Expository session on uncertainties

1. A group of students measures g, the acceleration due to gravity with a compound pendulum and obtained the following values in units of $\mathrm{m\,s^{-2}}$,

   9.81 9.79 9.84 9.81 9.75 9.79 9.83

   Calculate the mean and the residuals, hence estimate σ_m. Quote the best estimated value of g alongwith its uncertainty.

2. Suppose we wish to measure the spring constant k by timing the oscillations of a mass m fixed to its end. The time period for such an oscillation is,

$$T = 2\pi\sqrt{\frac{m}{k}},$$

   thus by measuring T and m, we can find k as,

$$k = \frac{4\pi^2 m}{T^2}.$$

   The data set for measurands is given in Table (I).

Mass, m (kg)	0.513	0.581	0.634	0.691	0.752	0.834
Time period, T (s)	1.24	1.33	1.36	1.44	1.50	1.59

TABLE I: Experimental data for mass and time period.

   (a) Calculate uncertainties in the independent and dependent variables. The mass is measured using a digital weighing balance (rating= 1%), while time period is measured using a stop watch (digital device with rating= 0).

   (b) Plot a graph with error bars, both in the dependent and independent variables.

   (c) Using transformation rule, transfer all the uncertainties to the dependent variable. Plot a graph with error bars only in the dependent variable.

   (d) Calculate the best estimated value of k using weighted fit of a straight line. Calculate uncertainty in k as well.

3. An experiment to test a relation between the resistance of a semiconductor and its temperature. The semiconductor is sample of fairly pure silicon, and a simple theory suggests that the resistance R depends on the thermodynamic temperature T according to the relation,

$$R = R_o \exp(T_o/T)$$

where R_o and T_o is the room temperature resistance and temperature, respectively. Variation of resistance of the sample with temperature is shown in Table (II).

Temperature, T (K)	570.6	555.9	549.4	544.1	527.3	522.2	513.1	497.6	484.9
Resistance $R\,(\Omega)$	148.1	202.6	227.1	255.1	362.0	406.1	502.5	750.1	1026.7

TABLE II: Experimental data for temperature and resistance.

(a) Calculate uncertainties in the independent and dependent variables. The resistance is measured using a digital multimeter (rating= 1%), while temperature is measured using an analog thermometer (rating= 0).

(b) Plot a graph with error bars, both in the dependent and independent variables.

(c) Using transformation rule, transfer all the uncertainties to the dependent variable. Plot a graph with error bars only in the dependent variable.

(d) Calculate the best estimated value of T_o using least squares fitting of a straight line. Calculate uncertainties in T_o as well.

(e) Find the bandgap ($E_g = 2k_BT_o$) of silicon.

Formula sheet:

Slope (m) and intercept (c) with equal weights:

$$m = \frac{\sum_i^N y_i(x_i - \bar{x})}{\sum_i^N (x_i - \bar{x})^2} \quad \text{or} \quad m = \frac{N\sum_i^N x_i y_i - \sum_i^N x_i \sum_i^N y_i}{N\sum_i^N x_i^2 - (\sum_i^N x_i)^2} \tag{1}$$

$$c = \bar{y} - m\bar{x} \quad \text{or} \quad c = \frac{\sum_i^N x_i^2 \sum_i^N y_i - \sum_i^N x_i \sum_i^N x_i y_i}{N\sum_i^N x_i^2 - (\sum_i^N x_i)^2}. \tag{2}$$

Uncertainty in slope m and intercept c is given as,

$$u_m = \sqrt{\frac{\sum_i^N d_i^2}{D(N-2)}}, \tag{3}$$

$$u_c = \sqrt{\left(\frac{1}{N} + \frac{\bar{x}^2}{D}\right)\left(\frac{\sum_i^N d_i^2}{(N-2)}\right)}, \tag{4}$$

where,

$$d_i = y_i - mx_i - c,$$

$$D = \sum_i^N (x_i - \bar{x})^2.$$

Slope m and intercept c with unequal weights

The weights are reciprocal squares of the total uncertainty (u_{Total}),

$$w = \frac{1}{u_{\text{Total}}^2}. \tag{5}$$

The mathematical relationships for slope (m) and intercept (c) are,

$$m = \frac{\Sigma_i w_i \, \Sigma_i w_i (x_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i y_i)}{\Sigma_i w_i \, \Sigma_i (w x_i^2) - (\Sigma_i w_i x_i)^2}, \tag{6}$$

$$c = \frac{\Sigma_i (w_i x_i^2) \, \Sigma_i (w_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i x_i y_i)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}, \tag{7}$$

where x is the independent variable, y is the dependent variable and w is the weight.

The expressions for the uncertainties in m and c are,

$$u_m = \sqrt{\frac{\Sigma_i w_i}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}, \tag{8}$$

$$u_c = \sqrt{\frac{\Sigma_i (w_i x_i^2)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}. \tag{9}$$