}, \\ \Delta(v_f - v_i) &= \sqrt{\Delta v_f{}^2 + \Delta v_i{}^2}, \\ &= \sqrt{(0.18)^2 + (0.14)^2}, \\ &= 0.23\,\text{cm/s}. \end{aligned}$$

and the resultant value can be written as,

$$v_f - v_i = (4.0 \pm 0.2)\,\text{cm/s}$$

Likewise for the second run yields,

2^{nd} run:

$$\begin{aligned} v_f - v_i &= [20.61 - 19.21] = 1.4\,\text{cm/s} \\ \Delta(v_f - v_i) &= \sqrt{\Delta v_f{}^2 + \Delta v_i{}^2} = \sqrt{(0.21)^2 + (0.19)^2} = 0.283\,\text{cm/s} \\ v_f - v_i &= (1.4 \pm 0.3)\,\text{cm/s}. \end{aligned}$$

The percentage uncertainties are,

1^{st} run:

$$\frac{0.2}{4.0} \times 100 = 5\%$$

2^{nd} run:

$$\frac{0.3}{1.4} \times 100 \; = \; 21\%$$

3. Suppose you are an experimental physicist and you measure three quantities x, y and z as follows,

$$x \; = \; 8.15, \qquad y = 5.1, \qquad z = 4.2$$

Compute the following expressions and in the true spirit of being faithful to the concept of precision,

(a) $q_1 = (x^2 y)/z$

(b) $q_2 = (xy)/\sqrt{z}$

(c) $q_3 = (x^2 - y^2)/z^2$

Solution:

Given values are:

$$x \; = \; 8.15, \qquad y = 5.1, \qquad z = 4.2$$

Precision of the primary measured quantities can be computed as,

$$\begin{aligned} x &= \frac{0.005}{8.15} \times 100 = 0.06\%, \\ y &= \frac{0.05}{5.1} \times 100 = 0.98\%, \\ z &= \frac{0.05}{4.2} \times 100 = 1.19\%. \end{aligned}$$

(b) The measured quantity q_1 can be computed as,

$$\begin{aligned} q_1 &= \frac{x^2\, y}{z} \\ &= 80.65589286 \end{aligned}$$

To quote the final value of q_1, we'll check the relative precision of q_1 first by taking it

Quoted value	Relative precision
80.6559	$\frac{0.00005}{80.6559} \times 100 = 0.00006\%$
80.656	$\frac{0.0005}{80.656} \times 100 = 0.0006\%$
80.66	$\frac{0.005}{80.66} \times 100 = 0.006\%$
80.6	$\frac{0.05}{80.6} \times 100 = 0.06\%$
81	$\frac{5}{81} \times 100 = 6.17\%$

TABLE I: Table for calculated precisions.

upto four decimal places and then rounding it off to achieve a final precision which is less than the least precise primary quantity (z). The process is shown in Table (I).

$$q_1 = 81 \quad \text{(Best estimated value)}.$$

(b) The calculator returns the value of q_2 as,

$$\begin{aligned} q_2 &= \frac{x\,y}{\sqrt{z}} \\ &= 20.28164327 \end{aligned}$$

The final value of q_2 can b chosen by looking at the relative precision of q_2 being rounded off. The results are tabulated in Table (II).

Quoted value	Relative precision
20.2816	$\frac{0.00005}{20.2816} \times 100 = 0.0002\%$
20.282	$\frac{0.0005}{20.282} \times 100 = 0.002\%$
20.28	$\frac{0.005}{20.28} \times 100 = 0.02\%$
20.2	$\frac{0.05}{20.2} \times 100 = 0.24\%$
20	$\frac{5}{20} \times 100 = 25\%$

TABLE II: Table for calculated precisions.

Hence, we can conclude,

$$q_2 = 20 \quad \text{(Best estimated value)}$$

(c) The quantity q_3 is given as,

$$q_3 = \frac{x^2 - y^2}{z^2}$$
$$= 2.29095805$$

The relative precision in tabulated in Table (III). The final value of q_2 is quoted as,

Quoted value	Relative precision
2.2909	$\frac{0.00005}{2.2909} \times 100 = 0.0002\%$
2.290	$\frac{0.0005}{2.291} \times 100 = 0.002\%$
2.29	$\frac{0.005}{2.29} \times 100 = 0.02\%$
2.3	$\frac{0.05}{2.3} \times 100 = 2.2\%$

TABLE III: Table for calculated precisions.

$$q_3 = 2.3 \quad \text{(Best estimated value)}$$

4. **(a)** Calculate the mean and standard deviation for the following 30 measurements of a time t (s),

8.16	8.14	8.12	8.16	8.18	8.10	8.18	8.18	8.18	8.24
8.16	8.14	8.17	8.18	8.21	8.12	8.12	8.17	8.06	8.10
8.12	8.10	8.14	8.09	8.16	8.16	8.21	8.14	8.16	8.13

TABLE IV: Model data of time t (s).

(b) After several measurements, we can expect about 68% of the observed values to be within σ_t of $< t >$ (i.e., $< t > \pm \sigma_t$).

(c) For the measurements of part (a),
How many would you expect to lie the range ($< t > \pm \sigma_t$)? How many actually do?

(d) How many would you expect to lie the range ($< t > \pm 2\sigma_t$)? How many actually do?

(e) What does $2\sigma_t$ correspond to?

Solution:

(a) The best approximated value of time t is,

$$< t > \; = \; \frac{\sum_{i=1}^{30} t_i}{n} = \frac{244.48}{30} = 8.15\,\text{s}.$$

The deviations d_i can be calculated as,

$$d_i \; = \; t_i - < t > .$$

The calculated deviations for the measured values of t are tabulated in Table (V).

Sr. No	Time t (s)	Deviations d_i (s)	d_i^2 (s^2)	Sr. No	Time t (s)	Deviations d_i (s)	d_i^2 (s^2)
1	8.16	0.0107	1.1378×10^{-4}	16	8.12	-0.0293	8.6044×10^{-4}
2	8.14	-0.0093	8.7111×10^{-5}	17	8.12	-0.0293	8.6044×10^{-4}
3	8.12	-0.0293	8.6044×10^{-4}	18	8.17	-0.0207	4.2711×10^{-4}
4	8.16	0.0107	1.1378×10^{-4}	19	8.06	-0.0893	8.000×10^{-3}
5	8.18	0.0307	9.4044×10^{-4}	20	8.10	-0.0493	2.400×10^{-3}
6	8.10	-0.0493	2.400×10^{-3}	21	8.12	-0.0293	8.6044×10^{-4}
7	8.18	0.0307	9.4044×10^{-4}	22	8.10	-0.0493	2.400×10^{-3}
8	8.18	0.0307	9.4044×10^{-4}	23	8.14	-0.0093	8.7111×10^{-5}
9	8.18	0.0307	9.4044×10^{-5}	24	8.09	-0.0593	3.500×10^{-3}
10	8.24	0.0907	8.200×10^{-3}	25	8.16	0.0107	1.1378×10^{-4}
11	8.16	0.0107	1.1378×10^{-4}	26	8.16	0.0107	1.1378×10^{-4}
12	8.14	-0.0093	8.7111×10^{-5}	27	8.21	0.0607	3.700×10^{-3}
13	8.17	0.0207	4.2711×10^{-4}	28	8.14	-0.0093	8.7111×10^{-5}
14	8.18	0.0307	9.4044×10^{-4}	29	8.16	0.0107	1.1378×10^{-4}
15	8.21	0.0607	3.700×10^{-3}	30	8.13	-0.0193	3.7378×10^{-4}

TABLE V: Table for calculated deviations.

The standard deviation is,

$$s \; = \; \sqrt{\frac{\sum_{i=1}^{30} d_i^2}{n}} = \sqrt{\frac{0.0448}{30}} = 0.0386\,\text{s}.$$

The standard uncertainty can be calculated as,

$$\sigma = \sqrt{\frac{n}{n-1}}\,(s) = \sqrt{\frac{30}{29}}\,(0.0386) = 0.0393 = 0.04\,\mathrm{s},$$

and standard uncertainty in the mean value is,

$$\begin{aligned}\sigma_m &= \frac{\sigma}{\sqrt{n}} = \frac{0.0393}{\sqrt{30}} = 0.0072,\ \mathrm{s} \\ &= 0.01\,\mathrm{s}.\end{aligned}$$

The best estimated value of time t can be quoted as,

$$t = (8.15 \pm 0.01)\,\mathrm{s}$$

(c) For type A evaluations, the probability distribution function associated with the measurement is a Gaussian probability distribution function as shown in Figure (1). The coverage probability or the confidence of interval for any arbitrary measurand μ corresponding to different standard uncertainties is given as,

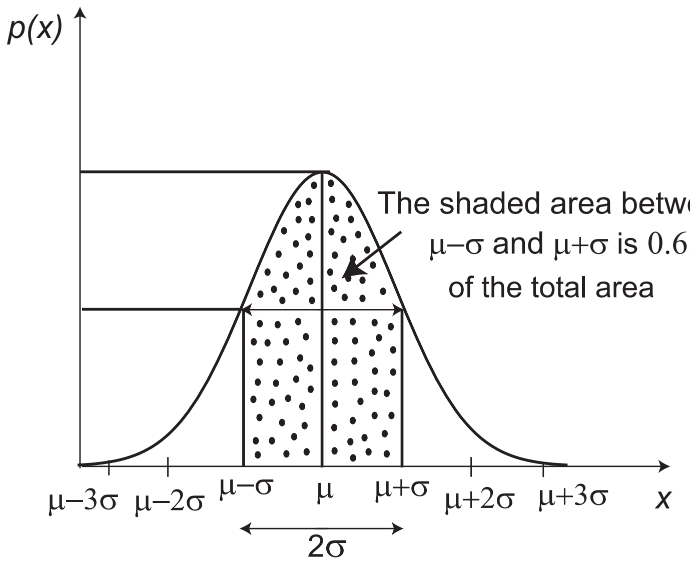

FIG. 1: A Gaussian probability distribution function.

$$(<\mu> \pm 1\sigma) \rightarrow (68\%)$$

$$(<\mu> \pm 2\sigma) \rightarrow (95\%)$$

$$(<\mu> \pm 3\sigma) \rightarrow (99\%)$$

Now as the above expressions predict, the 68% confidence of interval corresponds to $(< t > \pm 1\sigma)$. Considering this, the range of interval becomes,

$$
\begin{aligned}
t &= (8.15 \pm 0.04)\,\mathrm{s} \\
t &= [8.11,\ 8.19]\,\mathrm{s}.
\end{aligned}
$$

The values (in seconds s) that lie within 68% confidence of interval are,

8.16, 8.14, 8.12, 8.16, 8.18, 8.10, 8.18, 8.18, 8.18, 8.16, 8.14, 8.17, 8.18,
8.12, 8.12, 8.17, 8.10, 8.12, 8.10, 8.14, 8.16, 8.16, 8.14, 8.16, 8.13.

and values (in seconds s) which lie outside this interval or lying within 32% confidence of interval are,

8.24, 8.21, 8.06, 8.09, 8.21.

(d) The 95% confidence of interval corresponds to $(< t > \pm 2\sigma)$. The range of interval becomes,

$$
\begin{aligned}
t &= (8.15 \pm 2(0.04))\,\mathrm{s} = (8.15 \pm 0.08)\,\mathrm{s}, \\
t &= [8.07,\ 8.23]\,\mathrm{s}.
\end{aligned}
$$

The values (in seconds s) that lie within 95% confidence of interval are,

8.16, 8.14, 8.12, 8.16, 8.18, 8.10, 8.18, 8.18, 8.18, 8.16, 8.14, 8.17, 8.18, 8.12,
8.12, 8.17, 8.10, 8.12, 8.10, 8.14, 8.16, 8.16, 8.14, 8.16, 8.13, 8.09, 8.21, 8.21.

and values (in seconds s) which lie outside this interval or lying within 5% confidence of interval are,

8.06, 8.24.

(e) $2\sigma_t$ corresponds to the 95% confidence of interval. This means that there is 95% probability that the best approximated value of measurand lies somewhere within the interval $(< t > \pm 2\sigma)$ of standard uncertainty. Conversely, there is 5% probability that the best approximated value of the measurand lies outside the interval $(< t > \pm 2\sigma)$.

5. In an experiment, one measures values of current (I) for different applied voltages (V) across a light bulb, thus calculating the resistance (R) and power dissipation (P) in the bulb. The schematic of the setup is shown in Figure (2). The data collected during the experiment is shown in Table X.

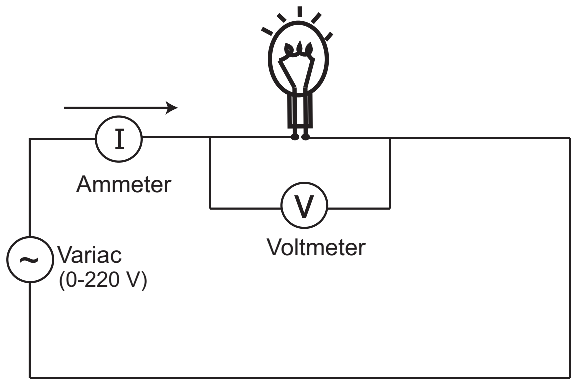

FIG. 2: A schematic diagram of the experiment.

Utilizing the relations for power dissipation ($P = I^2R$), emitted radiation by a black body ($P = A\sigma T^4$) and the power law ($T \propto R^\gamma$), we get,

$$P = \text{cosntant} * R^{4\gamma}. \tag{2}$$

(a) Find uncertainties in the dependent and independent variables by utilizing the fact that the voltage and current readings are taken by using a digital multimeter with a rating of 1%.

(b) Make a table quoting the uncertainties in V, I, R and P.

(c) Linearize Equation (2). Plot a graph between $\log(P)$ versus $\log(R)$. Calculate uncertainties in $\log(P)$ and $\log(R)$, also plot a graph with error bars, both in the dependent and independent variables.

(d) Transfer all the uncertainties in the dependent variable. Plot another graph with error bars in the dependent variable only.

(e) Calculate the values of the slope and intercept using the mathematical relationship for a weighted fit of a straight line.

V (V)	I (mA)	R (Ω)	P (W)
33	100.4	328.7	3.31
57	126.3	451.3	7.24
76	146.6	518.4	11.14
96	165.0	581.8	15.84
109	176.3	618.3	19.22
143	210.1	681.1	30.03
164	220.2	745.5	36.08
183	230.3	795.7	42.09
207	250.5	828.2	51.75

TABLE VI: Model table for experimental results.

Solution:

Solution

(a) The voltage V is measured using a digital multimeter and by looking at the data (shown in Table X), one can tell that the multimeter has a resolution of 1 V. Now the scale uncertainty can be calculated by associating a uniform probability distribution function with the reading and given as,

$$\begin{aligned} u_{\text{Scale}_{(V)}} &= \frac{\Delta}{\sqrt{3}} = \frac{0.5}{\sqrt{3}}\,\text{V}. \\ &= 0.3\,\text{V}. \end{aligned}$$

Since the rating of the digital multimeter is 1%, uncertainty associated with rating of the instrument is given as,

$$u_{\text{Rating}_{(V)}}\,@1\% \;=\; 0.01 \times \text{value of } (V),$$

while the combined uncertainty in voltage V can be calculated using the following expression,

$$u_V \;=\; \sqrt{(u_{\text{Scale}_{(V)}})^2 + (u_{\text{Rating}_{(V)}})^2}.$$

Likewise for the ammeter, the scale and combine uncertainty can be calculated as,

$$u_{\text{Scale}_I} = \frac{\Delta}{\sqrt{3}} = \frac{0.05}{\sqrt{3}}\,\text{mA} = 0.03 \times 10^{-3}\,\text{A}.$$
$$u_I = \sqrt{(u_{\text{Scale}_{(I)}})^2 + (u_{\text{Rating}_{(I)}})^2}.$$

The uncertainty in energy ($R = V/I$) can be calculated using Taylor series approximation,

$$\Delta R = \sqrt{\left(\frac{\partial R}{\partial V}\,\Delta V\right)^2 + \left(\frac{\partial R}{\partial I}\,\Delta I\right)^2}, \tag{3}$$

By looking at Equation (4), we can easily tell that the dependent variable is $\log(P)$, while the independent one is $\log(R)$. Now the question is how to propagate uncertainties from P and R to $\log(P)$ and $\log(R)$, respectively. For that we'll utilize the general rule of propagation,

$$u_x = \Delta(\log R) = \sqrt{\left(\frac{\partial(\log R)}{\partial R}\Delta R\right)^2} = \frac{\Delta R}{R},$$
$$u_y = \Delta(\log P) = \sqrt{\left(\frac{\partial(\log P)}{\partial P}\Delta P\right)^2} = \frac{\Delta P}{P}.$$

(b) Uncertainties in all the measured and inferred quantities are quoted in Table (VII).

(c) By taking log on both sides of Equation (2) yields a straight line equation given as,

$$\log(P) = 4\gamma \log(R) + \log(c), \tag{4}$$

where $(\log P)$ is the dependent variable, $(\log R)$ is the independent variable, (c=constant) and the value of γ can be calculated by finding the value of slope.

Graph is shown in Figure (3a).

(d) The mathematical expressions for transferring uncertainties to the dependent variable $(\log P)$ and for calculating the total uncertainty are,

$$u_{\text{Trans}} = \left(\frac{dy}{dx}\right) u_x = (3)\, u_x, \tag{5}$$

$$u_{\text{Total}} = \sqrt{(u_{\text{Trans}})^2 + u_y^2}. \tag{6}$$

Voltage V (V)	ΔV (V)	Current I $\times 10^{-3}$ (A)	ΔI $\times 10^{-3}$(A)	Resistance $R\,(\Omega)$	ΔR (Ω)	Power dissipation P (W)	ΔP (W)	$\log(R)$	$\Delta(\log R)$	$\log(P)$	$\Delta(\log P)$
33.0	0.4	100.4	1.0	328.7	5.5	3.3	0.1	5.79	0.02	1.19	0.03
57.0	0.6	126.3	1.3	451.3	6.8	7.2	0.2	6.11	0.02	1.97	0.03
76.0	0.8	146.6	1.5	518.4	7.6	11.1	0.3	6.25	0.01	2.41	0.02
96	1	165.0	1.7	581.8	8.4	15.8	0.4	6.37	0.01	2.76	0.02
109.0	1.1	176.3	1.8	618.3	8.9	19.2	0.5	6.43	0.01	2.95	0.02
143.0	1.4	210.0	2.1	681.0	9.7	30.0	0.7	6.52	0.01	3.40	0.02
164.0	1.7	220.0	2.2	745.5	10.6	36.1	0.9	6.61	0.01	3.59	0.02
183.0	1.9	230.0	2.3	795.7	11.3	42.0	1.0	6.68	0.01	3.74	0.02
207	2.	250.0	2.5	828.0	11.8	51.8	1.3	6.72	0.01	3.95	0.02

TABLE VII: Experimental data and calculated uncertainties.

The weights w are reciprocal squares of the total uncertainty being utilized in least-squares fitting of a straight line. The expression for calculating the weight w is,

$$w = \frac{1}{u^2_{\text{Total}}}\,. \tag{7}$$

The uncertainties calculated for the given data of independent ($\log R$) and dependent variables ($\log P$) are shown in Table (VIII).

Graph with error bars only in the dependent variable is shown in Figure (3b).

(e) The mathematical relationships for slope (m) and intercept (c) are,

$$m = \frac{\Sigma_i w_i \, \Sigma_i w_i (x_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i y_i)}{\Sigma_i w_i \, \Sigma_i (w x_i^2) - (\Sigma_i w_i x_i)^2}\,, \tag{8}$$

and,

$$c = \frac{\Sigma_i (w_i x_i^2) \, \Sigma_i (w_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i x_i y_i)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}, \tag{9}$$

where x is the independent variable ($\log R$ in our case), y is the dependent variable ($\log P$) and w is the weight.

The different terms in the numerator and denominator of Equations (8) and (9) are calculated separately and tabulated in Table (IX).

$\log(R)$	$\log(P)$	u_{Trans}	u_{Total}
5.79	1.19	0.050	0.057
6.11	1.97	0.045	0.052
6.25	2.41	0.044	0.051
6.37	2.76	0.044	0.050
6.43	2.95	0.043	0.050
6.52	3.40	0.043	0.049
6.61	3.59	0.043	0.049
6.68	3.74	0.043	0.049
6.72	3.95	0.0423	0.049

TABLE VIII: Calculated data for the transfer and total uncertainties.

w	wxy	wx	wy	wx^2
309.8185	0.2149×10^4	1.7954×10^3	0.3708×10^3	1.0405×10^4
373.4411	0.4506×10^4	2.2825×10^3	0.7372×10^3	1.3951×10^4
391.1746	0.5894×10^4	2.4451×10^3	0.9429×10^3	1.5284×10^4
400.3012	0.7040×10^4	2.5484×10^3	1.1058×10^3	1.6223×10^4
403.8517	0.7672×10^4	2.5955×10^3	1.1938×10^3	1.6681×10^4
409.1155	0.9080×10^4	2.6689×10^3	1.3919×10^3	1.7411×10^4
410.8932	0.9745×10^4	2.7177×10^3	1.4734×10^3	1.7975×10^4
412.0122	1.0292×10^4	2.7519×10^3	1.5408×10^3	1.8381×10^4
413.0174	1.0952×10^4	2.7751×10^3	1.6299×10^3	1.8646×10^4
$\sum w = 3523.6$	$\sum wxy = 67330$	$\sum wx = 22581$	$\sum wy = 10387$	$\sum wx^2 = 144960$

TABLE IX: Terms being utilized in finding the values of slope and intercept.

Substituting the calculated terms given in Table (IX) in Equation (8) yields,

$$\begin{aligned} m &= \frac{(3523.6)(67330) - (22581)(10387)}{(3523.6)(144960) - (22581)^2}, \\ &= 3.041 \end{aligned}$$

and the intercept (c) is given as,

$$c = \frac{(144960)(10387) - (22581)(67330)}{(3523.6)(144960) - (22581)^2}, \\ = -16.5381$$

The expressions for the uncertainties in slope (m) and intercept (c) are,

$$u_m = \sqrt{\frac{\Sigma_i w_i}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}\,, \tag{10}$$

$$u_c = \sqrt{\frac{\Sigma_i (w_i x_i^2)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}\,. \tag{11}$$

Substituting the values from Table (IX) in the above expressions yield,

$$u_m = \sqrt{\frac{(3523.6)}{(3523.6)(144960) - (22581)^2}} = 0.0629,$$

$$u_c = \sqrt{\frac{(144960)}{(3523.6)(144960) - (22581)^2}} = 0.4034.$$

Finally, the values of slope and intercept alongwith uncertainties are quoted as,

$$m = (3.04 \pm 0.06). \tag{12}$$

$$c = (-16.5 \pm 0.4). \tag{13}$$

Since the value of slope is equal to ($m = 4\gamma$), therefore we can write,

$$\gamma = \frac{m}{4} = 0.76.$$

Uncertainty in γ can be find out using Taylor series approximation, given as,

$$\Delta\gamma = \sqrt{\left(\frac{\partial \gamma}{\partial m}\,\Delta m\right)^2} = \sqrt{\left(\frac{\Delta m}{4}\right)^2} = 0.015$$

Finally, we can conclude,

$$\gamma = (0.76 \pm 0.02). \tag{14}$$

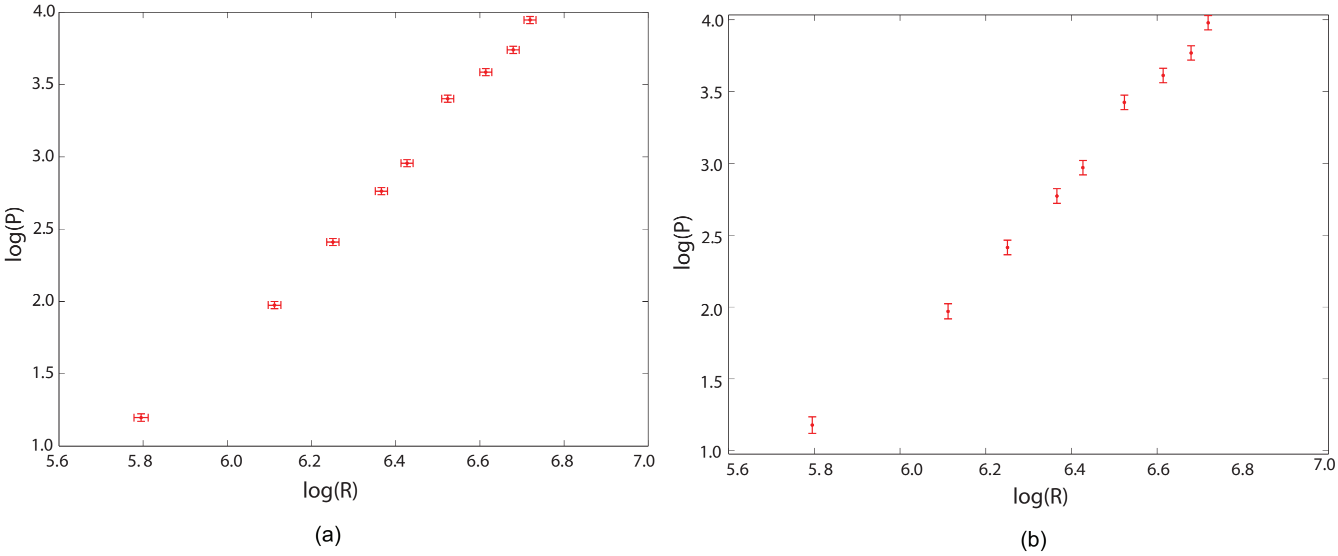

FIG. 3: Graph with errorbars: (a) Uncertainties in both the dependent and independent variables and (b) uncertainties are transformed to the dependent variable only.

6. In an experiment to find the acceleration of gravity (g) using a simple pendulum, a student records the following readings for the time period (T) and length of the pendulum (l),

Length, l (cm):	57.3	61.1	73.2	83.7	95.0
Time, T (s):	1.521	1.567	1.718	1.835	1.952

The length of the pendulum (l) is measured by a ruler, which is an analog device, and a time is measured using a digital stopwatch (rating= 0).
The student uses the following relation to calculate (g),

$$g = \frac{4\pi^2 l}{T^2}. \tag{15}$$

(a) Calculate the uncertainty in each reading of l and T using probability distribution function and record them in a table.

(b) The relation for the time period is given as,

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{16}$$

Using Matlab, draw a plot between T and $\sqrt{l}$, also calculate slope of the straight line using least squares curve fitting.

(c) Plot uncertainties in T and $\sqrt{l}$ along with errobars.

Solution:

(a) The uncertainty in the independent variable (length l) has a triangular probability distribution function associated with it and can be calculated using judgement. For the given data, the uncertainty would be,

$$u_y = \frac{\Delta}{\sqrt{6}} = \frac{0.1}{\sqrt{6}} = 0.041\,\text{cm}. \tag{17}$$

The dependent variable (time period of the pendulum (T)) is being measured using a digital stop watch and by looking at the data one can tell that the stop watch has a resolution of 0.001 s. Now the scale uncertainty can be calculated by associating a uniform probability distribution function with the reading,

$$u_{\text{Scale}} = \frac{\Delta}{\sqrt{3}} = \frac{0.0005}{\sqrt{3}}\,\text{s}. \tag{18}$$

$$= 0.0003\,\text{s}. \tag{19}$$

The uncertainties being quoted in independent and dependent variables are shown in Table (X).

Length l (cm)	$x = \sqrt{l}$ (cm)	u_x (cm)	Time period T (s)	u_y (s)
57.3	7.57	0.04	1.5210	0.0005
61.1	7.82	0.04	1.5670	0.0005
73.2	8.56	0.04	1.7180	0.0005
83.7	9.15	0.04	1.8350	0.0005
95.0	9.75	0.04	1.9520	0.0005

TABLE X: Results for calculated uncertainties.

(b) The independent and dependent row vectors are created as,

≫ **l=[57.3 61.1 73.2 83.7 95.0];**

≫ **T=[1.521 1.567 1.718 1.835 1.952];**

In least square curve fitting, we need to plot the data in order to know the relationship between independent and dependent quantities. For that we type,

$\gg$ **figure; plot(m, l, ‘ro’, ‘MarkerFaceColor’, ‘r’)**

The data plot shows that our best fit is of the form,

$$y = mx + c,$$

where m is the slope and c is the intercept.

Now we will make an M-file named **spring** depicted in Figure (4a).

Once the fitting function has been defined, we can find the least squares curve using the command,

$\gg$ **lsqcurvefit(@spring, [0.2 1.5], $\sqrt{\mathbf{l}}$, T)**

After the Matlab returns the values of the parameters, the output function is redefined as,

$\gg$ **cfit=0.1989*($\sqrt{\mathbf{l}}$)+0.0148;**

Now we plot the redefined function on the data points using the command,

$\gg$ **hold on**

$\gg$ **plot(m, cfit, ‘r’)** The built-in function of Matlab returns the value of slope

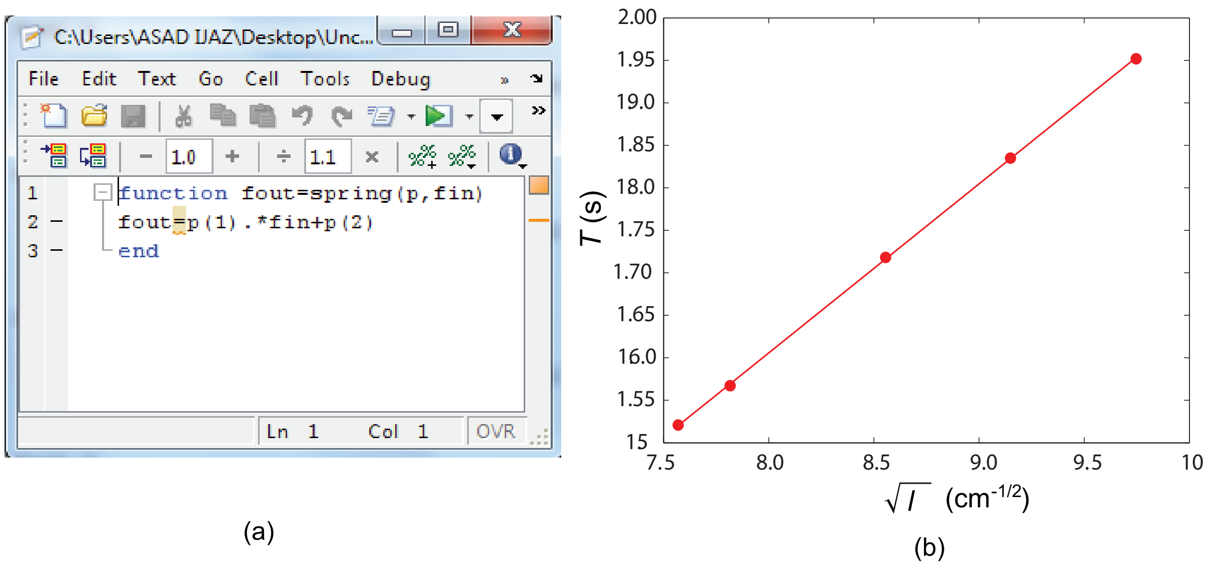

FIG. 4: (a) An M-file for a linear function, (b) Measured data: The initial data points are plotted in red circles while the curve fit is drawn as a solid line.

and intercept. Utilizing the first value yield,

$$\begin{aligned} \text{slope}\,(m) &= 0.1989\ \mathrm{s\,cm^{-1/2}} = (0.1989 \times 10)\ \mathrm{s\,m^{-1/2}}, \\ &= 1.989\ \mathrm{s\,m^{-1/2}} = 2\ \mathrm{s\,m^{-1/2}}. \end{aligned} \tag{20}$$

(c) Graph is shown in figure (5).

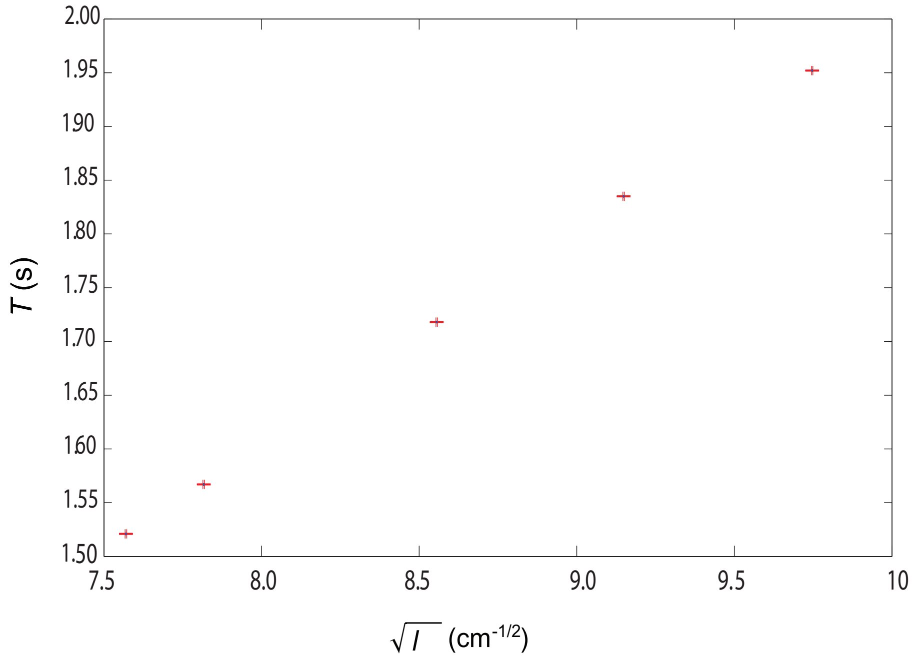

FIG. 5: (a) Graph with error bars both in independent and dependent variables.

Quiz 1(1): Experimental Physics Lab-I
Solution

1. Suppose you measure three independent variables as,

$$x = 20.2 \pm 1.5, \quad y = 8.5 \pm 0.5, \quad \theta = (30 \pm 2)^\circ.$$

Compute the following quantity,

$$q = x(y - \sin\theta).$$

Find the uncertainty in q and quote your results?

Solution:

Given values are:

$$x = 20.2 \pm 1.5, \quad y = 8.5 \pm 0.5, \quad \theta = (30 \pm 2)^\circ.$$

Using the values given above, q becomes,

$$\begin{aligned} q &= x(y - \sin\theta) = 20.2(8.5 - \sin 30), \\ &= 20.2(8.5 - 0.5) = 161.6 \end{aligned}$$

The uncertainty to the inferred quantity q can be find out using the following relationship,

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x}\,\Delta x\right)^2 + \left(\frac{\partial q}{\partial y}\,\Delta y\right)^2 + \left(\frac{\partial q}{\partial \theta}\,\Delta\theta\right)^2}. \tag{1}$$

The first expression on the R.H.S of Equation (1) can be obtained by differentiating q w.r.t x and multiplying with Δx,

$$\begin{aligned} \left(\frac{\partial q}{\partial x}\right)\Delta x &= (y - \sin\theta)\Delta x = (8.5 - \sin 30)(1.5), \\ &= (8.5 - 0.5)1.5 = 12 \end{aligned}$$

Likewise differentiating w.r.t y yields,

$$\left(\frac{\partial q}{\partial y}\right)\Delta y = x\Delta y = (20.2)(0.5) = 10.1$$

and w.r.t θ gives,

$$\begin{aligned}\left(\frac{\partial q}{\partial \theta}\right)\Delta\theta &= (-x\cos\theta)\Delta\theta = (-20.2\times 0.866)\left(\frac{2\pi}{180}\right),\\ &= 0.6103\end{aligned}$$

Substituting the above expressions in Equation (1) results in,

$$\begin{aligned}q &= \sqrt{(12)^2+(10.1)^2+(0.6103)^2},\\ &= 15.69\end{aligned}$$

Finally, the value of q alongwith its uncertainty can be quoted as,

$$q = (162 \pm 16).$$

2. A group of students measures g, the acceleration due to gravity, with a compound pendulum and obtain the following values in units of ms^{-2}.

$$9.81, 9.79, 9.84, 9.81, 9.75, 9.79, 9.82$$

(a) Calculate the mean and the standard deviation. Find the standard uncertainty and quote your results.

(b) How many measurements would you expect to lie outside the 95% confidence interval? How many actually do?

Solution:

(a) The best approximated value of acceleration due to gravity g is,

$$< g > = \frac{\sum_{i=1}^{7} g_i}{n} = \frac{68.61}{7} = 9.80\,\mathrm{m/s^2}. \tag{2}$$

The deviations d_i can be calculated as,

$$d_i = g_i - < g > .$$

The calculated deviations for the measured values of g are tabulated in Table (I).

g (m/s^2)	deviations d_i (m/s^2)	d_i^2 (m^2/s^4)
9.81	0.01	1×10^{-4}
9.79	-0.01	1×10^{-4}
9.84	0.04	1.6×10^{-3}
9.81	0.01	1×10^{-4}
9.75	-0.05	2.5×10^{-3}
9.79	-0.01	1×10^{-4}
9.82	0.02	4×10^{-4}

TABLE I: Table for calculated deviations.

The standard deviation is,

$$s = \sqrt{\frac{\sum_{i=1}^{7} d_i^2}{n}} = \sqrt{\frac{4.9 \times 10^{-3}}{7}} = 0.026\,\mathrm{m/s^2}.$$

The standard uncertainty can be calculated as,

$$\sigma = \sqrt{\frac{n}{n-1}}\,(s) = \sqrt{\frac{7}{6}}\,(0.0264) = 0.0285\,\mathrm{m/s^2},$$

and standard uncertainty in the mean value is,

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{0.0285}{\sqrt{7}} = 0.01\,\mathrm{m/s^2}.$$

To quote the final value of the acceleration due to gravity g alongwith its uncertainty, we will write the expression as,

$$g = (9.80 \pm 0.01)\,\mathrm{m/s^2}.$$

(b) For type A evaluations, the probability distribution function associated with the measurement is a Gaussian probability distribution function as shown in Figure (1). The coverage probability or the confidence of interval for any arbitrary measurand μ corresponding to different standard uncertainties is given as,

$$(<\mu> \pm 1\sigma) \rightarrow (68\%)$$
$$(<\mu> \pm 2\sigma) \rightarrow (95\%)$$
$$(<\mu> \pm 3\sigma) \rightarrow (99\%)$$

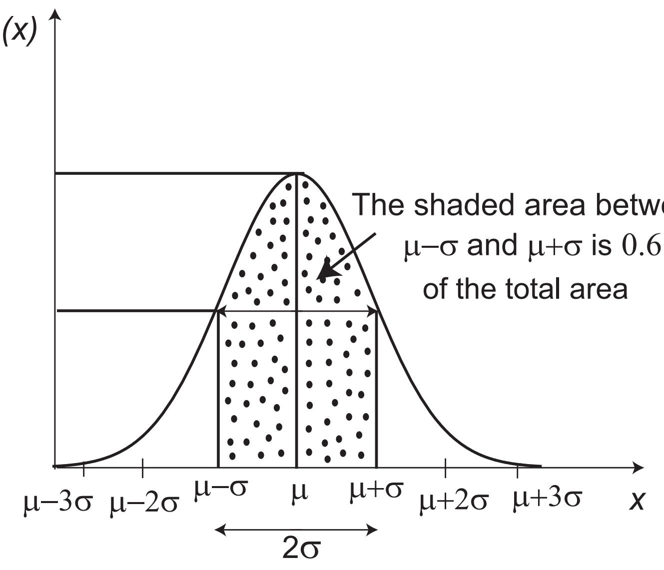

FIG. 1: A Gaussian probability distribution function.

Now as the above expressions predict, the 95% confidence of interval corresponds to $(< g > \pm 2\sigma)$. Considering this, the range of interval becomes,

$$g = (9.80 \pm 2(0.01))\,\mathrm{m/s^2} = (9.80 \pm 0.02)\,\mathrm{m/s^2},$$
$$g = [9.78,\ 9.82]\,\mathrm{m/s^2}.$$

The values (in $\mathrm{m/s^2}$) that lie within 95% confidence of interval are,

$$9.79, 9.81, 9.81, 9.79, 9.82,$$

and values (in $\mathrm{m/s^2}$) which lie outside this interval or lying within 5% confidence of interval are,

$$9.84, 9.75.$$

3. Suppose you measure the mass m of a steel ball (unit in g) using an analog weighing balance. The reading shown on the balance is depicted in Figure (2).

   (a) What is the best approximation of m?

   (b) Find the standard uncertainty associated with the reading?

   (c) What probability distribution function you will use?

   Solution:

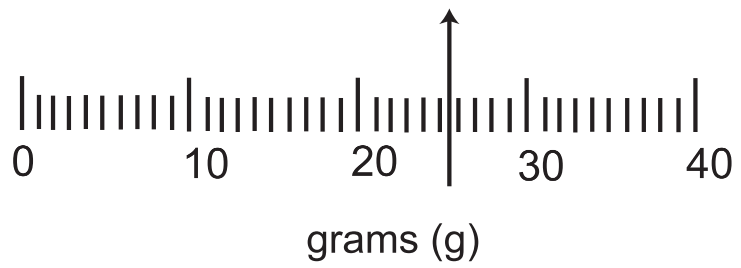

FIG. 2: An analog reading displayed on an analog balance.

(a) The uncertainty is so inherent in nature that even if you take a single reading and don't repeat the experiment, you get an uncertainty. In Figure (2), the scale is an analog scale and the probability distribution function associated with it is a triangular pdf. This is type B uncertainty which comes in due to the limited resolution of the measuring instrument. For evaluating type B uncertainties, one has to use his/her judgment.

Using this idea of judgement, the best approximated value of m being displayed on an analog weighing balance is 25.5 g.

(b) The standard uncertainty associated with the reading can be find out by calculating the second moment (variance) of the probability distribution function. The mathematical relationship for which is given as,

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx,$$

where μ is the mean value where the function is maximum. The standard uncertainty is just the square root of the variance i.e., ($u = \sqrt{\sigma^2}$).

The value of ($\Delta = 0.2\,\text{g}$) is chosen such that the value couldn't be greater than 25.70 g and less than 25.30 g. The scale uncertainty becomes,

$$u_{\text{scale}} = \frac{\Delta}{\sqrt{6}} = \frac{0.2}{\sqrt{6}} = 0.08\,\text{g}.$$

The final value of m alongwith its uncertainty can be quoted as,

$$g = (25.5 \pm 0.1)\, g.$$

(c) The probability distribution function (pdf) will be a triangular pdf with ($\Delta = 0.2$). This shows that probability is maximum at 25.50 g, gradually slopes down, eventually goes to zero at 25.30 g and 25.70 g. The (pdf) associated with reading of an analog weighing balance is sketched in Figure (3).

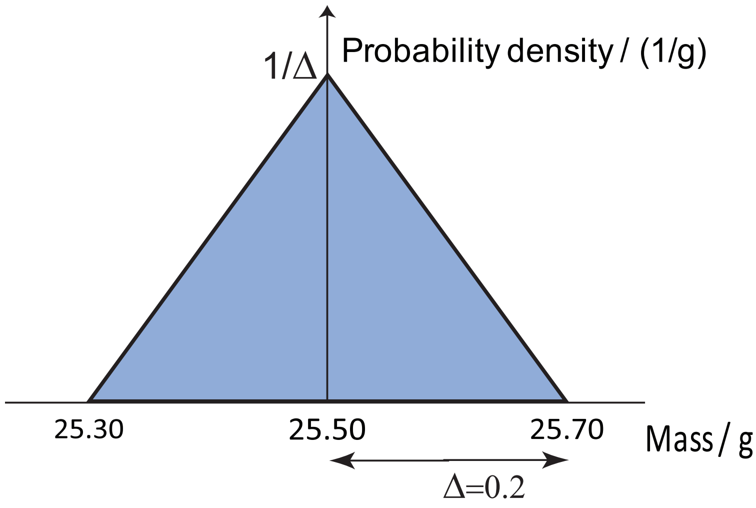

FIG. 3: The probability distribution function associated with an analog reading.

Quiz 1(2): Experimental Physics Lab-I
Solution

1. Suppose you measure three numbers as,

$$x = 20.2 \pm 1.5, \quad y = 8.5 \pm 0.5, \quad \theta = (30 \pm 3)^\circ.$$

Compute the following expression.

$$q = x - y \sin(\theta).$$

Find the uncertainty in q and quote your results.

Solution:

The value of q is,

$$q = [20.2 - (8.5)(\sin 30)] = 15.95 \tag{1}$$

The uncertainty in q can be calculated using the Taylor series approximation (which is a general rule),

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x}\Delta x\right)^2 + \left(\frac{\partial q}{\partial y}\Delta y\right)^2 + \left(\frac{\partial q}{\partial \theta}\Delta \theta\right)^2}. \tag{2}$$

Now differentiating q w.r.t x and multiplying with Δx yields,

$$\left(\frac{\partial q}{\partial x}\right)\Delta x = (1)(1.5) = 1.5 \tag{3}$$

Likewise, the second quantity of Equation (2) becomes,

$$\begin{aligned}\left(\frac{\partial q}{\partial y}\right)\Delta y &= -\sin\theta\Delta y = (-\sin 30)(0.5), \\ &= (-0.5)(0.5) = -0.25 \end{aligned} \tag{4}$$

and,

$$\begin{aligned}\left(\frac{\partial q}{\partial \theta}\right)\Delta \theta &= (-y\cos\theta)(\Delta\theta), \\ &= (-8.5 \times 0.866)\left(\frac{3\pi}{180}\right) = 0.3852 \end{aligned} \tag{5}$$

Substituting the values of Equations (3), (4), (5) in Equation (2) yields,

$$\Delta q \;=\; \sqrt{(1.5)^2 + (0.25)^2 + (0.3852)^2} = 1.56 \tag{6}$$

The final value of q can be quoted as,

$$q \;=\; (16.0 \pm 1.6).$$

2. To calibrate a prism spectrometer, a student sends light of 10 different known wavelengths λ through the spectrometer and measures the angle θ by which each beam is deflected. For one particular value of λ, the student measures θ seven times and obtains these results (in degrees).

$$52.5, 52.3, 52.6, 52.5, 52.7, 52.4, 52.5$$

(a) Find the best estimated value of θ and its standard uncertainty.

(b) How many measurements would you expect to lie outside the 95% confidence interval? How many actually do?

<u>Solution:</u>

(a) The mean value of the measured angle θ is,

$$<\theta> \;=\; \frac{\sum_{i=1}^{7}\theta_i}{n} = \frac{367.5^\circ}{7} = 52.5^\circ \tag{7}$$

The deviations d_i can be found out using the following relationship,

$$d_i \;=\; \theta_i - <\theta> . \tag{8}$$

The calculated deviations of all the measured angles are tabulated in Table (I). The standard deviation s is,

$$s \;=\; \sqrt{\frac{\sum_{i=1}^{7} d_i^2}{n}} = \sqrt{\frac{0.1}{7}} = 0.119^\circ$$

and sandard uncertainty is,

$$\sigma \;=\; \sqrt{\frac{n}{n-1}}\,(s) = \sqrt{\frac{7}{6}}\,(0.119^\circ) = 0.129^\circ$$

Angle θ (degrees)	Deviations d_i (degrees)	d_i^2(degrees^2)
52.5	0.0	0.00
52.3	-0.2	0.04
52.6	0.1	0.01
52.5	0.0	0.00
52.7	0.2	0.04
52.4	-0.1	0.01
52.5	0.0	0.00

TABLE I: Table for experimental results.

The standard uncertainty in the mean value is,

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = 0.048^\circ = 0.04^\circ$$

Finally, the best approximated value of angle θ alongwith its uncertainty is,

$$\theta = (52.50 \pm 0.04).^\circ \tag{9}$$

(b) For type A evaluations, the probability distribution function associated with measurement is a Gaussian probability distribution function as shown in Figure (1). The coverage probability or the confidence of interval for any arbitrary measurand μ corresponding to different standard uncertainties is given as,

$$(<\mu> \pm 1\sigma) \rightarrow (68\%)$$
$$(<\mu> \pm 2\sigma) \rightarrow (95\%)$$
$$(<\mu> \pm 3\sigma) \rightarrow (99\%)$$

For 95% coverage probability, we take twice of the uncertainty i.e., $(<\theta> \pm 2\sigma)$, therefore the range of our data that covers 95% confidence interval becomes,

$$\theta = 52.50 \pm 2(0.04) = (52.50 \pm 0.08),$$
$$\theta = [52.42, 52.58].$$

The values lying within 95% confidence of interval are,

$$52.50, 52.50, 52.50,$$

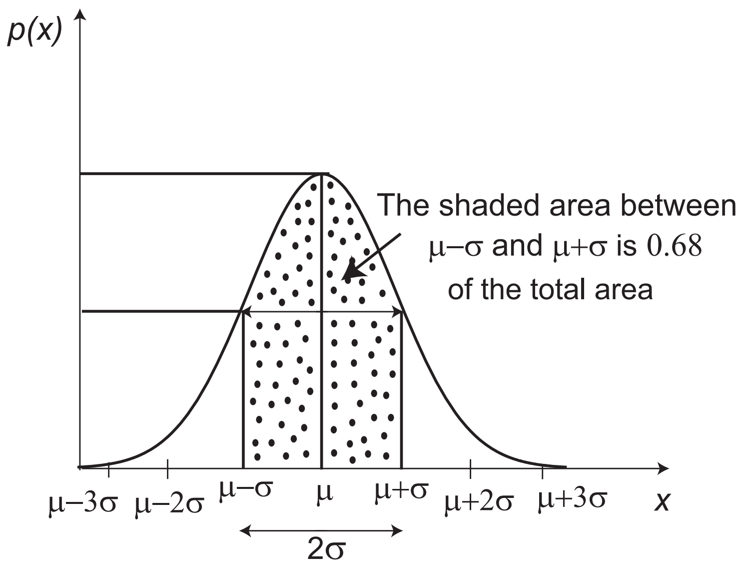

FIG. 1: A Gaussian probability distribution function.

and values lying outside 95% confidence of interval (or within 5%) are,

$$52.30, 52.40, 52.60, 52.70.$$

3. Suppose you measure the value of current I (in mA) using an analog ammeter. The reading displayed on an ammeter is shown in Figure (2).

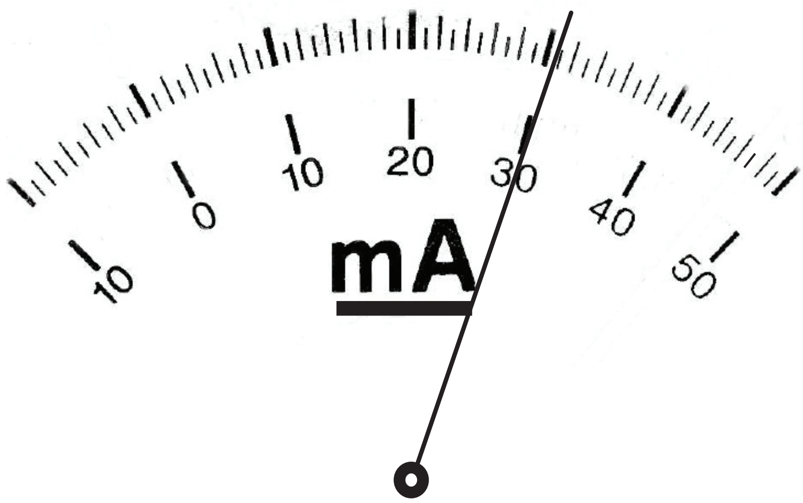

FIG. 2: An analog reading displayed on an analog ammeter.

(a) What is the best approximation of I?

(b) Find the standard uncertainty associated with the reading?

(c) What probability distribution function did you use to model your knowledge about I.

Solution:

(a) The uncertainty is so inherent in nature that even if you take a single reading and don't repeat the experiment, you get an uncertainty. The scale is an analog scale in Figure (2) with a triangular probability distribution function associated with it. This is type B uncertainty which comes in due to the limited resolution of the measuring instrument. For type B evaluations, one has to use his/her judgment. Using this idea of judgement, the best approximated value of I being displayed on an analog ammeter is 30.8 mA.

(b) The second moment (variance) of the probability distribution function is a measure of finding the standard uncertainty. The mathematical relationship for which is given as,

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx,$$

and the standard uncertainty is just the square root of the variance i.e., ($u = \sqrt{\sigma^2}$).

The value of Δ can be chosen such that the value couldn't be greater than 31.0 mA and less than 30.6 mA. The Δ for the reading being displayed in Figure (2) would be 0.2 mA. The scale uncertainty becomes,

$$u_{\text{scale}} = \frac{\Delta}{\sqrt{6}} = \frac{0.2}{\sqrt{6}} = 0.08\,\text{mA}.$$

The final value of I alongwith its uncertainty can be quoted as,

$$I = (30.8 \pm 0.1)\,\text{mA}.$$

(c) The probability distribution function (pdf) will be a triangular pdf with ($\Delta = 0.2$). This shows that the probability is maximum at 30.80 mA, gradually slopes down, eventually goes to zero at 30.60 mA and 31.0 mA. The (pdf) associated with the reading is drawn in Figure (3).

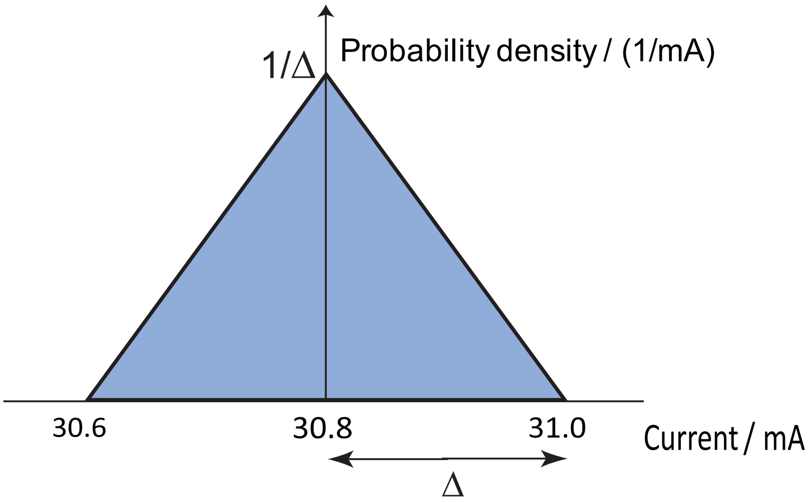

FIG. 3: The probability distribution function associated with an analog reading.

Quiz 2(1): Experimental Physics Lab-I
Solution

1. A student measures the area of a rectangle several times and concludes the standard deviation σ_{A} of the measurements is $\sigma_{\mathrm{A}} = 8\,\mathrm{cm}^2$. If all the uncertainties are truly random then the desired precision can be obtained by making enough measurements and averaging. How many measurements are needed to get a final uncertainty of $\pm 2\,\mathrm{cm}^2$. (10 points)

   Solution:

   The standard deviation of the measured data is $\sigma_{\mathrm{A}} = 8\,\mathrm{cm}^2$.
   The standard uncertainty in the mean value can be find out using the following relationship,

   $$\sigma_m = \frac{\sigma}{\sqrt{n}}. \tag{1}$$

   Rearranging the above expression yields,

   $$\begin{aligned} n &= \left(\frac{\sigma}{\sigma_m}\right)^2, \\ &= \left(\frac{8}{2}\right)^2 = 16. \end{aligned}$$

   Hence we conclude that we need to repeat the measurements 16 times to minimize the final uncertainty upto $\pm 2\,\mathrm{cm}^2$.

2. A students wants to measure the spring constant of a spring. For that the student loads it with various masses m and measures the corresponding lengths l. The mass is measured using a digital weighing balance (rating= 1%) while lengths is measured using a ruler. The results are shown in Table (I).

   The force is,

   $$mg = k(l - l_o), \tag{2}$$

   where l_o is the unstretched length of the spring and k is the spring constant. The data should fit a straight line.

Load m (grams)	200	300	400	500	600
Length l (cm)	5.0	5.5	6.1	6.8	7.4

TABLE I: Data for length versus load.

(a) Calculate uncertainties in the independent and dependent variables. (5 points)

(b) Transfer all the uncertainties in the dependent variable. (5 points)

(c) Calculate the best estimate of the unstretched length l_o and the spring constant k using weighted fit of a straight line. Find uncertainty in spring constant k and quote your final result. (5 points)

Solution:

(a) Equation (2) can be rearranged in the form of a straight line equation and given as,

$$l = \left(\frac{g}{k}\right) m + l_o, \tag{3}$$

where l is the dependent variable, m is the independent variable and the value of spring constant can be calculated by finding the value of slope.

The independent variable (load m) is being measured using a digital weighing balance and by looking at the data (shown in Table I) one can tell that the balance has a resolution of 1 g. Now the scale uncertainty can be calculated by associating a uniform probability distribution function with the reading,

$$u_{\text{Scale}} = \frac{\Delta}{\sqrt{3}} = \frac{0.5}{\sqrt{3}}\,\text{g}. \tag{4}$$

$$= 0.2887 = 0.29\,\text{g}. \tag{5}$$

The rating of the weighing balance is 1%, so the uncertainty associated with rating of the instrument is given as,

$$u_{\text{Rating}}\,@1\% = 0.01 \times \text{value of } (m), \tag{6}$$

while the combined uncertainty in the independent variable can be calculated using the following expression,

$$u_x = \sqrt{(u_{\text{Scale}})^2 + (u_{\text{Rating}})^2}. \tag{7}$$

The uncertainty in the dependent variable (length l) has a triangular probability distribution function associated with it and can be calculated using judgement. For the given data, the uncertainty would be,

$$u_y = \frac{\Delta}{\sqrt{6}} = \frac{0.1}{\sqrt{6}} \text{ cm.} \tag{8}$$

The uncertainties being quoted in independent and dependent variables are shown in Table (II).

(b) The mathematical expressions for transferring uncertainties to the dependent variable and for calculating the total uncertainty are,

$$u_{\text{Trans}} = \left(\frac{dy}{dx}\right) u_x = (0.006)\, u_x, \tag{9}$$

$$u_{\text{Total}} = \sqrt{(u_{\text{Trans}})^2 + u_y^2}. \tag{10}$$

The weights w are reciprocal squares of the total uncertainty which are utilized in least-squares fitting of a straight line. The expression for calculating the weight w is,

$$w = \frac{1}{u_{\text{Total}}^2}\,. \tag{11}$$

Now the uncertainties calculated for the given data of independent (load m) and dependent (length l) variables are shown in Table (II).

Load m (g)	u_x (g)	Length l (cm)	u_y (cm)	u_{Trans} (cm)	u_{Total} (cm)	Weights w (cm^{-2})
200	2	5.00	0.04	0.012	0.043	549.87
300	3	5.50	0.04	0.018	0.044	498.84
400	4	6.10	0.04	0.024	0.047	441.48
500	5	6.80	0.04	0.030	0.051	384.61
600	6	7.40	0.04	0.036	0.054	332.30

TABLE II: Results for calculated uncertainties and weights.

(c) The mathematical relationships for slope (m) and intercept (c) are,

$$m = \frac{\Sigma_i w_i\, \Sigma_i w_i(x_i y_i) - \Sigma_i(w_i x_i)\, \Sigma_i(w_i y_i)}{\Sigma_i w_i\, \Sigma_i(w x_i^2) - (\Sigma_i w_i x_i)^2}\,, \tag{12}$$

and,

$$c = \frac{\Sigma_i(w_i x_i^2)\,\Sigma_i(w_i y_i) - \Sigma_i(w_i x_i)\,\Sigma_i(w_i x_i y_i)}{\Sigma_i w_i\,\Sigma_i(w_i x_i^2) - (\Sigma_i w_i x_i)^2}, \tag{13}$$

where x is the independent variable (load m in our case), y is the dependent variable (length l) and w is the weight.

The different terms in the numerator and denominator of Equations (12) and (13) are calculated separately and tabulated in Table (III).

$w\,(\mathrm{cm}^{-2})$	$wml\,(\mathrm{cm}^{-1}\mathrm{g})$	$wm\,(\mathrm{cm}^{-2}\mathrm{g})$	$wl\,(\mathrm{cm}^{-1})$	$wm^2\,(\mathrm{cm}^{-2}\mathrm{g}^2)$
549.87	0.5499×10^6	1.0997×10^5	2.7494×10^3	0.2199×10^8
498.84	0.8231×10^6	1.4965×10^5	2.7436×10^3	0.4490×10^8
441.48	1.0772×10^6	1.7659×10^5	2.6930×10^3	0.7064×10^8
384.61	1.3077×10^6	1.9231×10^5	2.6154×10^3	0.9615×10^8
332.30	1.4754×10^6	1.9938×10^5	2.4590×10^3	1.1963×10^8
$\sum w = 2207.1$	$\sum wml = 5.2333 \times 10^6$	$\sum wm = 8.2790 \times 10^5$	$\sum wl = 1.3260 \times 10^4$	$\sum wm^2 = 3.5331 \times 10^8$

TABLE III: Terms being utilized in finding the values of slope and intercept.

Substituting the calculated terms given in Table (III) in Equation (12) yields,

$$\begin{aligned} m &= \frac{(2207.1)(5.2333 \times 10^6) - (8.2790 \times 10^5)(1.3260 \times 10^4)}{(2207.1)(3.5331 \times 10^8) - (8.2790 \times 10^5)^2}, \\ &= 0.0061\,\mathrm{cm\,g^{-1}}. \end{aligned}$$

and the intercept (c) is given as,

$$\begin{aligned} c &= \frac{(3.5331 \times 10^8)(1.3260 \times 10^4) - (8.2790 \times 10^5)(5.2333 \times 10^6)}{(2207.1)(3.5331 \times 10^8) - (8.2790 \times 10^5)^2}, \\ &= 3.73\,\mathrm{cm}. \end{aligned}$$

The expressions for the uncertainties in slope (m) and intercept (c) are,

$$u_m = \sqrt{\frac{\Sigma_i w_i}{\Sigma_i w_i\,\Sigma_i(w_i x_i^2) - (\Sigma_i w_i x_i)^2}}\,, \tag{14}$$

$$u_c = \sqrt{\frac{\Sigma_i(w_i x_i^2)}{\Sigma_i w_i\,\Sigma_i(w_i x_i^2) - (\Sigma_i w_i x_i)^2}}\,. \tag{15}$$

Substituting the values yield,

$$u_m = \sqrt{\frac{(2207.1)}{(2207.1)(3.5331 \times 10^8) - (8.2790 \times 10^5)^2}},$$
$$= 1.529 \times 10^{-4}\,\text{cm g}^{-1}. \quad (16)$$

and expression for finding the uncertainty in the intercept (c) value becomes,

$$u_c = \sqrt{\frac{(3.5331 \times 10^8)}{(2207.1)(3.5331 \times 10^8) - (8.2790 \times 10^5)^2}},$$
$$= 0.0612\,\text{cm}. \quad (17)$$

Finally, the values of slope and intercept alongwith uncertainties are quoted as,

$$m = (6.1 \pm 0.2) \times 10^{-3}\,\text{cm g}^{-1}. \quad (18)$$

$$c = (3.73 \pm 0.06)\,\text{cm}. \quad (19)$$

The spring constant can be find out utilizing the slope value calculated above and given as,

$$\text{Slope}\,(m) = 6.1 \times 10^{-3}\,\text{cm g}^{-1} = \left(\frac{g}{k}\right),$$

implying,

$$k = \left(\frac{g}{m}\right) = \left(\frac{9.8\,\text{m/s}}{(6.1 \times 10^{-3} \times 10^{-2} \times 10^3)\,\text{m kg}^{-1}}\right),$$
$$= 161\,\text{N/m}. \quad (20)$$

The uncertainty in spring constant k can be find out using Taylor series,

$$\Delta k = \sqrt{\left(\frac{\partial k}{\partial m}\Delta m\right)^2} = \sqrt{\left(\frac{-g\Delta m}{m^2}\right)^2},$$
$$= \sqrt{\left(\frac{-(9.8)(0.2 \times 10^{-2})}{(6.1 \times 10^{-2})^2}\right)^2} = 5.3\,\text{N/m}.$$

Hence the best estimated values of spring constant (k) and the unstretched length (l_o) are concluded as,

$$k = (161 \pm 5)\,\text{N/m} \quad (21)$$

$$l_o = (3.73 \pm 0.06) \times 10^{-2}\,\text{m}. \quad (22)$$

Quiz 2(2): Experimental Physics Lab-I
Solution

1. A student measures the speed of sound as $u = f\lambda$, where f is the frequency shown on the dial of an audio oscillator and λ is the wavelength measured by locating several maxima in the resonant air column. The best estimated value of λ is find out by repeating the measurements and evaluating them statistically. The value concluded by the student is $\lambda = 11.1 \pm 0.4\,\text{cm}$. Only one measurement of frequency $f = 3000\,\text{Hz}$ has been taken. Further, the student is told that the oscillator is 1% reliable.

   (a) What would be student's uncertainty in u? (5 points)

   (b) Is the possible 1% systematic uncertainty of the oscillator's calibration important? (5 points)

Solution:

(a) Given values:

$$\begin{aligned}\lambda &= (11.1 \pm 0.4)\,\text{cm},\\ f &= 3000\,\text{Hz}.\end{aligned}$$

The reliability of the oscillator is 1% for frequency measurement.
The speed of sound can be calculated by utilizing the given values of frequency and wavelength.

$$\begin{aligned}u &= (3000)(11.1) = 33300\,\text{cm/s},\\ &= 333\,\text{m/s}.\end{aligned}$$

The uncertainty in u can be calculated as,

$$\begin{aligned}\Delta u &= \sqrt{\left(\frac{\partial u}{\partial f}\Delta f\right)^2 + \left(\frac{\partial u}{\partial \lambda}\Delta\lambda\right)^2},\\ &= \sqrt{\left(\lambda\Delta f\right)^2 + \left(f\Delta\lambda\right)^2},\\ &= \sqrt{\left[(11.1\times10^{-2})\left(\frac{1}{100}\times 3000\right)\right]^2 + \left[(3000)(0.4\times10^{-2})\right]^2},\\ &= 12.4\,\text{m/s}\end{aligned}$$

Hence the value of speed of sound alongwith its uncertainty can be quoted as,

$$u = (333 \pm 12)\,\text{m/s} \tag{1}$$

(b) The frequency measured by the audio oscillator is reliable upto 1% (i.e $\Delta f = 1\%$), while the fractional uncertainty of the measured wavelength λ can be calculated as,

$$\Delta\lambda = \left(\frac{0.4}{11.1}\right) \times 100 = 4\%,$$

which is greater than the 1% systematic uncertainty of the oscillator's calibration. Hence the 1% systematic uncertainty in frequency f is negligible beside the 4% uncertainty in wavelength λ.

2. Steel balls of different masses m are dropped from different heights into a container of sand. The impact of the ball on sand in called a crater and its diameter is measured using a plastic rule. The relationship between the diameter of the crater and the kinetic energy of the impacting object is given as,

$$D = c\,E^n, \tag{2}$$

where c is a constant, D is the diameter and E is the kinetic energy that can be calculated by assuming that all the kinetic energy possessed by a ball at a height h is converted into potential energy before impact. The data is given in Table (II).

Mass m (g)	**Height h (cm)**	**Crater diameter D (cm)**
8.4	26	3.9
28.2	26	5.2
66.8	26	6.3
66.8	68	7.9
66.8	150	10.2

TABLE I: Experimental data for crater formation.

(a) Calculate uncertainties in the independent and dependent variables. The mass is measured using a digital weighing balance (rating= 1%), the height and diameter is measured using a ruler (an analog device). (5 points)

(b) Using the transformation rule, transfer all the uncertainties to the dependent variable. (5 points)

(c) Calculate the best estimate of n using weighted fit of a straight line. Calculate uncertainty in n as well. (5 points)

Solution

(a) By taking log on both sides of Equation (2) yields a straight line equation given as,

$$\log(D) \;=\; n\log(E) + \log, \tag{3}$$

where $(\log D)$ is the dependent variable, $(\log E)$ is the independent variable and the value of n which is energy dissipation mechanism, can be calculated by finding the value of slope.

The mass m of each steel ball is measured using a digital weighing balance and by looking at the data (shown in Table I), one can tell that the balance has a resolution of 0.1 g. Now the scale uncertainty can be calculated by associating a uniform probability distribution function with the reading and given as,

$$u_{\text{Scale}} \;=\; \frac{\Delta}{\sqrt{3}} = \frac{0.05}{\sqrt{3}}\,\text{g}. \tag{4}$$

$$= \; 0.0289 = 0.03\,\text{g}. \tag{5}$$

Since the rating of the weighing balance is 1%, uncertainty associated with rating of the instrument is given as,

$$u_{\text{Rating}}\,@1\% \;=\; 0.01 \times \text{value of } (m), \tag{6}$$

while the combined uncertainty in mass m can be calculated using the following expression,

$$u_m \;=\; \sqrt{(u_{\text{Scale}})^2 + (u_{\text{Rating}})^2}. \tag{7}$$

The uncertainty in height h has a triangular probability distribution function associated with it and can be calculated using judgement. For the given data, the uncertainty would be,

$$u_h \;=\; \frac{\Delta}{\sqrt{6}} = \frac{1}{\sqrt{6}} = 0.4082 = 0.41\text{cm}. \tag{8}$$

The uncertainty in energy ($E = mgh$) can be calculated using Taylor series approximation,

$$\Delta E = \sqrt{\left(\frac{\partial E}{\partial m}\Delta m\right)^2 + \left(\frac{\partial E}{\partial h}\Delta h\right)^2},$$

while the uncertainty associated with the diameter of the crater D is,

$$u_D = \frac{\Delta}{\sqrt{6}} = \frac{0.1}{\sqrt{6}} = 0.0408 = 0.04\text{cm}$$

By looking at Equation (3), we can easily tell that the dependent variable is $\log(D)$, while the independent one is $\log(E)$. Now the question is how to propagate uncertainties from D and E to $\log(D)$ and $\log(E)$, respectively. For that we'll utilize the general rule of propagation,

$$u_x = \Delta(\log E) = \sqrt{\left(\frac{\partial(\log E)}{\partial E}\Delta E\right)^2} = \frac{\Delta E}{E},$$

$$u_y = \Delta(\log D) = \sqrt{\left(\frac{\partial(\log D)}{\partial E}\Delta D\right)^2} = \frac{\Delta D}{D}.$$

Uncertainties in all the measured and inferred quantities are quoted in Table (II).

Mass m (g)	Δm (g)	Height h (cm)	Δh (cm)	Energy E (J)	ΔE (J)	Crater diameter $D(\times 10^{-2})$ (m)	ΔD $\times 10^{-2}$ m	$\log(E)$	$\Delta(\log E)$	$\log(D)$	$\Delta(\log D)$
8.4	0.1	26.0	0.4	0.0214	0.0004	3.90	0.04	-3.844	0.018	-3.244	0.010
28.2	0.2	26.0	0.4	0.072	0.001	5.20	0.04	-2.633	0.018	-2.956	0.008
66.8	0.6	26.0	0.4	0.170	0.003	6.30	0.04	-1.771	0.018	-2.765	0.006
66.8	0.6	68.0	0.4	0.445	0.005	7.90	0.04	-0.809	0.011	-2.538	0.005
66.8	0.6	150.0	0.4	0.982	0.010	10.20	0.04	-0.018	0.010	-2.283	0.004

TABLE II: Experimental data and calculated uncertainties for crater formation.

(b) The mathematical expressions for transferring uncertainties to the dependent variable ($\log D$) and for calculating the total uncertainty are,

$$u_{\text{Trans}} = \left(\frac{dy}{dx}\right) u_x = (0.25)\, u_x, \tag{9}$$

$$u_{\text{Total}} = \sqrt{(u_{\text{Trans}})^2 + u_y^2}. \tag{10}$$

The weights w are reciprocal squares of the total uncertainty being utilized in least-squares fitting of a straight line. The expression for calculating the weight w is,

$$w = \frac{1}{u_{\text{Total}}^2} . \tag{11}$$

The uncertainties calculated for the given data of independent ($\log E$) and dependent variables ($\log D$) are shown in Table (III).

$\log(E)$	$\log(D)$	u_{Trans}	u_{Total}
-3.844	-3.244	0.004	0.011
-2.633	-2.956	0.004	0.009
-1.771	-2.765	0.004	0.008
-0.809	-2.538	0.002	0.005
-0.018	-2.283	0.002	0.004

TABLE III: Calculated data for transfer and total uncertainties.

(c) The mathematical relationships for slope (m) and intercept (c) are,

$$m = \frac{\Sigma_i w_i \, \Sigma_i w_i (x_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i y_i)}{\Sigma_i w_i \, \Sigma_i (w x_i^2) - (\Sigma_i w_i x_i)^2} , \tag{12}$$

and,

$$c = \frac{\Sigma_i (w_i x_i^2) \, \Sigma_i (w_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i x_i y_i)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}, \tag{13}$$

where x is the independent variable ($\log E$ in our case), y is the dependent variable ($\log D$) and w is the weight.

The different terms in the numerator and denominator of Equations (12) and (13) are calculated separately and tabulated in Table (IV).

Substituting the calculated terms given in Table (IV) in Equation (12) yields,

$$\begin{aligned} m &= \frac{(107640)(324940) - (-112310)(-275910)}{(107640)(263010) - (1.2613 \times 10^{10})}, \\ &= 0.254 \end{aligned}$$

and the intercept (c) is given as,

$$\begin{aligned} c &= \frac{(263010)(-275910) - (-112310)(324940)}{(107640)(263010) - (1.2613 \times 10^{10})}, \\ &= -2.298 \end{aligned}$$

w	wxy	wx	wy	wx^2
0.7577×10^4	9.4498×10^4	-2.9128×10^4	-0.2458×10^5	1.1198×10^5
1.1996×10^4	9.3386×10^4	-3.1587×10^4	-0.3547×10^5	0.8317×10^5
1.5708×10^4	7.6896×10^4	-2.7814×10^4	-0.4343×10^5	0.4925×10^5
2.8393×10^4	5.8329×10^4	-2.2980×10^4	-0.7207×10^5	0.1860×10^5
4.3967×10^4	0.1827×10^4	-0.0800×10^4	-1.0037×10^5	0.0001×10^5
$\sum w = 107640$	$\sum wxy = 324940$	$\sum wx = -112310$	$\sum wy = -275910$	$\sum wx^2 = 263010$

TABLE IV: Terms being utilized in finding the values of slope and intercept.

The expressions for the uncertainties in slope (m) and intercept (c) are,

$$u_m = \sqrt{\frac{\Sigma_i w_i}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}\,, \tag{14}$$

$$u_c = \sqrt{\frac{\Sigma_i (w_i x_i^2)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}\,. \tag{15}$$

Substituting the values from Table (IV) in the above expressions yield,

$$u_m = \sqrt{\frac{(107640)}{(107640)(263010) - (1.2613 \times 10^{10})}} = 0.0026, \tag{16}$$

$$u_c = \sqrt{\frac{(263010)}{(107640)(263010) - (1.2613 \times 10^{10})}} = 0.0041. \tag{17}$$

Finally, the values of slope and intercept alongwith uncertainties are quoted as,

$$m = (0.254 \pm 0.002). \tag{18}$$

$$c = (-2.298 \pm 0.004). \tag{19}$$

Since the value of slope directly corresponds to energy dissipation mechanism n, hence we can conclude,

$$n = (0.254 \pm 0.002). \tag{20}$$

Quiz 3: Experimental Physics Lab-I
Solution

1. Suppose we have to measure the area A of a rectangular plate. We make several measurements of the length l and width w of the plate at different positions and results are shown in Table (I).

l (mm)	24.25	24.26	24.22	24.28	24.24	24.25	24.22
w (mm)	50.36	50.35	50.41	50.37	50.36	50.32	50.39

TABLE I: Measured values of length and width in (mm).

(a) Based on the data provided in Table (I), find the best estimated values of length l and width w along with uncertainties in the mean. (5 points)

(b) Suppose we have been told that the vernier calliper used to measure the length l and width w have systematic uncertainties upto 1%. Calculate the systematic uncertainty in l and w. Quote your final answer for the length l and width w. (5 points)

(c) Based on your results from the previous two parts, calculate the best estimate for area ($A = lw$) and its uncertainty. (5 points)

Solution:

(a) In any experiment, the measured values are repeated to reduce random uncertainty. This is type A uncertainty with a Gaussian probability distribution function associated with it and can be evaluated statistically.
The best approximated value of length l and width w are calculated as,

$$< l > = \frac{\sum_{i=1}^{7} l_i}{n} = \frac{169.72}{7} = 24.24\,\text{mm}, \tag{1}$$

$$< w > = \frac{\sum_{i=1}^{7} w_i}{n} = \frac{352.56}{7} = 50.36\,\text{mm}. \tag{2}$$

The deviations d_i can be calculated as,

$$d_i = l_i - < l > .$$

Likewise for the width w. The calculated deviations for the measured values of length l and width w are tabulated in Table (II).

Length l (mm)	Deviations of length l d_i (mm)	d_i^2 (length) (mm^2)	Width w (mm)	Deviations of width w d_i (mm)	d_i^2 (width) (mm^2)
24.25	0.01	0.0001	50.36	0.00	0.0000
24.26	0.02	0.0004	50.35	-0.01	0.0001
24.22	-0.02	0.0004	50.41	0.05	0.0025
24.28	0.04	0.0016	50.37	0.01	0.0001
24.24	0.00	0.0000	50.36	0.00	0.0000
24.25	0.01	0.0001	50.32	-0.04	0.0016
24.22	-0.02	0.0004	50.39	0.03	0.0009

TABLE II: Results for calculated deviations in length and width.

The standard deviation for length measurements is,

$$\begin{aligned} s_{\text{length}} &= \sqrt{\frac{\sum_i d_i^2}{n}} = \sqrt{\frac{0.0030}{7}}, \\ &= 0.021\,\text{mm}, \end{aligned}$$

and for width is,

$$\begin{aligned} s_{\text{width}} &= \sqrt{\frac{\sum_i d_i^2}{n}} = \sqrt{\frac{0.0052}{7}}, \\ &= 0.027\,\text{mm}. \end{aligned}$$

The standard error σ can be find out using the following relationship,

$$\sigma = \sqrt{\frac{n}{n-1}}\,(s).$$

However for the set of measured data, the standard error is,

$$\begin{aligned} \sigma_{\text{length}} &= \sqrt{\frac{7}{7-1}}\,(0.021), \\ &= 0.023\,\text{mm}. \end{aligned}$$

$$\sigma_{\text{width}} = \sqrt{\frac{7}{7-1}}\,(0.027), \\ = 0.029\,\text{mm}.$$

The standard error in the mean is,

$$\sigma_m(\text{length}) = \frac{\sigma}{\sqrt{n}} = \frac{0.023}{\sqrt{7}}, \\ = 0.008\,\text{mm}.$$

$$\sigma_m(\text{width}) = \frac{0.029}{\sqrt{7}}, \\ = 0.011\,\text{mm}.$$

Therefore, the best estimated values of length and width alongwith their uncertainties are,

$$l = (24.240 \pm 0.008)\,\text{mm}, \\ b = (50.36 \pm 0.01)\,\text{mm}.$$

(b) The systematic uncertainties in length l and width w are upto 1%, hence we can deduce,

$$u_{\text{Systematic}\,(l)} = 0.01 \times 24.24 = 0.2424\,\text{mm}, \\ u_{\text{Systematic}\,(w)} = 0.01 \times 50.36 = 0.5036\,\text{mm},$$

Since both type A and type B uncertainties contribute towards the total uncertainty therefore they must be combined in quadrature according to the prescription,

$$u_{\text{Total}} = \sqrt{u_A^2 + u_B^2}.$$

The total uncertainty in length l and width w can be calculated as,

$$u_{\text{Total}}\,(\text{length}\,l) = \sqrt{\sigma_l^2 + u_{\text{Systematic}\,(l)}^2}, \\ = \sqrt{(0.008)^2 + (0.2424)^2}, \\ = 0.2425\,\text{mm}.$$

Likewise for width w is,

$$\begin{aligned} u_{\text{Total}}\,(\text{width}\,w) &= \sqrt{\sigma_l^2 + u^2_{\text{Systematic}\,(w)}}\,, \\ &= \sqrt{(0.01)^2 + (0.5036)^2}\,, \\ &= 0.5037\,\text{mm}. \end{aligned}$$

Finally we can conclude that,

$$\begin{aligned} l &= (24.24 \pm 0.24)\,\text{mm}, \\ w &= (50.36 \pm 0.50)\,\text{mm}. \end{aligned}$$

(c) The area of a rectangular plate is,

$$\begin{aligned} A &= l \times w, \\ &= (24.24)(50.36), \\ &= 1220.7\,\text{mm}^2. \end{aligned}$$

The uncertainty in area A can be calculated using Taylor series approximation,

$$\begin{aligned} \Delta A &= \sqrt{\left(\frac{\partial A}{\partial l}\,\Delta l\right)^2 + \left(\frac{\partial A}{\partial w}\,\Delta w\right)^2}\,, \\ &= \sqrt{(w\Delta l)^2 + (l\Delta w)^2}, \\ &= \sqrt{(50.36 \times 0.24)^2 + (24.24 \times 0.50)^2}, \\ &= 17.12\,\text{mm}^2. \end{aligned}$$

Finally, the best estimated value of area can be quoted as,

$$A = (1221 \pm 17)\,\text{mm}^2.$$

2. The radioactivity is statistical in nature and follows a Poisson distribution. The rate of decay of a radioactive material decreases exponentially. To measure this rate a Geiger Muller tube is place near the radioactive sample and the GM counter records the number of decays. The data is shown in Table (III).

Elapsed time t (min)	10	20	30	40	50
Total number of decays since the total elapsed time ($v(t)$)	409	304	260	192	170

TABLE III: The data for the number of decays versus time elapsed.

The sample decays exponentially and the number of decays in the elapsed time are,

$$v(t) \;=\; v_o\, e^{-t/\tau}, \tag{3}$$

where τ is the mean life and v_o is an unknown constant.

(a) Calculate the best estimate for mean life τ using the mathematical expression of least-squares fit of a straight line. (10 points)

(b) Calculate uncertainty in τ and quote your final result. (5 points)

Solution:

(a) To find the value of the mean life τ, we need to linearize Equation (3) given as,

$$\log v(t) \;=\; \log(v_o) - \frac{t}{\tau}\,. \tag{4}$$

which is a straight line equation with $(\log v(t))$ as a dependent variable, elapsed time t as an independent variable, $(1/\tau)$ is the slope and $\log(v_o)$ is the intercept. The best approximated value of slope can be deduced using the relationship,

$$m \;=\; \frac{\sum_i^N y_i(x_i - \bar{x})}{\sum_i^N (x_i - \bar{x})^2}, \tag{5}$$

where x is the independent variable (elapsed time t) and y is the dependent one (decays $\log v(t)$).

The expression for calculating the intercept c is,

$$c \;=\; \bar{y} - m\bar{x}. \tag{6}$$

Here,

$$\bar{x} = \frac{1}{n}\sum_i x_i, \qquad \bar{y} = \frac{1}{n}\sum_i y_i.$$

This means that the straight line passes through the points $\bar{x}$ and $\bar{y}$ which are the center of gravity of all the points. Now we will tabulate the different terms being utilized in the above expressions and shown in Table (IV).

Time t (min)	Decays $v(t)$	$\log v(t)$	$(t_i - \bar{t})$	$y_i(t_i - \bar{t})$	$(t_i - \bar{t})^2$	d_i	d_i^2
10	409	6.0137	-20	-120.2743	400	0.0337	0.0011
20	304	5.7170	-10	-57.1703	100	-0.0415	0.0017
30	260	5.5607	0	0	0	0.0237	0.0006
40	192	5.2575	10	52.5750	100	-0.0579	0.0034
50	170	5.1358	20	102.7160	400	0.0419	0.0018
$\bar{t}$=30	$\bar{v(t)}$=267	$\bar{\log v(t)}$=5.5369	avg=0	$\sum=-22.1537$	$\sum$=1000	$\sum=-8.88\times 10^{-16}$	$\sum$=0.0085

TABLE IV: Terms being utilized in finding the values of slope and intercept.

The mean values of measured elapsed time t and decays $\log v(t)$ is,

$$\begin{aligned}
\bar{t} &= \frac{\sum_i t_i}{n} = \frac{10+20+30+40+50}{5} \\
&= 30\,\text{min}. \\
[\bar{\log v(t)}] &= \frac{\sum_i [\log v(t)]_i}{n}, \\
&= \frac{6.014+5.717+5.561+5.258+5.136}{5} \\
&= 5.537.
\end{aligned}$$

The best value of slope can be deduced by substituting values from Table (IV) in expression (5),

$$m = \frac{(-22.1537)}{1000} = -0.022\,\text{min}^{-1}.$$

After calculating the slope value, we can easily find out the intercept value using Equation (6),

$$\begin{aligned}
c &= [\bar{\log v(t)}] - m\,\bar{t} = 5.5369 - (-0.0222\times 30), \\
&= 6.203.
\end{aligned}$$

The value of the slope is related to the mean life τ through the following relationship,

$$\text{Slope}\,(m) = \frac{1}{\tau},$$

implying,

$$\tau = \frac{1}{(\text{Slope m})} = \frac{1}{0.022} = 45.45\,\text{min}.$$

(b) The uncertainty in slope can be calculated as,

$$\Delta m = \sqrt{\frac{\sum_i^N d_i^2}{D(N-2)}},$$

where $(d_i = y_i - mx_i - c)$ is the deviation of each point from the best fit straight line called residuals, and $(D = \sum_i^N (x_i - \bar{x})^2)$.
Substituting values from Table (IV) into the above expression yields,

$$\Delta m = \sqrt{\frac{0.0085}{(1000)(5-2)}} = 0.002\,\text{min}^{-1}.$$

Now the uncertainty in the mean life τ can be find out by Taylor series approximation,

$$\Delta\tau = \sqrt{\left(\frac{\partial\tau}{\partial m}\Delta m\right)^2} = \sqrt{\left(\frac{-\Delta m}{m^2}\right)^2},$$

$$= \sqrt{\left(\frac{0.002}{(0.022)^2}\right)^2} = 4.13\,\text{min}$$

Hence the final value of mean life τ can be quoted as,

$$\tau = (45 \pm 4)\,\text{min}.$$

3. Neutrons reflected by a crystal obey Bragg's law $n\lambda = 2d\sin\theta$, where λ is the de Broglie wavelength of the neutrons $(\lambda \propto \sin\theta)$, d is the spacing between the reflecting planes of atoms in the crystal, θ is the angle between the incident (or reflected) neutrons and the atomic planes, and n is an integer. If n and d are known, the measured value of θ for a beam of monochromatic neutrons determines λ, and hence the kinetic energy $E\,(\propto (\text{momentum})^2 \propto (1/\lambda^2))$ of the neutrons. If $(\theta = (18 \pm 1)^\circ)$, what is the fractional uncertainty in E? (5 points)

<ins>**Solution:**</ins>

The wavelength of the neutrons (assuming, n, $d = 1$) is given as,

$$\begin{aligned}\lambda \propto \sin\theta &= \sin\theta = \sin(18), \\ &= 0.3090\,\text{m}.\end{aligned}$$

The uncertainty in λ can be calculated using the generic rule,

$$\begin{aligned}\Delta\lambda &= \sqrt{\left(\frac{\partial\lambda}{\partial\theta}\,\Delta\theta\right)^2} = (\cos\theta)\Delta\theta, \\ &= \cos(18)\left(\frac{1(\pi)}{180}\right) = (0.9511)(0.0174), \\ &= 0.0166\,\text{m}.\end{aligned}$$

The calculated value of λ can be quoted as,

$$\lambda = (0.31 \pm 0.02)\,\text{m}.$$

The fractional uncertainty in λ is,

$$\frac{\Delta\lambda}{\lambda} = \frac{0.02}{0.31} \times 100 = 6.5\%. \tag{7}$$

The kinetic energy of the neutrons is

$$\begin{aligned}E &\propto (\text{momentum})^2 \propto \left(\frac{1}{\lambda^2}\right), \\ &= \left(\frac{1}{\lambda^2}\right) = \frac{1}{(0.31)^2}, \\ &= 10.41\,\text{J}.\end{aligned}$$

The uncertainty in E can be find out using Taylor series and given as,

$$\begin{aligned}\Delta E &= \sqrt{\left(\frac{\partial E}{\partial\lambda}\,\Delta\lambda\right)^2} = \sqrt{\left(\frac{-2\Delta\lambda}{\lambda^3}\right)^2} \\ &= \sqrt{\left(\frac{-2(0.02)}{(0.31)^3}\right)^2} = 1.346\,\text{J}.\end{aligned}$$

Hence we conclude,

$$E = (10.41 \pm 1.3)\,\text{J}.$$

The fractional uncertainty in energy E is given as,

$$\frac{\Delta E}{E} = \frac{1.3}{10.41} \times 100 = 12.5\%. \tag{8}$$

Comparing Equations (7) and (8) yields that fractional uncertainty in E is almost double than that in the wavelength λ.

Assignment : Error analysis and data processing

(Due date: October. 7, 2011, 11 am)

1. Suppose you measure four numbers as:

$$x = 200 \pm 2, \quad y = 50 \pm 2, \quad z = 20 \pm 1, \quad u = 3 \pm 0.1,$$

where the uncertainties are independent and random. What would you give values to the following quantities with their uncertainties?

(a) $q = x/(y - z)$.

(b) $p = e^u$.

(c) $r = x(y - z\sin(u))$.

2. A student measures g, the acceleration of gravity, using a simple pendulum. The period is well known to be $T = 2\pi\sqrt{l/g}$, where l is the length of this pendulum. If l and T are measured as,

$$l = 92.95 \pm 0.01 \text{ cm},$$
$$T = 1.936 \pm .004 \text{ s},$$

calculate the best estimate of g and its uncertainty.

3. The resistance of a coil is measured in ohms (Ω), and the following set of data is obtained,

$$4.615, 4.638, 4.597, 4.634, 4.613, 4.623, 4.659, 4.623.$$

Find the best estimated value and standard error in the mean.

4. Suppose we have to measure accurately the area A of a rectangular plate approximately 2.5 cm×5 cm. We make several measurements of the length l and breadth b of the plate at different positions. We make 10 measurements for length and breadth and the results are shown in Table (I).

Find the best estimated values of length l and breadth b along with standard error in the mean. Calculate the best estimate for area ($A = lb$) and its uncertainty.

l **(mm)**	24.25	24.26	24.22	24.28	24.24	24.25	24.22	24.26	24.23	24.24
b **(mm)**	50.36	50.35	50.41	50.37	50.36	50.32	50.39	50.38	50.36	50.38

TABLE I: Measured values of length and breadth in (mm).

5. In an experiment of measuring absolute zero with a constant volume gas thermometer, and if the volume of an ideal gas is kept constant, the relationship between temperature and pressure is,

$$T = mP + c,$$

where c is the temperature at which the pressure drops to zero, called the absolute zero of temperature. A set of five measurements of temperatures T with different pressure P is taken as given in Table (II).

Pressure (mm of mercury)	65	75	85	95	105
Temperature (°C)	−20	17	42	94	127

TABLE II: Pressure and temperature of a gas at constant volume.

Calculate the best estimate for slope and intercept using mathematical expressions.

6. Answer the following questions for the array shown below. How will you enter your commands in Matlab.

$$\mathbf{b} = \begin{pmatrix} 1.1 & 0.0 & 2.1 & -3.5 & 6.0 \\ 0.0 & 1.1 & -6.6 & 2.8 & 3.4 \\ 2.1 & 0.1 & 0.3 & -0.4 & 1.3 \\ -1.4 & 5.1 & 0.0 & -1.1 & 0.0 \end{pmatrix}.$$

(a) What is the dimension of matrix **b**?

(b) Write down commands to extract the elements 2.1, 5.1, 0.3 and -6.6.

(c) Define a 2×5 matrix named **ext** having all elements of second and third rows.

(d) How would you access first, third and fifth entry of third row to get a row vector of size 1×3?

7. Use the **for** loop to compute the following expressions,

   (a) $10!$

   (b) $1 + 1/2^2 + 1/3^2 + 1/4^2.......1/10^2$.

8. Suppose we have two functions,

$$y = 3.5^{-0.5x}\cos(6x),$$
$$z = \sin(4x).$$

   Draw graphs for both functions simultaneously on the same plot for the range $-4 \leq x \leq 2$.

9. An exponentially decaying sine function is defined as,

$$y = e^{-0.4x}\sin(x),$$

   for $0 \leq x \leq 4\pi$. Draw graphs by taking 10 and 100 points in the interval. The plot with 10 points should be a solid line joining data points in circles. Plot both graphs simultaneously, one on top of another.

10. A student aims to find the spring constant of a spring, he loads it with various masses m and measures the corresponding lengths l. The force acting on the spring is $mg = k(l - l_o)$, where l_o is the unstretched length of the spring. The results are shown in Table (III).

Mass (g)	200	300	400	500	600	700	800	900
Length (cm)	5.1	5.5	5.9	6.8	7.4	7.5	8.6	9.4

TABLE III: Length versus load for a spring.

   Plot a graph for the data given in Table (III), and fit that on a straight line $l = l_o + (g/k)m$. Make a least-squares fit to this line, and find the best estimates for the unstretched length l_o and the spring constant k. The M-file must be attached.

11. The rate at which a radioactive sample emits radiation decreases exponentially as the the sample is depleted. To record the number of decays, a Geiger counter is placed near the source and data is given in Table (IV).

Elapsed time (min)	10	20	30	40	50
Counts detected	409	304	260	192	170

TABLE IV: Number of counts detected versus elapsed time.

If the sample decays exponentially, the number $v(t)$ can be written as,

$$v(t) = v_o e^{-t/\tau}, \tag{1}$$

where τ is the mean life of the sample and v_o is the number at time $t = 0$.
Plot the data, fit it to the function (1) using least-squares fitting and find the mean life τ. Submit your M-file for function (1).

12. Suppose we directed a sinusoidal AC voltage into the computer using a data acquisition system. The hardware acquires voltage by taking one sample in 50 ms and saves the first 21 points. The time sampling information is stored in a row vector t with an increment of 0.05 s. The voltage data is taken as,

5.4792	7.4488	7.5311	5.7060	2.4202	-1.5217	-5.1546
-7.5890	-8.2290	-6.9178	-3.9765	-0.1252	3.6932	6.5438
7.7287	6.9577	4.4196	0.7359	-3.1915	-6.4012	-8.1072

TABLE V: The voltage measurement (Volts).

Fit the data given above on a sinusoidal function $V = A\sin(\omega t + \phi)$ using least squares fitting technique and find the best estimates of amplitude A, angular frequency ω and phase ϕ. The M-file must be attached.

Solution key for error analysis and data processing problems

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October 10, 2011

Q.No.1

Given values of $x = 200 \pm 2$, $y = 50 \pm 2$, $z = 20 \pm 1$ and $u = 3 \pm 0.1$.

(a) The value of q is,

$$q = \frac{x}{(y-z)} = \frac{200}{(50-20)} = 6.66.$$

Let's define a function a,

$$a = y - z.$$

The uncertainty in a is,

$$\begin{aligned}\Delta a &= \sqrt{(\Delta y)^2 + (\Delta z)^2}, \\ &= \sqrt{(2)^2 + (1)^2}, \\ &= 2.24 \approx 2.\end{aligned}$$

Therefore,

$$a = (30 \pm 2).$$

The error in q is,

$$\begin{aligned}\frac{\Delta q}{q} &= \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta a}{a}\right)^2}, \\ \Delta q &= q\sqrt{\left(\frac{2}{200}\right)^2 + \left(\frac{2}{30}\right)^2}, \\ &= 0.45.\end{aligned}$$

Hence,

$$q = (6.6 \pm 0.5).$$

(b) Many calculations require complicated operations such as computation of sine, cosine, exponential or square root. In such cases the uncertainties can be calculated by taking the partial derivative and multiplying with the uncertainty.

Suppose we have a function p,

$$p = e^u.$$

The uncertainty in p can be calculated using the relationship,

$$(\Delta p)^2 = \left(\frac{dp}{du}\right)^2 (\Delta u)^2.$$

Implying,

$$\begin{aligned} \Delta p &= e^u \Delta u, \\ &= e^3(0.1), \\ &= 2.01. \end{aligned}$$

The value of p is,

$$p = e^3 = 20.08.$$

Now we can write,

$$p = (20 \pm 2)$$

(c) The value of r is,

$$\begin{aligned} r &= x\,(y - z\sin(u)), \\ &= 200\,(50 - (20\sin(3))), \\ &= 97990.6 \end{aligned}$$

Let's define a function b as,

$$b = \sin(u).$$

The uncertainty in b is,

$$\begin{aligned} \Delta b &= \left(\frac{db}{du}\right)\Delta u, \\ &= \cos(u)\Delta u, \\ &= \cos(3)(\frac{0.1}{57.3}) = 1.74 \times 10^{-3}, \\ &= 0.002. \end{aligned}$$

The value of b is,

$$b = 0.052.$$

So,

$$b = (0.052 \pm 0.002).$$

Suppose we take,

$$c = z\,b,$$

and its value is,

$$c = (20)(0.052) = 1.04.$$

The error in c is,

$$\begin{aligned}
\left(\frac{\Delta c}{c}\right) &= \sqrt{\left(\frac{\Delta z}{z}\right)^2 + \left(\frac{\Delta b}{b}\right)^2}, \\
\Delta c &= c\sqrt{\left(\frac{1}{20}\right)^2 + \left(\frac{0.002}{0.052}\right)^2} = (1.04)(0.063), \\
&= 0.07.
\end{aligned}$$

Therefore,

$$c = (1.04 \pm 0.07).$$

Now,

$$d = y - c.$$

The value of c is calculated as,

$$d = 50 - 1.04 = 48.96.$$

The uncertainty in d is,

$$\begin{aligned}
\Delta d &= \sqrt{(\Delta y)^2 + (\Delta c)^2}, \\
&= \sqrt{(2)^2 + (0.07)^2}, \\
&= 2.
\end{aligned}$$

Implies that,

$$d = (49 \pm 2).$$

Now the function r can be written as,

$$r = x\,d.$$

The uncertainty in r is,

$$\begin{aligned}
\left(\frac{\Delta r}{r}\right) &= \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta d}{d}\right)^2}, \\
\Delta r &= r\sqrt{\left(\frac{2}{200}\right)^2 + \left(\frac{2}{49}\right)^2}, \\
&= 4115.6.
\end{aligned}$$

Hence we can write,

$$r = (97991 \pm 4116).$$

Q.No.2

The time period of a simple pendulum is given as,

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{1}$$

The best estimates of l and T are,

$$\begin{aligned} l &= 92.95 \pm 0.01\,\text{cm}, \\ T &= 1.936 \pm 0.004\,\text{s}. \end{aligned}$$

We can find g by rearranging Equation (1),

$$g = 4\pi^2 l/T^2,$$

and its best estimated value is,

$$g = \frac{4\pi^2 \times (92.95\,\text{cm})}{(1.936\,\text{s})^2} = 979\,\text{cm/s}^2.$$

If the various uncertainties are independent and random, the fractional uncertainty in g is just the quadratic sum of the fractional uncertainties of l and T, given as,

$$\begin{aligned} \left(\frac{\Delta g}{g}\right) &= \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(2\,\frac{\Delta T}{T}\right)^2}, \\ \Delta g &= g\sqrt{\left(\frac{0.01}{92.95}\right)^2 + \left(2\,\frac{0.004}{1.936}\right)^2}, \\ &= 4.04 \end{aligned}$$

Thus, based on these measurements, our final answer is,

$$g = (979 \pm 4)\,\text{cm/s}^2$$

Q.No.3

The mean of the resistance of a coil is,

$$\begin{aligned} \bar{x} &= \frac{\sum_i x_i}{n} \\ &= \frac{4.615 + 4.638 + 4.597 + 4.634 + 4.613 + 4.623 + 4.659 + 4.623}{8} \\ &= 4.625\,\Omega. \end{aligned}$$

Deviation from the mean value is,

$$d_i = x_i - \bar{x},$$

$d_i\,(\Omega)$	$(d_i\,(\Omega))^2$
-0.01	1×10^{-4}
0.013	1.69×10^{-4}
-0.028	7.84×10^{-4}
0.009	8.1×10^{-5}
-0.012	1.44×10^{-4}
-0.002	4×10^{-6}
0.034	1.156×10^{-3}
-0.002	4×10^{-6}

so, for the measured data, the deviations are,

The square of standard deviation, s, is,

$$\begin{aligned} s &= \sqrt{\frac{\sum_i d_i^2}{n}} = \sqrt{\frac{0.0024}{8}} \\ &= 0.017\,\Omega. \end{aligned}$$

The standard error, σ, can be find out using the following relationship,

$$\begin{aligned} \sigma &= \sqrt{\frac{n}{n-1}}\, s \\ &= 0.018\,\Omega. \end{aligned}$$

As expected the standard error is approximately equal to the standard deviation, $\sigma \approx s$.

The standard error in the mean is,

$$\begin{aligned} \sigma_m &= \frac{\sigma}{\sqrt{n}} \\ &= 0.006\,\Omega. \end{aligned}$$

Therefore, the best estimated value of R, along with its uncertainty is,

$$R = (4.625 \pm 0.006)\,\Omega.$$

There is only one significant figure in the uncertainty and the least significant digit of the mean value has the same position as the non-zero digit of the uncertainty.

Q.No.4

The mean values of of length and breadth are,

$$\begin{aligned} \bar{l} &= \frac{\sum_i l_i}{n} \\ &= \frac{24.25 + 24.26 + 24.22 + 24.28 + 24.24 + 24.25 + 24.22 + 24.26 + 24.23 + 24.24}{10} \\ &= 24.25\,\text{mm}, \end{aligned}$$

$$
\begin{aligned}
\bar{b} &= \frac{\sum_i b_i}{n} \\
&= \frac{50.36 + 50.35 + 50.41 + 50.37 + 50.36 + 50.32 + 50.39 + 50.38 + 50.36 + 50.38}{10} \\
&= 50.37\,\mathrm{mm}.
\end{aligned}
$$

For the measured data, the deviations are,

d$_i$ for length (l)	**(d$_i$ (mm))^2**	**d$_i$ for breadth (b)**	**(d$_i$ (mm))^2**
0	0	-0.01	1×10^{-4}
0.01	1×10^{-4}	-0.02	4×10^{-4}
-0.03	9×10^{-4}	0.04	1.6×10^{-3}
0.03	9×10^{-4}	0	0
-0.01	1×10^{-4}	-0.01	1×10^{-4}
0	0	-0.05	2.5×10^{-3}
-0.03	9×10^{-4}	0.02	4×10^{-4}
0.01	1×10^{-4}	0.01	1×10^{-4}
-0.02	4×10^{-4}	-0.01	1×10^{-4}
-0.01	1×10^{-4}	0.01	1×10^{-4}

The standard deviation for length measurements is,

$$
\begin{aligned}
s_{\text{length}} &= \sqrt{\frac{\sum_i d_i^2}{n}} = \sqrt{\frac{0.0035}{10}} \\
&= 0.019\,\mathrm{mm},
\end{aligned}
$$

and for breadth is,

$$
\begin{aligned}
s_{\text{breadth}} &= \sqrt{\frac{\sum_i d_i^2}{n}} = \sqrt{\frac{0.0054}{10}} \\
&= 0.023\,\mathrm{mm}.
\end{aligned}
$$

The standard error, σ, can be find out using the following relationship,

$$
\sigma = \sqrt{\frac{n}{n-1}}\; s
$$

However for the set of data measured, the standard error is,

$$
\begin{aligned}
\sigma_{\text{length}} &= \sqrt{\frac{10}{10-1}}\,(0.019), \\
&= 0.02\,\mathrm{mm}.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\text{breadth}} &= \sqrt{\frac{10}{10-1}}\,(0.023), \\
&= 0.023\,\mathrm{mm}.
\end{aligned}
$$

The standard error in the mean is,

$$\begin{aligned}\sigma_m(\text{length}) &= \frac{\sigma}{\sqrt{n}} = \frac{0.02}{\sqrt{10}},\\ &= 0.006 \approx 0.01\,\text{mm}\end{aligned}$$

$$\begin{aligned}\sigma_m(\text{breadth}) &= \frac{0.023}{\sqrt{10}},\\ &= 0.007 \approx 0.01\,\text{mm}\end{aligned}$$

Therefore, the best estimated values of length and breadth, along with their uncertainties are,

$$\begin{aligned}l &= (24.25 \pm 0.01)\,\text{mm},\\ b &= (50.37 \pm 0.01)\,\text{mm}.\end{aligned}$$

The are of a rectangular plate is,

$$\begin{aligned}A &= l \times b,\\ &= (24.25)(50.37),\\ &= 1221.5\,\text{mm}^2.\end{aligned}$$

The uncertainty in area A can be calculated using the relationship,

$$\begin{aligned}\left(\frac{\Delta A}{A}\right) &= \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{\Delta b}{b}\right)^2},\\ \Delta A &= A\sqrt{\left(\frac{0.01}{24.25}\right)^2 + \left(\frac{0.01}{50.37}\right)^2},\\ &= 0.56 \approx 0.6\,\text{mm}.\end{aligned}$$

Therefore the best estimated value of area along with its uncertainty is,

$$A = (1221.5 \pm 0.6)\,\text{mm}.$$

Q.No.5

Given data is,

Pressure (mm of mercury)	65	75	85	95	105
Temperature (°C)	−20	17	42	94	127

Table 1: Pressure and temperature of a gas at constant volume.

The relationship between temperature and pressure is,

$$T = mP + c,$$

where m is the slope and c is the intercept.

The mean values of measured pressure and temperature is,

$$\begin{aligned}\bar{P} &= \frac{\sum_i P_i}{n} \\ &= \frac{65+75+85+95+105}{5} \\ &= 85\,\text{mm of mercury.}\end{aligned}$$

$$\begin{aligned}\bar{T} &= \frac{\sum_i T_i}{n} \\ &= \frac{-20+17+42+94+127}{5} \\ &= 52\,^{\circ}\text{C}.\end{aligned}$$

The best value of slope can be deduced using the relationship,

$$m = \frac{\sum_i^N y_i(x_i - \bar{x})}{\sum_i^N (x_i - \bar{x})^2}.$$

However in our case,

$$m = \frac{\sum_{i=1}^5 T_i(P_i - \bar{P})}{\sum_{i=1}^5 (P_i - \bar{P})^2},$$

Implying,

$$\begin{aligned}m &= \frac{(-20(65-85)+17(75-85)+42(85-85)+94(95-85)+127(105-85))}{((65-85)^2+(75-85)^2+(85-85)^2+(95-85)^2+(105-85)^2)}, \\ &= \frac{3710}{1000}, \\ &= 3.71\,(^{\circ}\text{C/mm of mercury}).\end{aligned}$$

After calculating the slope value, we can easily find out the intercept value using the following relationship,

$$c = \bar{y} - m\bar{x}.$$

Hence,

$$\begin{aligned}c &= \bar{T} - m\bar{P}, \\ &= 52 - (3.71)(85), \\ &= -263.35\,^{\circ}\text{C}.\end{aligned}$$

Q.No.6

(a) The size of matrix b can be find out using the Matlab command,

$\gg$ **size(b)**

and displayed output is,

ans =

4 5

where 4 and 5 represents the number of rows and the number of columns of matrix b, respectively.

(b) The following Matlab commands are used to extract different elements:

$\gg$ **b(1,3)**

ans =

2.1000

$\gg$ **b(3,1)**

ans =

2.1000

$\gg$ **b(4,2)**

ans =

5.1000

$\gg$ **b(3,3)**

ans =

0.3000

$\gg$ **b(2,3)**

ans =

-6.6000

(c) The elements of second and third rows can be accessed using the command,

$\gg$ **ext=b(2:3,:)**

ext =

0 1.1000 -6.6000 2.8000 3.4000

2.1000 0.1000 0.3000 -0.4000 1.3000

(d) A row vector having first, third and fifth entry of third row can be obtained by typing the following command in the Command Window,

≫ **b(3, [1 3 5])**

ans =

2.1000 0.3000 1.3000

Q.No.7

(a) The Matlab code to compute '**for**' loop of 10! is,

```
>> f=1;
>> for n = 2:10
f = f*n
end
```

(b) The Matlab code is,

```
>> k=0;
>> for i = 1:10
k = k + 1/(i .^ 2)
end
```

Q.No.8

The input vectors are,

```
>> x = -4 : 0.01 : 2 ;
>> y=3.5.^ (-0.5*x) .*cos(6*x);
>> y1=sin(4*x);
```

We can plot graphs of both functions simultaneously on the same plot using the following commands,

```
>> figure ; plot(x, y, 'r.')
>> hold on
>> plot(x, y1, '.')
```

The graph is depicted in Figure (1).

Q.No.9

The input vectors with 10 points in the interval are,

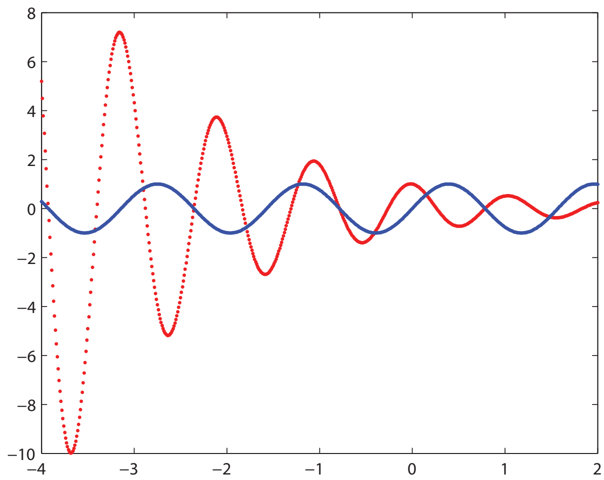


Figure 1: Overlaid plots. The red dots represent the output of an exponential function while the blue line corresponds to sin function.

≫ **x=0:1.256:4*pi;**

≫ **z=exp(-0.4.*x).*sin(x);**

For 100 points interval, we type,

≫ **y=0:0.1256:4*pi;**

≫ **z1=exp(-0.4.*y).*sin(y);**

To plot both the functions, we use,

≫ **figure;plot(x,z,'r-o','MarkerFaceColor','r')**

≫ **hold on**

≫ **plot(y,z1)**

The final result is shown in Figure (2)

Q.No.10

The independent and dependent row vectors are created as,

≫ **m=[200 300 400 500 600 700 800 900];**

≫ **l=[5.1 5.5 5.9 6.8 7.4 7.5 8.6 9.4];**

In least square curve fitting, we need to plot the data in order to know the relationship between independent and dependent quantities. For that we type,

≫ **figure;plot(m,l,'ro')**

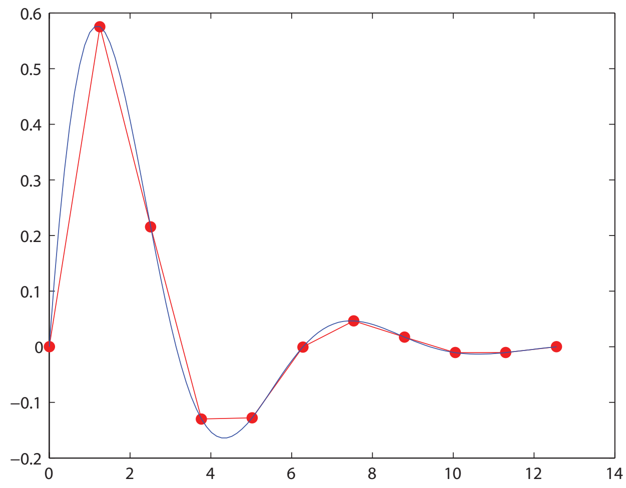


Figure 2: An exponentially decaying function. The red line with circles is for 10 points in the given interval while blue one is for 100 points.

The data plot shows that our best fit is of the form,

$$y = mx + c,$$

where m is the slope and c is the intercept.

Now we will make an M-file named **spring** depicted in Figure (3a).

Once the fitting function has been defined, we can find the least squares curve using the command,

≫ **lsqcurvefit(@spring, [3 6.428e-3], m, l)**

After the Matlab returns the values of the parameters, the output function is redefined as,

≫ **cfit= 3.6857+0.0061*m;**

Now we plot the redefined function on the data points using the command,

≫ **hold on**

≫ **plot(m,cfit)**

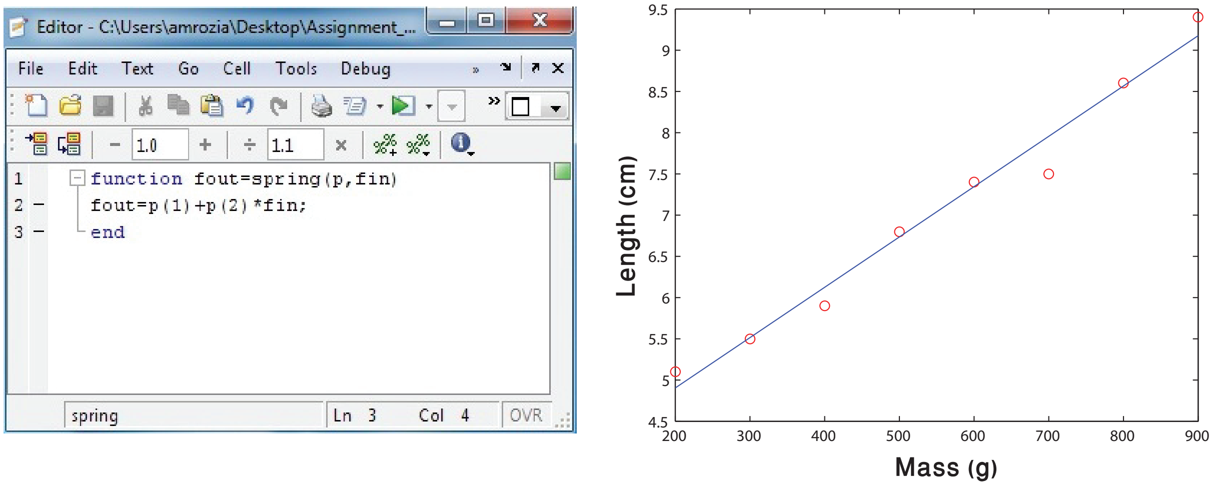
Q.No.11

The input vectors are defined as,

≫ **time=[10 20 30 40 50];**

≫ **counts=[409 304 260 192 170];**

The experimental data is plotted using the command,



(a) (b)

Figure 3: (a) An M-file for a linear function, (b) Measured data: The initial data points are plotted in red circles while the curve fit is drawn as a solid blue line.

≫ **figure;plot(time,counts,'ro')**

The graph shows that our best fit is an exponentially decaying function,

$$v(t) \quad = \quad v_o\, e^{(-t/\tau)}, \tag{2}$$

where v_o is the number at time $t = 0$ and τ is the mean life time.
In order to know the best estimated values of v_o and τ, we will make an M-file named **exponential** for (2) as shown in Figure (4a).

The function file can be accessed in the Command Window using the command,

≫ **lsqcurvefit(@exponential,[410 30],time,counts)**

The resolution of the best fit line can be enhance by increasing the step size. For that we will take the independent vector as,

≫ **time1=[10:0.1:50];**

Now the output function turns out to be,

≫ **cf=505.3005*exp(-time1./43.3921);**

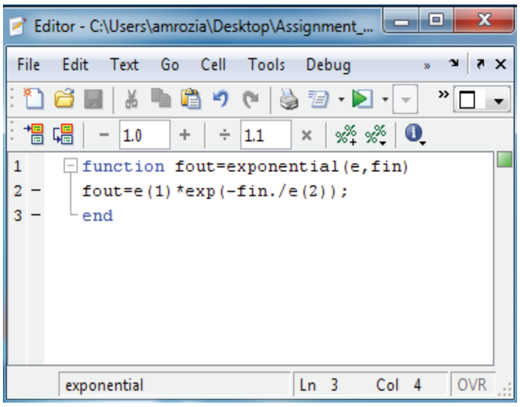
Now we plot the best fit line on our data plot using the following set of commands,

≫ **hold on**

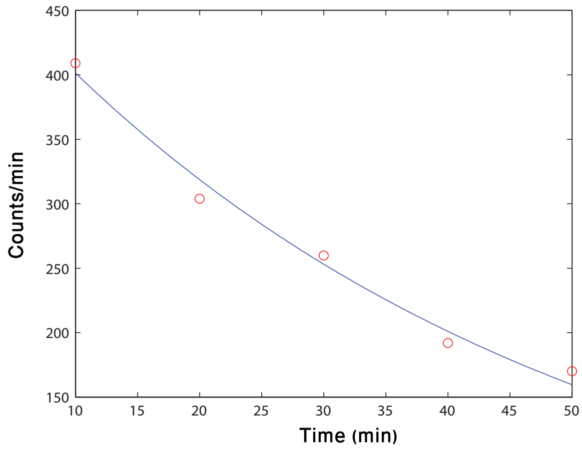
≫ **plot(time1,cf)**

Q.No.12

The time and voltage vectors are defined as,



(a)



(b)

Figure 4: (a) An M-file for an exponentially decaying function, (b) A plot of initial data points overlaid with curve fitted line.

≫ **t=[0:0.05:1];**

≫ **v=[5.4792 7.4488 7.5311 5.7060 2.4202 -1.5217 -5.1546 -7.5890 -8.2290 -6.9178 -3.9765 -0.1252 3.6932 6.5438 7.7287 6.9577 4.4196 0.7359 -3.1915 -6.4012 -8.1072];**

Given data is plotted as,

≫ **figure;plot(t,v,'ro')**

Our best fit is of the form,

$$v = A\sin(\omega t + \phi).$$

In order to get the best values of amplitude A, angular frequency and phase, we use least square curve fitting technique. The M-file named **sinusoid** is shown in Figure (5a).

After the fitting function is defined, we type in the Command Window,

≫ **lsqcurvefit(@sinusoid,[8 10 0],t,v)**

We can get a higher resolution curve fit by increasing the step size,

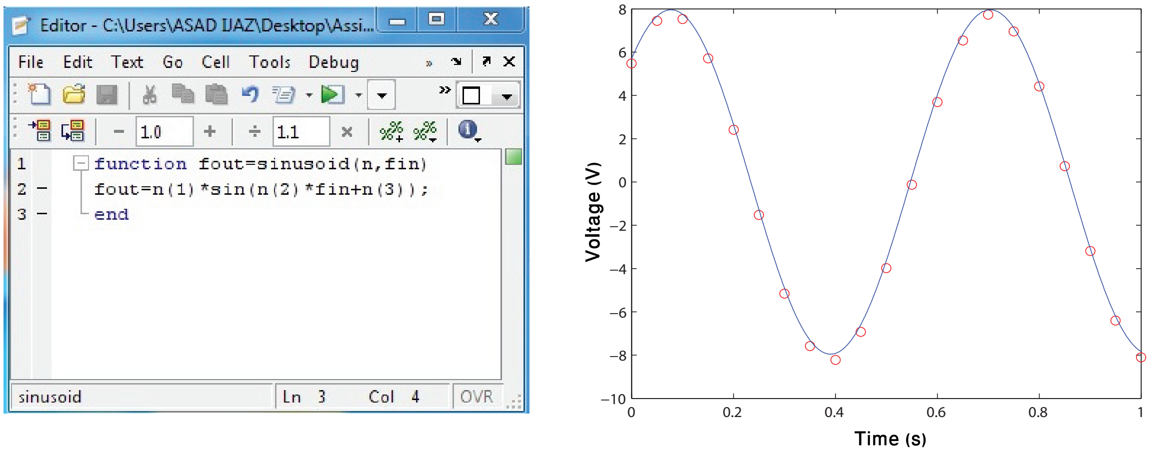
≫ **t1=0:0.001:1;**

The fitting function having best values of unknown parameters is created as,

≫ **final=7.9549*sin(10.0253*t1+0.7973);**

Now the data points and high resolution curve fit is plotted using the following commands,

≫ **hold on**



(a) (b)

Figure 5: (a) An M-file for a sinusoidal function, (b) Acquired voltage samples: initial data points circles whereas the curve fit is drawn as a solid line.

$\gg$ **plot(t1,final)**

The final result is shown in Figure (5b).