Moment of Inertia of a Tennis Ball

Amrozia Shaheen and Muhammad Sabieh Anwar

LUMS School of Science and Engineering

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KEYWORDS

Rigid body · Moment of inertia · Torque · Torsional pendulum · Simple harmonic motion.

APPROXIMATE PERFORMANCE TIME 2 **Hours**

1 Conceptual Objectives

In this experiment, we will,

1. appreciate and practice the process of calibration,
2. learn how error propagates from a set of readings to inferred quantities,
3. learn how to compare theoretical prediction with experimental observations,
4. run through a complete cycle of experiment, data generation, analysis and presentation, and,
5. measure a property of interest in the rotational dynamics of rigid bodies.

2 Introduction

2.1 Torsional pendulum

The torsional pendulum exhibits simple harmonic motion. The Cavendish balance that was used to determine the value of the gravitational constant G, was a torsional pendulum. Torsional pendulums are also used in mechanical watches and mousetraps.

Q 1. Learn how Cavendish used a torsional pendulum to determine G.

Cavendish's torsion balance [1].

A torsional pendulum consists of a disk suspended by a wire attached to the center of mass of the disk. The other end of the wire is fixed to a solid support or clamp. In such a pendulum, unlike a simple pendulum, the disk twists about the vertically held string, instead of swinging in a vertical plane. When the disk is in the equilibrium position, a radial line can be drawn, say, from its center to a point P. When a slight twist is given to the disk, the line moves to the position OQ, as shown in Figure (1a). The twisted wire will exert a restoring torque on the disk trying to bring it back towards its equilibrium position. The interested student is referred to pages 322-323 of [2] for further details.

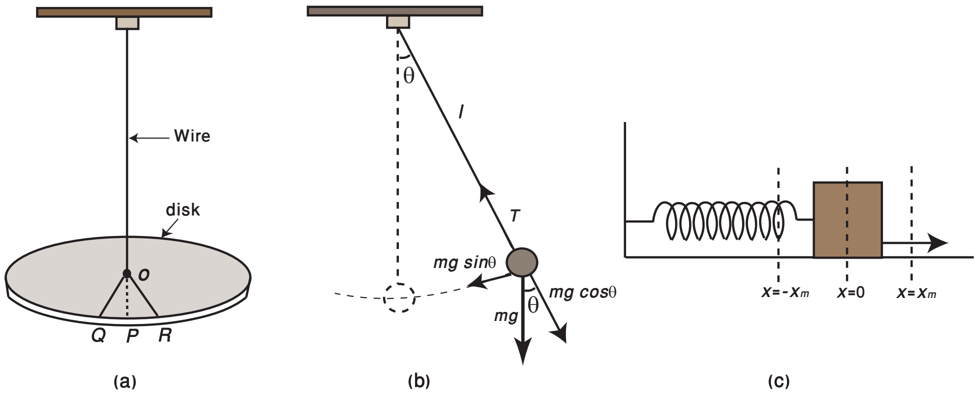


Figure 1: (a) The torsional pendulum, compared with a (b) simple pendulum. Analogy with a (c) mass attached to a spring.

For small twists, the restoring torque τ is proportional to the angular displacement θ. This is also called Hook's law,

$$\tau = -\kappa\,\theta, \tag{1}$$

where κ is the torsional constant depending on the properties of the wire. This relation is perfectly analogous to the restoring force $F = -kx$ on a mass-spring system that has been displayed through a distance, x, shown in Figure (1c). The minus sign shows that the torque is directed opposite to the angular displacement θ.

The equation of motion based on angular form of Newton's second law is,

$$\tau = I\alpha = I\,\frac{d^2\theta}{d\,t^2}, \tag{2}$$

where I is the moment of inertia and $\alpha = d^2\theta/dt^2$ is the angular acceleration. Notice the analogy of Equation (2) with its linear counterpart $F = ma = m(d^2x/dt^2)$.

Comparing Equations (1) and (2), we obtain,

$$-\kappa\,\theta \;=\; I\,\frac{d^2\theta}{d\,t^2},$$

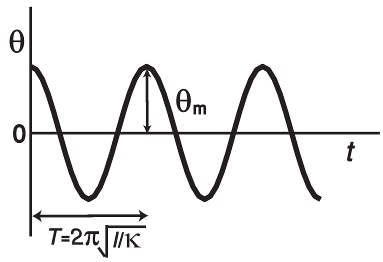
which after rearrangement becomes,

$$\frac{d^2\theta}{d\,t^2} \;=\; -\left(\frac{\kappa}{I}\right)\theta. \tag{3}$$

The above equation is typical of simple harmonic motion and the solution is given by,

$$\theta(t) \;=\; \theta_m \cos(\omega t + \phi),$$

where $\omega = \sqrt{\kappa/I}$ is the angular frequency, θ_m is the maximum angular displacement and ϕ is the phase constant.



The position of the particle executing SHM.

The period of the oscillation is,

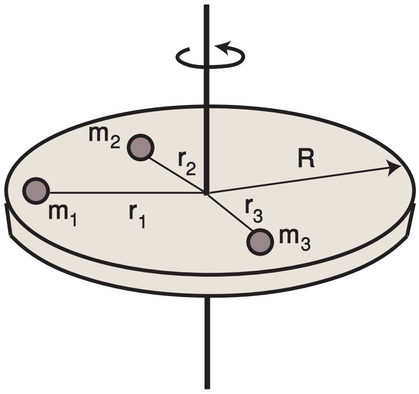
$$T \;=\; \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}. \tag{4}$$

Therefore, if κ is known and T is experimentally measured, the rotational inertia of any oscillating rigid body can be determined. Alternatively, if I is known and T is measured, the torsional constant of any wire can be calculated [3, 4].

2.2 Moment of inertia

The moment of inertia characterizes the distribution of the mass throughout an object from a certain axis of rotation. The moment of inertia is the rotational analog of mass just as the angular velocity is the rotational analog of linear velocity. For further details, you may like to look up our experiment "Rotational dynamics, moment of inertia, torque and rotational friction", numbered 1.2.

Consider a simple case of a point particle of mass m rotating about a fixed axis at a certain distance r. The moment of inertia of this particle is $I = mr^2$. But if the object is extended



A disk can be conceptulized as comprising of a large number of particles.

such as a circular disk comprising a large number of point particles, then the moment of inertia can be find out by summing the moments of each particle within the disk resulting in,

$$I = \sum_i I_i = \sum_i m_i r_i^2, \tag{5}$$

where the sum is taken over i individual particles, each of mass m_i and at a distance r_i from the axis of rotation. If we consider a body as being made up of large number of discrete particles, then we can use Equation (5) for calculating rotational inertia. If, however, we regard it as a continuous distribution of matter, then we can divide it into a large number of small mass elements having each δm_i at a particular perpendicular distance r_i from the axis of rotation. The rotational inertia in this case is,

$$I = \sum_i r_i^2 \delta m_i. \tag{6}$$

In the limit of infinitesimally small δm_i, the sum becomes an integral,

$$I = \lim_{\delta m_i \to 0} \sum r_i^2 \delta m_i = \int r^2 dm. \tag{7}$$

Using this recipe, the moment of inertia of a hollow sphere (Figure 3b, page 6) of internal radius R_{int}, external radius R_{ext} and mass M is,

$$I = \frac{2}{5} M \frac{R_{\text{ext}}^5 - R_{\text{int}}^5}{R_{\text{ext}}^3 - R_{\text{int}}^3}, \tag{8}$$

and for a composite cylinder (3c, page 6), the expression is,

$$I = \frac{1}{2}\pi\rho(h_1 R_1^4 + h_2 R_2^4 - h_2 R_1^4), \tag{9}$$

where ρ is the density of the material ($2.70\,\text{g}\,\text{cm}^{-3}$ for aluminum).

The detailed calculations have been spelled out for you in the Appendix.

3 Apparatus

1. Tennis ball,
2. Composite cylinder made of aluminum of the shape shown in Figure (3c), page 6,
3. Stainless steel wire (45 cm long with diameter 0.3 mm (28 AWG)),
4. Stand,
5. Vernier callipers,
6. Stop watch.

4 Experimental Method

In this experiment, the moment of inertia of a tennis ball will be determined experimentally and compared with the theoretical prediction, given in Equation (8). The necessary apparatus is provided to you and photographed in Figure (2).

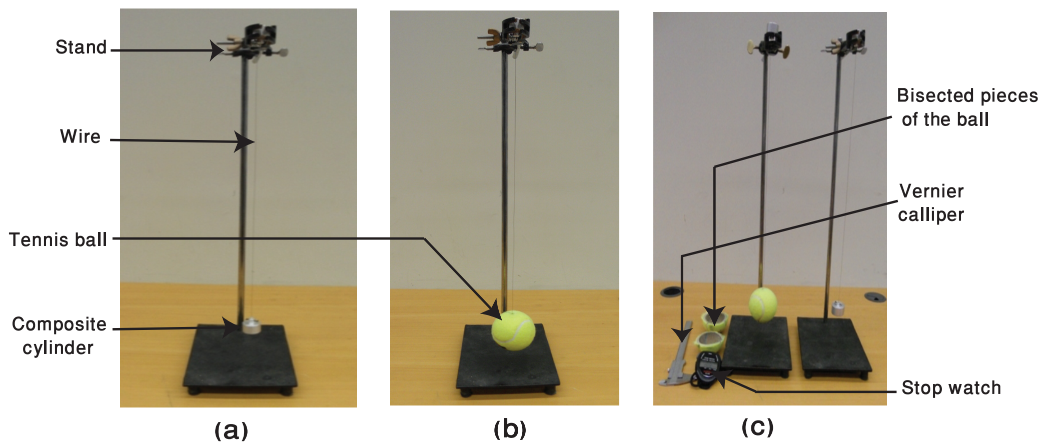


Figure 2: (a) Setup for calibration, (b) tennis ball attached to a fixed support, and (c) the provided apparatus.

The tennis ball is a hollow sphere!

We are leaving out the details of this experiment, which you will need to figure out on your own. Generate your own data, analyze and quote uncertainties. You should sketch or log the procedure appropriately in your notebooks. You have to keep the data in (*cm, grams and seconds*) throughout the experiment.

★ **Q** 2. Predict the moment of inertia of the tennis ball using its particular geometry. Quote the uncertainty.

★ **Q** 3. Measure the torsional constant κ of the wire using the provided composite cylinder. What is the uncertainty in κ?

★ **Q** 4. Determine the moment of inertia of the tennis ball and its uncertainty.

5 Appendix: Calculating the moment of inertia of selected objects

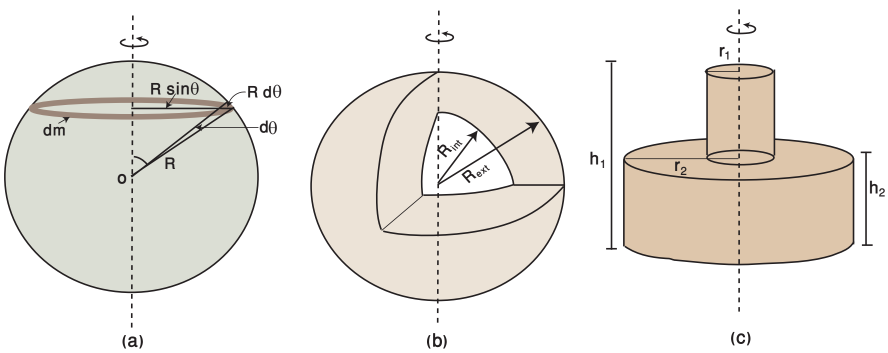


Figure 3: (a) Spherical shell, (b) hollow sphere, and (c) composite cylinder.

Hollow sphere: A hollow sphere can be considered to be composed of infinite number of rings of variable radius. Let us consider one such ring as a small element, situated at a distance R from the center of the sphere and positioned at an angle θ from the axis of rotation as shown in Figure (3a).

Let's take a small mass of area dA of the ring thickness of $Rd\theta$, situated at a distance R, from the center. The area density is,

$$dm = \sigma dA = \left(\frac{M}{4\pi R^2}\right)(2\pi R \sin\theta R d\theta), \tag{10}$$

$$= \left(\frac{M}{2}\right)\sin\theta d\theta. \tag{11}$$

As the radius of elemental ring about the axis is $R\sin\theta$, therefore, the moment of inertia is,

$$\begin{aligned} I &= \int r^2 dm = (R\sin\theta)^2\left(\frac{M}{2}\right)\sin\theta d\theta, \\ &= \left(\frac{MR^2}{2}\right)\int_0^\pi \sin^3\theta d\theta, \\ &= \left(\frac{MR^2}{2}\right)\int_0^\pi (1-\cos^2\theta)\sin\theta d\theta, \\ &= -\left(\frac{MR^2}{2}\right)\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^\pi, \\ &= \frac{2}{3}MR^2. \end{aligned}$$

Now, if the hollow sphere is of reasonable thickness, then dividing the hollow sphere into a series of infinitesimal spherical shells and integrating these infinitesimal moments yields,

$$I = \int_{R_{\text{int}}}^{R_{\text{ext}}} \frac{2}{3} r^2 dm. \tag{12}$$

Substituting

$$dm = \rho dV,$$

where $\rho = M/V = M/(4/3)\pi(R^3_{\rm ext} - R^3_{\rm int})$ and dV is surface area times infinitesimal thickness. Therefore, the integral 12 becomes,

$$\begin{aligned} I &= \int_{R_{\rm int}}^{R_{\rm ext}} \frac{2}{3} r^2 \rho dV, \\ &= \int_{R_{\rm int}}^{R_{\rm ext}} \frac{2}{3}\, r^2 \frac{M}{(4/3)\pi(R^3_{\rm ext} - R^3_{\rm int})}\, 4\pi r^2 dr, \\ &= \frac{2M}{R^3_{\rm ext} - R^3_{\rm int}} \left.\frac{r^5}{5}\right|_{R_{\rm int}}^{R_{\rm ext}}, \\ &= \frac{2}{5} M \frac{R^5_{\rm ext} - R^5_{\rm int}}{R^3_{\rm ext} - R^3_{\rm int}}. \end{aligned}$$

Composite cylinder: Suppose that the axis of rotation passes through the center of composite cylinder as shown in Figure (3c).

The moment of inertia for the tall section is,

$$\begin{aligned} I &= \int_0^{r_1} r^2 dm = \int_0^{r_1} r^2 \rho dV, \\ &= \int_0^{r_1} r^2 \rho (2\pi r h_1) dr, \\ &= 2\pi\rho h_1 \left.\frac{r^4}{4}\right|_0^{r_1}, \\ &= \frac{1}{2}\pi\rho h_1 r_1^4, \end{aligned}$$

and for the short section is,

$$\begin{aligned} I &= \int_{r_1}^{r_2} r^2 \rho dV = \int_{r_1}^{r_2} r^2 \rho (2\pi r h_2) dr, \\ &= 2\pi\rho h_2 \left.\frac{r^4}{4}\right|_{r_1}^{r_2}, \\ &= \frac{1}{2}\pi\rho h_2 (r_2^4 - r_1^4). \end{aligned}$$

The total moment of inertia is,

$$I_{\rm Total} = \frac{1}{2}\pi\rho(h_1 r_1^4 + h_2 r_2^4 - h_2 r_1^4).$$

References

[1] http://www.ssplprints.com/image.php?id=94282&idx=0&keywords=1977-0417&filterCategoryId=&fromsearch=true.

[2] R. Resnick, D. Halliday, K. S. Krane, "*Introduction to Physics*", John Wiley & Sons., (1992), pp. 315-325.

[3] X-S. Cao, "*Moment of inertia of a Ping-Pong ball*", Phys. Teach, Vol. 50, 292 (2012).

[4] H. Brody, "*The moment of inertia of a tennis ball*", Phys. Teach. Vol. 43, 503-505 (2005).

[5] http://physics.info/rotational-inertia/practice.shtml.