Tracking a Harmonic Oscillator using a webcam*

Sohaib Shamim, Wasif Zia and
Muhammad Sabieh Anwar
LUMS School of Science and Engineering

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Is there any thing that in this universe that does not oscillate? Look at things around you. From galaxies and stars in the skies to the inner workings of an atom, we observe never-ending motion. In this experiment we are going to study the simplest form of periodic motion—namely simple harmonic motion (SHM) using a webcam and image processing algorithms.

KEYWORDS Simple Harmonic Motion · Amplitude · Acceleration ·Angular Frequency · Damping

1 Conceptual Objectives

In this experiment, we will,

1. understand simple harmonic motion and its damping under various conditions;
2. start making simple measurements of lengths and understand the parallax error;
3. understand how errors propagate from an observed to an inferred quantity;
4. identify dimensions and units for simple physical quantities and transform between physical and logical coordinates;
5. fit experimentally observed curves with mathematically modelled solutions;
6. perform simple image processing and computational tasks on the personal computer; and last,
7. understand the formation and display of colour on TV and computer screens.

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2 Experimental Objectives

The aim of the present experiment is to examine the amplitude of a freely oscillating as well as an *underdamped harmonic oscillator*. The experiment requires measuring the damping constant and then make a quantitative comparison of theory with experimental results.

3 Theoretical Introduction

Stand against a light source. Tie a tennis ball to a string and whirl it over your head in a horizontal circle. Observe the shadow on the wall that is opposite to the light source. You will see linear motion of the shadow which is slowest on the edges (where it turns around) and fastest in the center. This behavior, slowing down when moving away from the center and speeding up when approaching the center, is the signature of *simple harmonic motion* (SHM).

3.1 Characteristic Equation of SHM

Now let's analyze the above example in somewhat mathematical detail. The motion of the shadow of the ball cast on the wall can be described by,

$$x(t) = x_0 \cos(\omega t + \phi) \tag{1}$$

where $x(t)$ is the position of the shadow on the wall, x_0, called the amplitude, is the maximum distance of the shadow from the center, ω the rate at which you are rotating the ball, t is the time and ϕ, the phase, is the deviation of the wave from a reference. (Also see Fig. 1.)

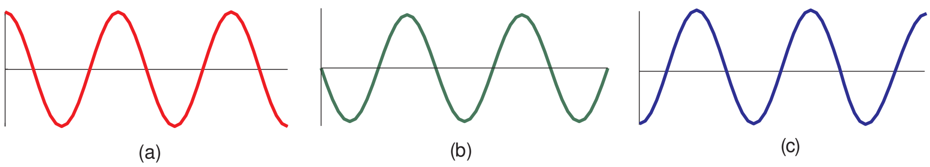

Figure 1: Position, velocity and acceleration in SHM. (a) The position x of the shadow, (b) the slope of x gives the velocity v and (c) the slope of v gives the acceleration a.

The gradient (slope) of equation (1) gives the velocity of the shadow and the sign shows the direction of the velocity, the negative means that as x increases, v decreases. So the shadow has maximum velocity at $x = 0$ and minimum velocity (in fact momentarily zero) at $x = x_0$.

$$v(t) = \frac{dx(t)}{dt} = -\omega x_0 \sin(\omega t + \phi). \tag{2}$$

This notion of changing velocity can also be expressed in terms of the acceleration, given by the gradient (slope) of equation (2).

$$a(t) = \frac{d^2x(t)}{dt^2} = \frac{d\,v(t)}{dt} = -\omega^2 x_0 \cos(\omega t + \phi). \tag{3}$$

We can replace $x_0 \cos(\omega t + \phi)$ in the above equation with equation (1) and as a result obtain,

$$a(t) = \frac{d^2x(t)}{dt^2} = -\omega^2 x(t). \tag{4}$$

This is the characteristic differential equation describing SHM.

Q 1. Write down the equation for the motion of the particle which is released from rest at 2 meters from origin, O in the positive direction and first returns to this position after 4 seconds [1].

Q 2. A balloon performs SHM in a vertical line with a period of 40 seconds. Its height varies between 800 and 850 meters. Find the speed of the balloon when it is at 820 meters [1].

3.2 Mass-Spring System

Consider a mass attached to a spring. Now combining *Newton's law*,

$$F = ma \tag{5}$$

and *Hooke's law*,

$$F = -kx \tag{6}$$

where k is the *spring constant*, we get

$$a = -\frac{k}{m}x. \tag{7}$$

This can also be written as,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \tag{8}$$

with the definition,

$$\omega_0^2 = \frac{k}{m}. \tag{9}$$

Equation (8) is identical to equation (4). Hence the periodic motion of a mass attached to a spring is simple harmonic. Now let's analyze the energy of this system. The total mechanical energy (E_T) of the mass-spring system is the sum of *kinetic* and *potential* energies, E_K and E_P,

$$E_T = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \tag{10}$$

that can be re-written in light of equation (2) as,

$$E_T = \frac{1}{2}m\Big(\frac{dx}{dt}\Big)^2 + \frac{1}{2}kx^2. \tag{11}$$

The *conservation of energy* ensures that the total energy is unchanging. As a result, kinetic and potential energies inter-convert keeping the sum constant. At the point of maximum displacement, the energy is wholly potential ($E_K = 0, E_T = E_P$) and at the centre, all the energy is kinetic ($E_P = 0, E_T = E_K$).

Q 3. What are the units of ω, k and m?

Q 4. Sketch a graph between E_T and time. Now on the same graph sketch curves for E_k and E_P.

Q 5. When the undamped spring is stretched and released, show that the total energy E_T at any given time is,

$$E_T = \frac{1}{2}mx_0^2\omega_0^2 = \frac{1}{2}kx_0^2. \tag{12}$$

HINT: Use $v = -x_0\omega sin(\omega t)$ and Equation (10).

Q 6. A spring of natural length 0.6 meters is attached to a fixed point A on a smooth horizontal table in the presence of air. A force of $10\,x$ N is needed to keep the spring extended by x meters. A block of mass 0.1 kg is attached to the other end. The block is pulled away from A until it is 0.75 meters from A, and then let go. Describe the subsequent motion [1].

3.3 The Decay of Vibrations

Equation (4) represents the ideal situation where there is no damping. A mass set in motion once will continue displaying its oscillations for all times to come. However, as we all very well know, this is never the case. The amplitude of the oscillating mass keeps on decreasing, till it eventually comes to a stop. We now investigate this behaviour, called the *damping effect*.

Every medium (such as air, water) exerts a frictional force on any object moving inside it. This resistive force slows down the object, eventually bringing it to rest. In our case, the medium is air, slowing down the oscillating mass-spring system. One form of Newton's second law that models this frictional effect is

$$F = -bv, \tag{13}$$

where v is the velocity of the particle and b is the *drag coefficient*.

Combining the two forces (-kx and -bv), we obtain

$$F = -bv - kx = ma \tag{14}$$

$$\implies a + \frac{b}{m}v + \frac{k}{m}x = 0. \tag{15}$$

This can be re-written as,

$$\frac{d^2x(t)}{dt^2} + \frac{b}{m}\frac{dx(t)}{dt} + \frac{k}{m}x(t) = 0, \tag{16}$$

substituting,

$$\frac{b}{m} = \gamma, \tag{17}$$

$$\frac{k}{m} = \omega_0^2, \tag{18}$$

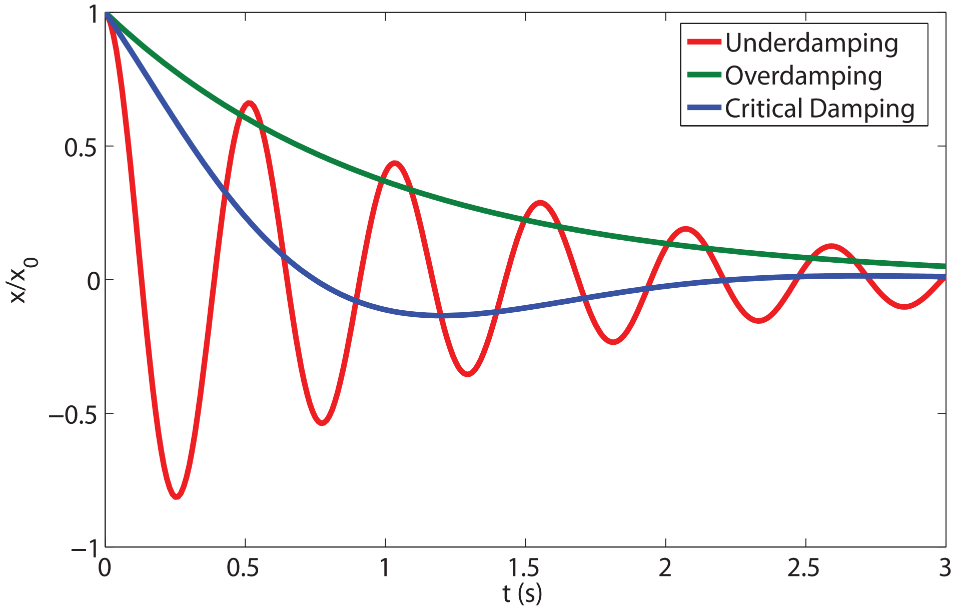

Figure 2: Decaying oscillations for underdamped, overdamped and critically damped harmonic oscillations. Our experiment is performed in the underdamped regime.

the differential equation now becomes,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0. \tag{19}$$

Q 7. What is the difference between Equations (8) and (19)?

In this particular case, the damping is characterized by γ, while ω_0 represents the *natural angular frequency* of the system *if damping were absent.*

In the current experiment, we will consider only one kind of damping: *underdamped* SHM. As the term underdamped suggests, the oscillations die away, albeit slowly. We will not go into the details of the various kinds of damping. However Figure 2 gives a good intuitive feel of how these damped motions look like.

The solution to Equation (19), for the underdamped oscillations, is given by ($\phi = 0$),

$$x(t) = A\exp{(-\frac{\gamma}{2}t)}cos(\omega_1 t), \tag{20}$$

where A is the maximum amplitude at $t = 0$ and,

$$\omega_1 = \sqrt{{\omega_0}^2 - \frac{\gamma^2}{4}}. \tag{21}$$

The solution, Equation (20) is the mathematical representation of the underdamped oscillation of the mass-spring system, it shows how the position $x(t)$ varies as time t. This solution can be divided into two parts, $A\exp(-\frac{\gamma}{2}t)$ represents the damping of the amplitude (also called the envelope) whereas $\cos(\omega_1 t)$ represents the oscillations of the mass, with a frequency ω_1. Note that $\omega_1 \neq \omega_0$.

4 Apparatus

Our damped harmonic oscillator apparatus consists of a set of masses (locally fabricated) attached to the end of a helical spring. The mass is displaced slightly and the resulting motion is recorded using a web camera that is attached to a computer. The data processing is performed in Matlab that is equipped with the *Image Processing* toolbox. The stands holding the mass-spring system are manufactured in house. Fig. 3 shows the schematic setup of the experiment.

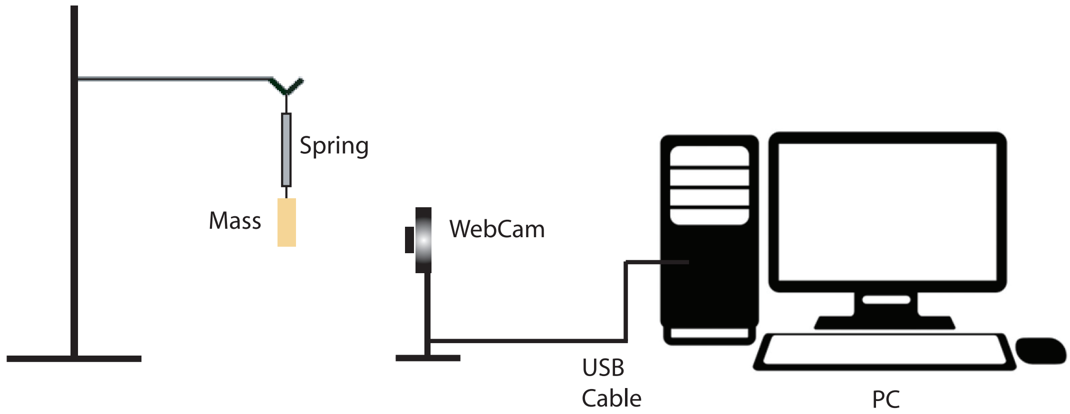

Figure 3: Schematic diagram of the experimental setup.

5 The Experiment

5.1 Determining the Spring Constant

Q 8. You have been provided with the mass hanger (20 ± 1) g along with different mass blocks.

Q 9. Set up the apparatus as shown in the schematic diagram. Attach the mass hanger and note the extension, using the attached meter rule. What are the units of the extension?

Q 10. Add weights of equal mass and measure the extension each time a weight is added.

Q 11. Plot a graph and intelligently use your experimental data to obtain the spring constant k.

Q 12. What is the error in k?

Q 13. Calculate an expected value of ω_0 using equation (18). Also predict the period of free oscillations. What is the propagated uncertainty in the time period?

5.2 Free Oscillations

Remove the meter rule and start Matlab. All the function files that you have downloaded from our website must be in the current directory of the Matlab.

Ensure that the webcam video is close before you type the following command. Pull the mass (60 g) down by about 5 cm and release it, then type,

$\gg$ **webcam**

The command above activates a live video preview window and you can see the mass oscillating on your computer screen. Readjust the apparatus to obtain a vertically oscillating mass appearing on the computer screen.

In order to acquire frames, you need to type,

$\gg$ **start(vid)**

This will acquire 300 frames. Observe the parameters updating themselves in the bottom of the video preview window. In order to store these frames in the memory, you have to wait for 30 sec, then type,

$\gg$ **savedata**

This will create two arrays named **frames** and **time** and store the acquired frames and their time stamped values.

$\gg$ **size(frames)**

★ **Q** 14. You see 4 numbers. Write down these values and explain what they represent.

The first frame can be viewed by typing in the following commands.

$\gg$ **firstframe = frames(: , : , : , 1)**

$\gg$ **imtool(firstframe)**

★ **Q** 15. What is the size of the first frame?

Try viewing the second, third and the fourth frame.

Next you need to crop the acquired frames so that all the irrelevant information is filtered out. Matlab has a built-in command known as **imcrop** which crops an image to a specified rectangle. You can crop the first frame by typing

$\gg$ **rect**$=$ $[x_{min} \quad y_{min} \quad \underbrace{(x_{max} - x_{min})}_{width} \quad \underbrace{(y_{max} - y_{min})}_{height}]$

HINT: Use the data cursor to scan the x and y coordinates for their maximum and minimum values.

$\gg$ **cropregion = imcrop(firstframe,rect)**

★ **Q** 16. How will you view the cropregion?

Now we need a Matlab code that crops all the 300 frames in one go and saves the cropped array with the name **regions**. For that you need to type a code, already written for you,

$\gg$ **harmonic**

★ **Q** 17. What is the size of the array **regions**?

$\gg$ **imageprocess**

will process all the cropped images inside the data set named **regions** and will calculate the center of mass of the oscillating masses. In order to calculate the displacement co-ordinates for the center of mass, use the command,

$\gg$ **masscentre**

The x and y coordinates of the center of mass through the 300 frames can be viewed by typing

$\gg$ **x = centre(:,1);** and

$\gg$ **y = centre(:,2);**

Q 18. Plot a graph of the x coordinates against time. What does this graph physically represent?

Q 19. What happens to the x coordinates and explain why?

Q 20. Now plot a graph of the y coordinates with time. Record the approximate time period of the oscillation in seconds.

Q 21. Compare this result with the time period from Question 14.

5.3 Damped Oscillations

Q 22. We now move on to see how the energy changes with time in an underdamped system with $\omega_0 \gg \gamma$. The energy at any subsequent time is given by,

$$E(t) \approx E_0 exp(-\gamma t) \tag{22}$$

where,

$$E_0 = \frac{1}{2} m x^2 \omega_0^2 = \frac{1}{2} k x^2. \tag{23}$$

Q 23. Place the beaker containing water under the mass-spring system. Pull the mass down so that it is completely immersed in water. Repeat the experiment as in Section 5.2.

Q 24. Plot the amplitude against time. What do you observe?

Q 25. From the graph, record the value of amplitude of each cycle and the corresponding time at which this happens.

Q 26. Plot a graph of $\ln [\frac{E(t)}{E_0}]$ versus time, where E_0 represents the energy of the first oscillation that you took into consideration and $E(t) = \frac{1}{2} k x_t^2$ represents the energy at any given time t thereafter.

Q 27. Why do you think plotting a graph of $\ln[\frac{E(t)}{E_0}]$ against time is more appropriate than a graph of ln[E(t)] against time?

Q 28. Fit the plot and calculate the best estimated value of γ. Also calculate the values of ω_1 and b.

6 Idea Experiments

1. Use glycerine instead of water for damping. Can you determine the viscosity of glycerine? HINT: Use Stokes Law.
2. Solve the differential equation (16) using the quadratic formula and analyze the discriminant for less than, equal to and greater than zero.
3. Try horizontal mass-spring system instead of gravity driven. Discuss friction.
4. Try to find out if friction is surface area dependent or not. HINT:Slide blocks down a plank and observe using a webcam.
5. Study the diffusion of ink in water using webcam [2].
6. Demonstrate the damping of a pendulum [3].

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