Uncertainties and Measurements in Experimental Physics

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In science, the word 'uncertainty' does not mean a mistake. In fact, the term refers to the fact that we cannot make measurements to infinite accuracy and precision and we cannot eliminate them just by being more careful, smart or using more expensive equipment. The best we can do is to ensure that uncertainties are as small as reasonably possible and more importantly, to have a reliable quantitative estimate of how large they are.

1 Measurement matters

Measurement is an essential part of science and without measurement scientific models and theories can not be implemented. Careful measurement with properly identified uncertainties can lead to a new discovery. For an engineer, the accurate measurement may lead to improved complex systems e.g., space shuttles, while in medical metrology accurate measurement of blood pressure lessens the risk of misdiagnosis and disease.

The world of physics revolves around fundamental constants such as the speed of light c, Planck's constant h, the fine-structure constant α and the gravitational constant G. All these constants were measured with some uncertainty because a measurement is meaningless without uncertainty. The values of some fundamental constants are shown in Table (1).

The values enclosed in brackets are uncertainties in the respective measured constants. For example, the value of h is $6.62606957\,(29) \times 10^{-34}$ Js tells that the Planck's constant is measured with an uncertainty lying between $6.62606929 \times 10^{-34}$ Js and $6.62606957 \times 10^{-34}$ Js. It can also be written as

Fundamental constant	**Symbol**	**Measured value**
Planck's constant	h	$6.62606957\,(29) \times 10^{-34}$ Js
Boltzmann conatant	k_B	$1.3806488\,(13) \times 10^{-23}$ J-K^{-1}
Charge of the electron	e	$1.602176565\,(35) \times 10^{-19}$ C
Stefan-Boltzmann constant	σ	$5.670373\,(21) \times 10^{-8}$ Wm^{2}K^{-1}

Table 1: Values of some fundamental constants.

$[6.62606929, 6.62606957] \times 10^{-34}$ Js which shows that the number is between this range.

1.1 Rounding off and significant figures

When a number has too many significant figures then the number of significant figures can be reduced by a method called 'rounding'. For example, if the distance is measured as 2.1451 m with five significant figures, and it is required to quote the precision to three significant figures, the value becomes 2.15 m. Since 2.15 m is more closer to 2.1451 m, therefore rounding it to 2.14 m would be incorrect.

The question is what if the original figure ends in 5 or greater than 5. The most suitable selection of rounding is to retain an even value but increase the value of an odd digit by one. Some rounded values are shown in Table (2). Doing this consistently ensures that we do not introduce any bias in our results.

Observed value	**Rounded value**
3.05	3.0
3.15	3.2
3.25	3.2
3.35	3.4
3.33	3.3
3.36	3.4

Table 2: Examples of rounding off of some values.

In physics the precision of the numerical values is an important concept. We will say more about this now. From the point of view of experimental sciences, the three numbers 3, 3.0, 3.00 are different from significant figure and precision perspective. For example, the value 3.00 tells that the number could be some number between 3.005 and 2.995. The relative precision of this number is,

$$\text{Relative precision} = \frac{0.005}{3.00} \times 100 = 0.17\%.$$

Similarly the relative precisions of the other two numbers are shown in Table (5).

Value	**Relative precision**
3	$(0.5/3) \times 100 = 17\%$
3.0	$(0.05/3) \times 100 = 1.7\%$
3.00	$(0.005/3.00) \times 100 = 0.17\%$

Table 3: Relative precision of some numbers.

By looking at the precision of these numbers we can say that the number 3.00 is much more precise (smaller percentage means precise measurement). This also shows that the experimentalist has an instrument that can achieve this much precision.

Likewise if we measure the mass of a wooden block using two different electronic weighing balances which display 4.1 g and 4.12 g, respectively. The relative precision of the reading 4.1 g is 1.2% while the value 4.12 g is 0.12% precise. We can therefore say that the second balance is more precise than the first one.

2 An introduction to uncertainty

We live in an uncertain world. For example, will the weather be suitable to have a barbecue at the weekend? Is the investment we made a wise decision or not? Uncertainties are everywhere. In some cases, it is possible to reduce these uncertainties and we can always quantify them.

Remember not to use the word ‘error’ as errors are mistakes, idealized and can never be known completely. Always use the word ‘uncertainty’ because uncertainties are quantifiable and transferrable.

There are two kinds of uncertainties: Type A and type B.

2.1 Type A uncertainties

These uncertainties are random fluctuations in the measured values and can easily be identified by repeating the experiment. The reliability of a measurement can be accessed by repeating a measurement several times. The analysis for a sequence of repeated measurements that results in slightly different values can be done by calculating the mean and then finding individual differences from the mean. The scatter of these individual differences correspond to the uncertainty of the measurement, the greater the scatter the more uncertain the measurement [1], [2].

Suppose a student measures g, the acceleration due to gravity five times and finds the following results (in ms^{-2}),

$$9.9, 9.6, 9.5, 9.7, 9.8.$$

The first question addresses that what would be the best estimate of g? The statistical method for finding the best value for a measurement is to repeat the measurement many times and then taking the average value. The readings are recorded in Table (4).

Value of g (m/s^2)	**Deviations (m/s^2)**	**Square deviation (m^2/s^4)**
9.9	0.2	0.04
9.6	-0.1	0.01
9.5	-0.2	0.04
9.7	0.00	0.00
9.8	0.1	0.01
average: 9.7	avg deviation:0.0	avg square deviation: 0.02

Table 4: The standard deviation in a measurand.

It turns out that the average deviation is zero! For random uncertainties, we are as likely to overestimate a value as underestimate it. A much more useful quantity is the square of the deviation. The sum of the square of deviation is 0.1, a non zero number! We can now take the average of this number and square root it to get the uncertainty. This final answer is known as standard deviation.

The standard deviation in mathematical form is,

$$\text{Standard deviation}\,(s) \;=\; \sqrt{\frac{d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2}{N}} = \sqrt{\frac{\Sigma d_i^2}{N}}\,.$$

where $d_i = x_i - \bar{x}$ represents the deviation from the average value.

But, there is a problem with this equation. What if we make only one measurement? The standard deviation will be equal to zero in that case, which clearly is an absurd statement. The uncertainty can never be zero. We overcome this problem by introducing the expression,

$$\text{Standard uncertainty}\,(\sigma) \;=\; \sqrt{\frac{n}{n-1}}\,(s)\,. \tag{1}$$

Finally, the uncertainty in the final answer is given by the standard uncertainty of the mean which is given by:

$$\sigma_{mean} = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n-1}} s .$$

For the example at hand, the calculator returns 0.0707. Now the question is how to quote the uncertainty? The relative precision of the best approximated value is ($0.05/9.7 \times 100 = 0.52\%$), while the relative precision of the calculated uncertainty is,

Value of g in (m/s^2)	**Uncertainty**	**Relative precision**
9.70	0.07	$0.07/9.7 \times 100 = 0.72\%$
9.7	0.1	$0.1/9.7 \times 100 = 1.03\%$

Table 5: Relative precision of the uncertainty of a measurand (the acceleration due to gravity g).

Eyeballing the third column in the table above, the precision for uncertainty cannot be better than the best estimate. Hence we choose ($9.7 \pm 0.1\,\mathrm{m/s^2}$).

2.2 Type B uncertainties

Type B uncertainties involve specific information regarding the measurand that can be found in the calibration report. The calibration report gives the estimated uncertainty of the measurand. Another kind of type B uncertainties involves those due to the finite resolution of the measurement scale.

The uncertainty of a single measurement is restricted by the accuracy and precision of the measuring instrument and also depends on some factors that affect the ability of the experimenter to make a measurement [4]. We separately discuss type B uncertainties due to the finitude of the measuring scale in digital and analog instruments as well as type B uncertainties due to the instrument's rating and accuracy.

2.2.1 A digital reading

Suppose you measure a voltage value using a digital device. The question is how would you calculate the standard uncertainty associated with the scale of the reading? The key is to assign a probability distribution with each measurement.

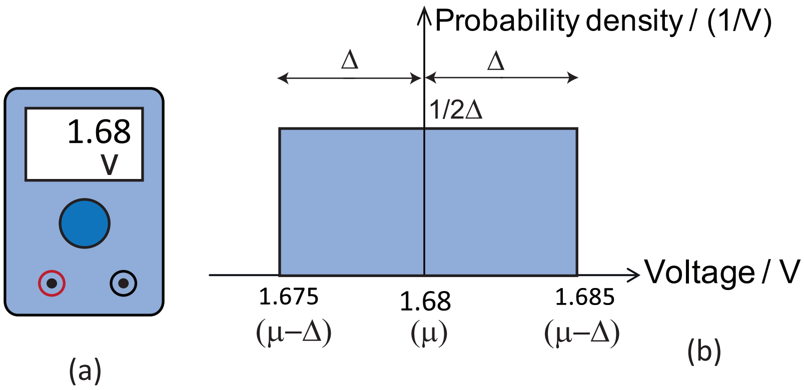


Figure 1: (a) A digital reading displayed on a digital voltmeter and (b) the associated probability distribution function assigned to this reading.

The determination of the standard uncertainty is based on the idea of mathematical moments. A moment is a quantitative measure of the shape of a set of data and the second moment characterizes the width of the probability density function. The calculation of the standard uncertainty is based on the second moment, also called the variance.

The mathematical expression for the second moment of a function $f(x)$ is,

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx, \tag{2}$$

where μ is the mean value. For example in Figure (1b) the value of μ is 1.68 V.

For a rectangular probability distribution function, the value of the function at the center is $(f(x) = 1/2\Delta)$. Substituting this value in the above expression yields,

$$\sigma^2 = \int_{\mu-\Delta}^{\mu+\Delta} (x-\mu)^2 \left(\frac{1}{2\Delta}\right) dx. \tag{3}$$

Let's substitute,

$$z = x - \mu \quad \text{and} \quad dz = dx.$$

The substitution yields,

$$\sigma^2 = \int_{\mu-\Delta}^{\mu+\Delta} z^2 \left(\frac{1}{2\Delta}\right) dz = \frac{\Delta^2}{3}. \tag{4}$$

Since the standard uncertainty is the square root of the variance $(u = \sqrt{\sigma^2})$, hence we can define the uncertainty for a rectangular pdf as,

$$u_{\text{rectangular}} = \frac{\text{Half of the length of the interval}}{\sqrt{3}}. \tag{5}$$

Suppose you measure the voltage across resistor using a digital voltmeter as shown in Figure (1a). The best estimate of the voltage is 1.68. The last digit 8 represents the interval 1.675 to 1.785. All we can assume is that the voltage value is distributed between 1.675 and 1.685 with equal probability. The probability function would be a rectangular function with limits 1.675 to 1.685 as shown in Figure (1b).

So for the above example, the uncertainty becomes,

$$\begin{aligned}\text{Standard uncertainty in voltage}\,(u_{\text{scale}}) &= \frac{\frac{1}{2}(1.685 - 1.675)}{\sqrt{3}}, \\ &= 0.0029\,\text{V}\,. \qquad (6)\end{aligned}$$

The uncertainty in the above equation is due to the resolution of the measuring device.

2.2.2 An analog reading

To determine the best approximation of a single measurement while using an analog device is slightly complicated because it relies to a larger extent on your judgement.

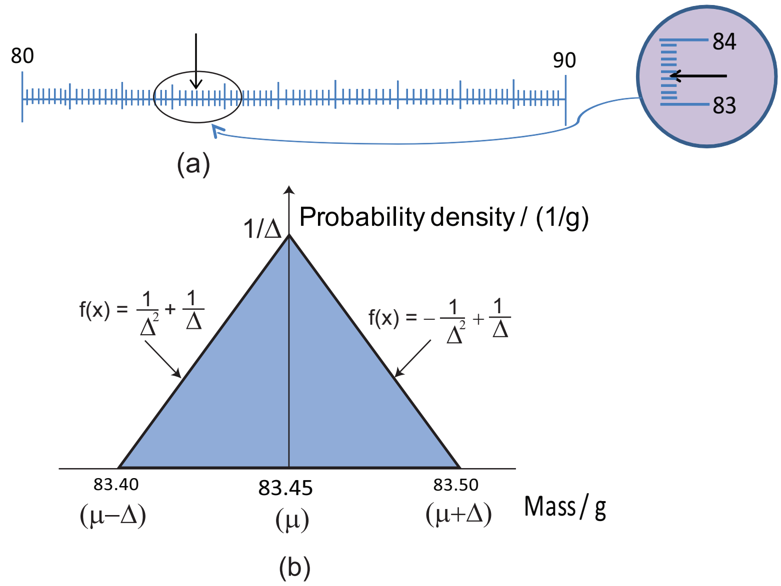


Figure 2: (a) An analog reading displayed on an analog balance and (b) associated probability distribution function.

Assume that you are measuring the mass of a can using an analog balance and the reading apparent on the scale is shown in Figure (2a). The reading you might take as the best approximation is 83.45 g but it could be a little bit larger or smaller than the observed value. Now, in this situation you need to make a judgement in assigning a probability distribution. For example, you know one thing for sure that is you can declare that the probability of the values being 83.40 g or 83.50 g is precisely zero. So you proceed from the best approximated value towards the unlikely value and assign a probability distribution function which is triangular shaped with extremities at 83.40 g and 83.50 g, as shown in Figure (2b).

The standard uncertainty of a triangular probability distribution can be calculated using Equation (2). Assuming $\mu = 0$, the expression becomes,

$$\begin{aligned} \sigma^2 &= \int_{-\Delta}^{\Delta} x^2 f(x) dx, \\ &= \int_{-\Delta}^{0} x^2 \left(\frac{1}{\Delta^2} x + \frac{1}{\Delta} \right) dx + \int_{0}^{\Delta} x^2 \left(-\frac{1}{\Delta^2} x + \frac{1}{\Delta} \right) dx, \qquad (7) \\ &= -\frac{\Delta^2}{2} + \frac{2\Delta^2}{3} = \frac{\Delta^2}{6}. \qquad (8) \end{aligned}$$

You can also repeat the integration assuming a non-zero μ. Hence we can define the standard uncertainty associated with a analog reading as,

$$u_{\text{triangular}} = \frac{\text{Half of the length of the interval}}{\sqrt{6}} = \frac{\Delta}{\sqrt{6}}. \qquad (9)$$

For the measured mass displayed in Figure (2), the uncertainty becomes,

$$\text{Standard uncertainty in mass} = \frac{\frac{1}{2}(83.50 - 83.40)}{\sqrt{6}} = 0.02\,\text{g}, \qquad (10)$$

and finally, the best approximated value for mass is,

$$\text{Mass of the can} = (83.45 \pm 0.02)\,\text{g}.$$

2.2.3 Rating or accuracy of the instrument

Now, in a comprehensive probabilistic approach, then we also need to consider the uncertainty associated with rating of any digital device. For example, if the digital voltmeter's rating is 1 % of the displayed reading, then the associated uncertainty of the reading (1.68 V) being measured on a digital voltmeter is,

$$u_{\text{instrument}} \ @1\% = 0.01 \times 1.68 = 0.0168\,\text{V},$$

and the combined uncertainty for this digital device turns out to be,

$$\begin{aligned} u_{\text{combined}} &= \sqrt{u_{\text{scale}}^2 + u_{\text{instrument}}^2}, \\ &= \sqrt{(0.0029)^2 + (0.0168)^2} = 0.0170\,\text{V}, \end{aligned}$$

where u_{scale} is the type B uncertainty taken from Equation (6).
Hence the best approximation of voltage measurement alongwith associated uncertainty is,

$$\text{Voltage across resistor} = (1.68 \pm 0.02)\,\text{V}. \tag{11}$$

Note that we have rounded off the uncertainty value to the same number of decimal places as the best estimate.

2.3 Combining type A and type B uncertainties

The type A and B uncertainties must be combined as both of them contribute towards the combined uncertainties.

Suppose u_A are the uncertainties measured by repeating the measurements and u_B either using digital, analog devices or from the rating of the instrument. The expression for total uncertainty then becomes,

$$u_{\text{total}} = \sqrt{u_A^2 + u_B^2}\,.$$

If you have a large number of uncertainties u_i where $i = 1,\ 2\,3, ..., n$, they are combined in quadrature according to the prescription,

$$u_{\text{total}} = \sqrt{\sum u_i^2}\,. \tag{12}$$

3 A probabilistic approach towards measurement

3.1 Probability density function

The previous discussion should have convinced you that a probabilistic approach had been adopted in metrology. This has also been advocated by the International Standards Organization (ISO) in 1993. This methodology has also been accepted by other standards such as IUPAP (International Union of Pure and Applied Physics, IUPAC (International Union of Pure and Applied Chemistry) and BIMP (International Bureau of Weights and Measures) and affects the way in which measurements and uncertainties are reported in scientific work. We have directly introduced this approach into all the experimental work being performed at the physics lab at *LUMS*.

Each measurement has a probability distribution function associated with it. A probability distribution function (pdf) is a way of describing the data being collected either from a single measurement or multiple measurements.

Probability density is simply the probability of a variable lying between two values bounded by an interval. The area under the pdf is always 1 or 100%. The shape and size of this probability density function depends on the kind of uncertainty coupled with logical reasoning and some subjective judgement. These pdfs model all the information that we have for a particular measurand.

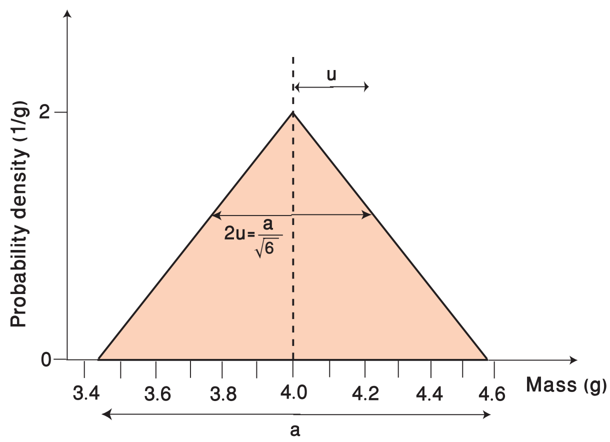


Figure 3: A triangular probability distribution function.

Consider a triangular pdf that describes the mass of an object being measured on an analog measuring instrument. The centre of the pdf corresponds to the most probable value of the measurand and is shown in Figure (3). Surely the area under the curve is one but that doesn't tell about how fat or thin the triangle is. The average width of the triangle is a measure of uncertainty in the measurand and referred as the 'standard uncertainty'. A higher spread of the pdf is associated with large standard uncertainty.

The final result of an experiment can be communicated by describing the best approximation of the measurand (the centre of the pdf) and the standard uncertainty (which is calculated based upon the shape of the pdf). A table summarizing the most often used pdfs in measurement science (metrology) are shown in Table (4).

A large number of probability density functions are useful in a variety of applications. However in physical measurements, three continuous pdfs are most often used. A Gaussian pdf is associated with type *A* evaluations of uncertainty involving a set of repeated measurements of a measurand with some scatter in the readings. The type *B* evaluations involve a uniform pdf associated with a digital scale while a triangular one with an analog scale.

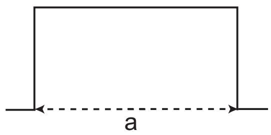
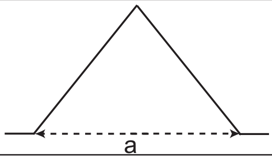
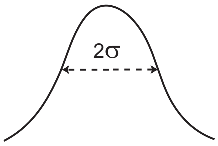
Evaluation type	Probability distribution function type		Standard uncertainty (u)	Extended Uncertainty
Type B A single digital reading	Rectangular		$u = \frac{a/2}{\sqrt{3}}$	1u = 58%
				1.65u = 95%
				1.73u = 100%
Type B A single analog reading	Triangular		$u = \frac{a/2}{\sqrt{6}}$	1u = 65%
				1.81u = 95%
				2.45u = 100%
Type A For a set of repeated measurements	Gaussian		$u = \frac{\sigma}{\sqrt{n}}$	1u = 68%
				2u = 95%
				3u = 99%

Figure 4: (a) Commonly used probability distribution functions associated with measurements.

3.2 Coverage intervals

The coverage interval is an interval within which the true value of the measurand lies with high probability, usually 95%. This interval is very often symmetrical about the best approximated value and correspond to percentages of the area of the normal density lying within the defined limits. For example, a 68% coverage interval tells that 68% area of the normal probability distribution function is within one standard uncertainty.

Suppose we measure the mass of a can and the result is quoted with 68% confidence or in a 68% confidence interval is (83.45 ± 0.34 g). This means that there is 68% probability that the best approximated value of mass lies somewhere within the interval (83.45 ± 0.34 g) of one standard uncertainty. Conversely, there is 32% probability that the best approximated value of the measurand lies outside the interval (83.45 ± 0.34 g).

Consider a Gaussian probability distribution function. The general equation of a Gaussian function is,

$$p(x) \quad = \quad \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \qquad (13)$$

where μ is the expectation value and σ is the standard deviation that gives the width of the Gaussian pdf. The standard deviation σ can also be determined by considering the point where the height of the probability distribution function drops to $1/\sqrt{e} = 0.61 = 61\%$ of the maximum value and half of this width is called the standard deviation σ.

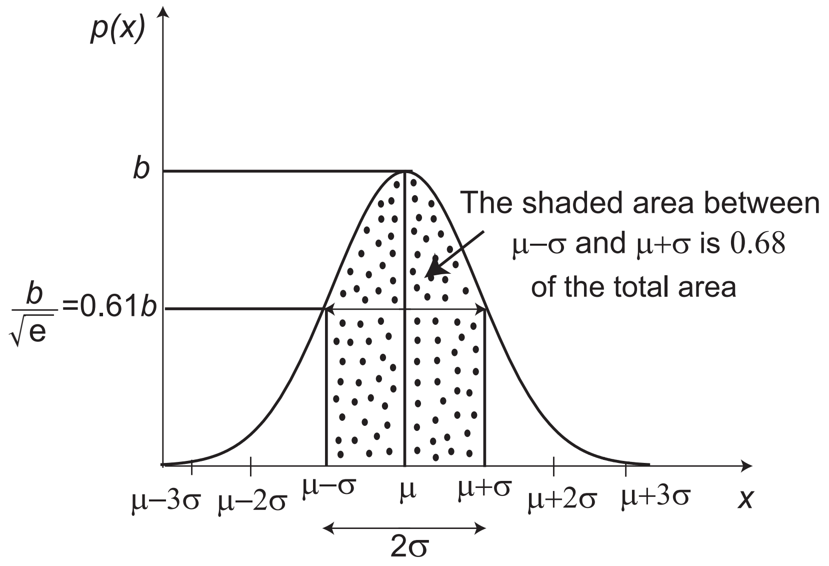


Figure 5: A Gaussian probability distribution function, where $p(x)dx$ is the probability of finding a value in the range x and $x + dx$.

The Gaussian probability distribution function is again shown in Figure (5). The shaded area is between $\mu - \sigma$ and $\mu + \sigma$ corresponds to the probability of the measurand lying within one standard uncertainty of the best approximated value [3]. We say that with a confidence level of 68%, our measurand has a value in the range $\mu - \sigma$ and $\mu + \sigma$.

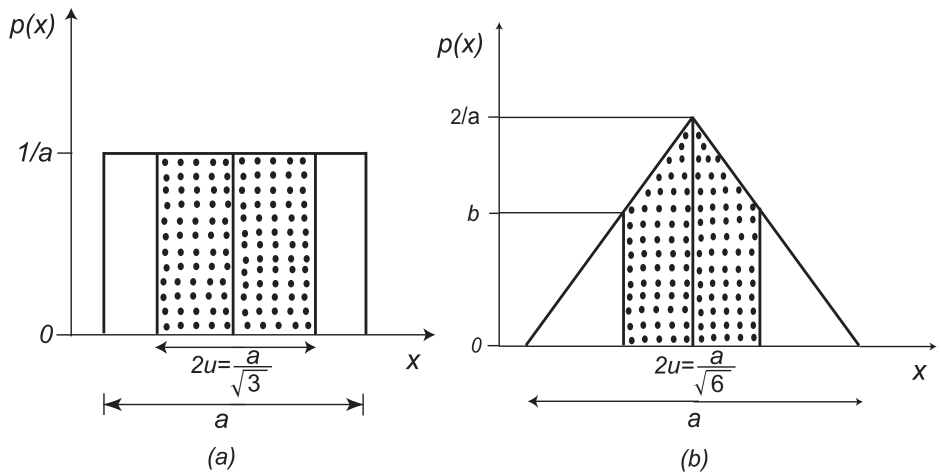


Figure 6: Coverage probability. (a) Rectangular pdf and (b) triangular pdf.

Further, the confidence of intervals for a rectangular pdf covers 58% area of the distribution function within one standard while for a triangular pdf, it encloses almost 65% area of the probability distribution function. The confidence intervals for rectangular or triangular pdfs are shown by the shaded regions in Figure (6).

4 Least squares fitting of a straight line function

Many physical laws imply that one quantity is proportional to another. Many experiments in the teaching laboratories are designed to check this kind of proportionality. To test whether a certain quantity y is proportional to another variable x, we can plot a graph of y against x and see if the points lie on a straight line. Because a straight line is so easily recognizable even visually, this method is a simple and effective way to check for proportionality.

4.1 Calculation of the slope and intercept

Now the goal is to find the best straight line through the n pairs $(x_1, y_1),\ldots,(x_N,y_N)$ of measurements as shown in Figure (7). At the start we assume that the uncertainties are only in the dependent variable. This assumption is quite reasonable because generally uncertainties in one variable are larger than in the other, and can be safely ignored in the more precisely measured quantity.

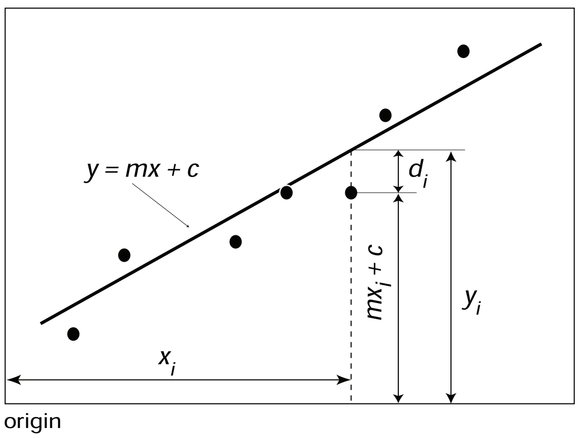


Figure 7: Setting for the least squares best fit.

For a given pair of slope and intercept, the deviations are defined as,

$$d_i = y_i - m\,x_i - c.$$

The best value of m and c, which are the properties of the best fit line, can be find out by taking the minimum of the sum of the squares of the deviations,

$$S = \Sigma(y_i - m x_i - c)^2.$$

This method is called the least squares fitting. Minimizing the squares of the deviations yields,

$$\frac{\partial S}{\partial m} = -2\Sigma x_i(y_i - m x_i - c) = 0,$$
$$\frac{\partial S}{\partial c} = -2\Sigma(y_i - m x_i - c) = 0,$$

Rewriting the above equations for m and c,

$$m\Sigma x_i^2 + c\Sigma x_i = \Sigma x_i y_i,$$
$$m\Sigma x_i + cn = \Sigma y_i.$$

Finally, solving for the constants m and c, we get,

$$m = \frac{N\Sigma(xy) - \Sigma x \Sigma y}{N\Sigma x^2 - (\Sigma x)^2},$$
$$c = \frac{\Sigma x^2 \Sigma y - \Sigma x \Sigma(xy)}{N\Sigma x^2 - (\Sigma x)^2},$$

which are recipes for determining the best fit line. We have so far assumed that all the data points have equal weights.

4.2 Example: Least squares fitting and transferring uncertainties to the dependent variable

Now consider an experiment in which a spring-mass system is used to find the spring constant. We are now exploring what happens if the uncertainty in the independent variable cannot be ignored. According to Hooke's law,

$$F = -kx, \tag{14}$$

where k is the spring constant. Combining Newton's law and Hooke's law,

$$mg = -kx. \tag{15}$$

The above equation is a simple linear equation and can be written as,

$$y = mx + c, \tag{16}$$

where m is the slope and c is the intercept.

Mass (g)	**Extension (cm)**
20	3.0
42	6.8
64	10.5
86	14.0
108	17.5

Table 6: Model table for experimental results.

Suppose we hang a mass hanger on a stand and measure its mass using an electronic weighing balance. Now adding weight into it, we see that the length of the spring increases. We note down this extension using a meter rule. The obtained data is summarized in Table (6).

The weighing balance has a digital scale while the scale on the meter rule is an analog one, both involve type B uncertainties. The uncertainty in mass limited by resolution of the digital weighing balance and due to its rating is given,

$$\begin{aligned} u_{m(\text{scale})} &= \frac{\Delta}{\sqrt{3}} = \frac{0.5}{\sqrt{3}}\,\text{g} = 0.3\,g, \\ u_{m(\text{rating})} \; @\; 1\% &= 0.01 \times (\text{each value of mass}). \end{aligned}$$

The total uncertainty in the independent variable (mass) can be calculated using the following expression,

$$\begin{aligned} u_x &= \sqrt{(u_{m(\text{scale})})^2 + (u_{m(\text{rating})})^2}, && (17) \\ &= 0.4\,\text{g} \qquad (\text{for } 20\,\text{g mass}) && (18) \end{aligned}$$

the value of u_x will be different as the $u_{m(\text{rating})}$ is different for each mass.

The uncertainty in the dependent variable (extension) is,

$$u_y = u_{\text{extension (scale)}} = \frac{\Delta}{\sqrt{6}} = \frac{0.5}{\sqrt{6}}\,\text{cm} = 0.2\,\text{cm}.$$

The data has uncertainties both in dependent and independent variables and the least-squares require uncertainty only in the dependent variable. In order to address this issue we will transform the uncertainty from independent to dependent variable by employing the transformation rule,

$$u_{\text{trans}} = \frac{dy}{dx}\,u_x. \qquad (19)$$

The trivial step is the determination of the slope (dy/dx) which can be done using either of the following approaches,

- tracing the best straight line passing through the experimental data,
- numerically estimating the slope value by looking at adjacent points,

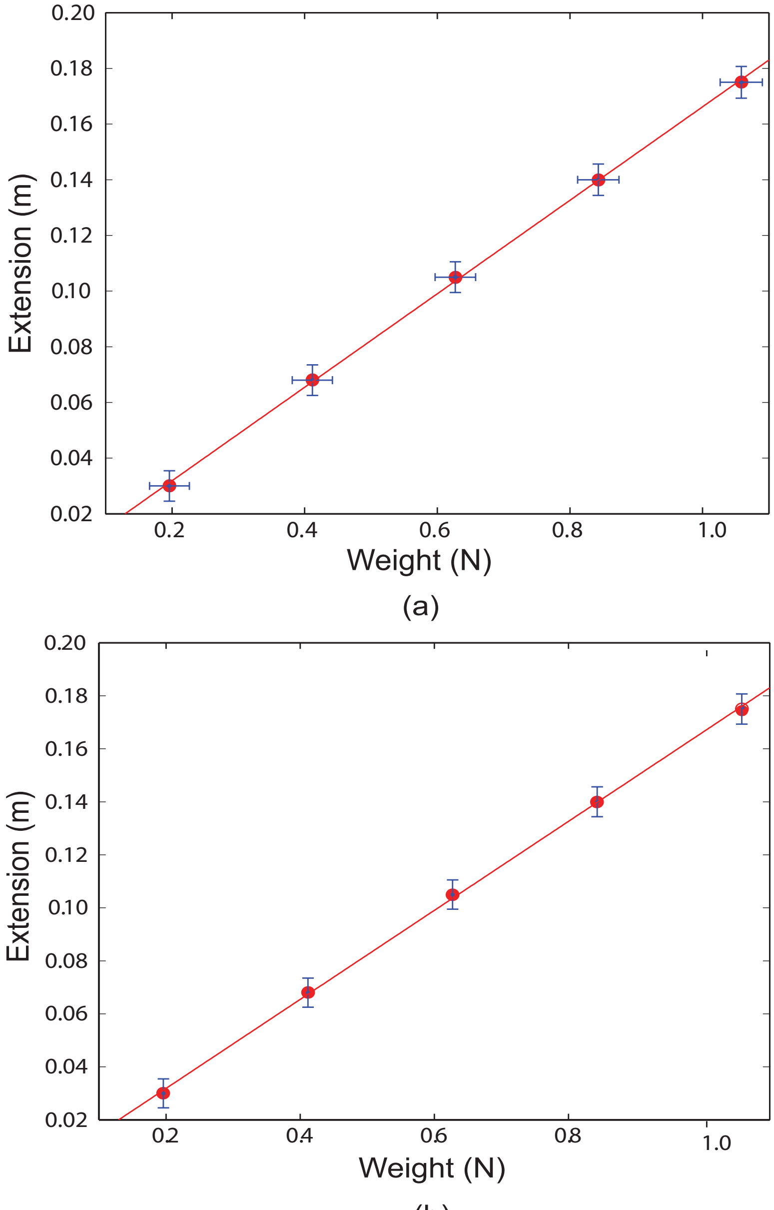


Figure 8: Graph with errorbars: (a) Uncertainties in both the dependent and independent variables and (b) uncertainties are transformed to the dependent variable only.

By adopting the first method, we obtain the slope,

$$\frac{dy}{dx} = 0.1679\,\mathrm{m/N},$$

and the total uncertainty in the dependent variable becomes,

$$u_T = \sqrt{u_{\text{trans}}^2 + u_y^2}. \tag{20}$$

The graph with uncertainties both in dependent and independent variables is shown in Figure (8a) while the graph for transformed uncertainty is plotted in Figure (8b). u_T is the total uncertainty which includes its inherent uncertainty u_y as well as the portion u_{trans} that it has inherited from the independent variable. From this point onwards, we proceed with the assumption that no uncertainty remains in the independent variable; we have properly accounted for it by reflecting its effect into the dependent variable.

Now introducing weights for each experimental point,

$$w_i = \frac{1}{u_{T,i}^2}. \tag{21}$$

The weights are associated with the reciprocal square of uncertainties and mean that any measurement which is less precise contributes very little to the total uncertainty. For example, if a measurement is three times less precise than the rest, its weight is 9 times less than the other weights and for many purposes this can simply be ignored. The mathematical relations for finding the values of the slope m and intercept c in this approach of weighted uncertainties are,

$$m = \frac{\Sigma w\, \Sigma w(xy) - \Sigma wx\, \Sigma(wy)}{\Sigma w\, \Sigma(wx^2) - (\Sigma wx)^2}, \tag{22}$$

$$c = \frac{\Sigma(wx^2)\, \Sigma(wy) - \Sigma(wx)\, \Sigma(wxy)}{\Sigma w\, \Sigma(wx^2) - (\Sigma wx)^2}, \tag{23}$$

where x is the independent variable, y is the dependent variable and w is the weight.
finally, the expressions for the uncertainties in m and c are,

$$u_m = \sqrt{\frac{\Sigma w}{\Sigma w\, \Sigma(wx^2) - (\Sigma wx)^2}}, \tag{24}$$

$$u_c = \sqrt{\frac{\Sigma(wx^2)}{\Sigma w\, \Sigma(wx^2) - (\Sigma wx)^2}}. \tag{25}$$

5 Propagation of Uncertainties

Most physical quantities cannot be measured directly. First, we measure some quantities directly that can be directly measured and then infer from these the quantity of interest. For example, the velocity v of a car is measured by measuring the time it takes to travel a particular distance and then calculating the speed by using $v = d/t$. We must first estimate the uncertainty in the measured quantity and then figure out how these uncertainties "propagate" through the calculations to produce an uncertainty in the final deduced answer. Given below are some rules for finding propagated uncertainties, the particular rules followed by a general rule.

5.1 Uncertainty in sums and differences

If $q = x + y$ or $z = x - y$, the uncertainty in q is given by:

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2},$$

and the uncertainties always add, no matter whether we are adding or subtracting the measured quantities. In this section, we are using $delta$ to denote the uncertainty.

5.2 Uncertainty in products and quotients

If the equation is $q = xy$ or $q = \frac{x}{y}$, then the uncertainty in q is:

$$\delta q = q\sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$

5.3 Uncertainty in a power

If the equation is $q = x^m$, the uncertainty is given by:

$$\delta q = q\sqrt{\left(\frac{m\delta x}{x}\right)^2}.$$

5.4 General formula for uncertainty propagation

We already have seen uncertainties both independent and propagate through sums, differences, products and quotients. However, many calculations involve one or more complicate functions. The question is how do uncertainties propagate in these functions. For example, what if the quantity of interest is $q(x) = 1/\sin(x)$ or $q(x) = \sqrt{x}$. In such cases the best approach is to draw a graph of $q(x)$ as shown in figure (9).

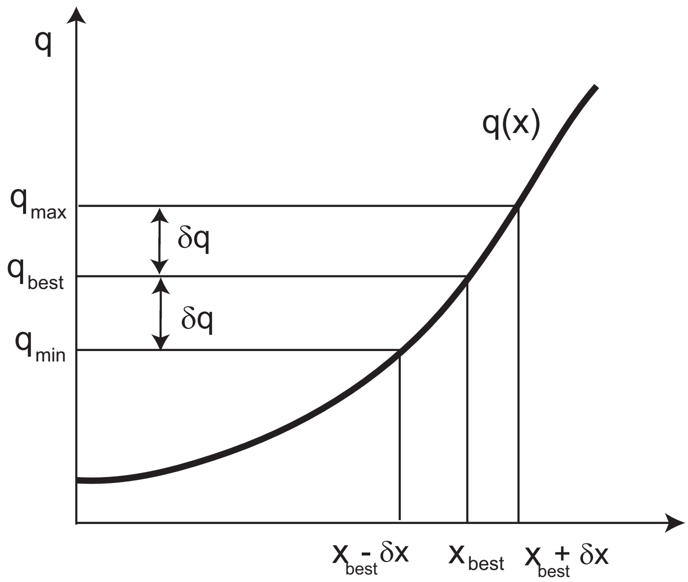


Figure 9: Graph of $q(x)$ versus x. If x is measured as $x_{\text{best}} \pm \delta x$ then the best estimate for $q(x)$ is q_{best}. The largest and smallest values of $q(x)$ which correspond to $x_{\text{best}} + \delta x$.

Suppose δx is the uncertainty in x and defines the extremity or range of variable x and is a measure of the uncertainty in x. The largest probable value of x is $x_{\text{best}} + \delta x$ and the corresponding largest value of q is q_{max}. Likewise the minimum probable value of x is $x_{\text{best}} - \delta x$ and the smallest value of q is q_{min} shown in Figure (9). If we assume the uncertainty δx is small, we can take the section of the graph under consideration to be approximately straight with q_{max}, q_{min} equally spaced and lying on either side of the q_{best}. The uncertainty δq can be estimated as,

$$\delta q = q(x_{\text{best}} + \delta x) - q(x_{\text{best}}). \tag{26}$$

Now using the fundamental approximation of of calculus that, for any function $q(x)$ with sufficiently small increment v

$$q(x + v) - q(x) = \frac{dq}{dx} v, \tag{27}$$

assuming the uncertainty δx is small and equating Equations (26) and (27) yields,

$$\delta q = \frac{dq}{dx} \delta x. \tag{28}$$

Thus the value of uncertainty in $q(x)$ is found by multiplying the derivative of q with respect to x with the uncertainty in x. You can convince yourself that the uncertainties calculated above for sum, difference and quotient etc. are in complete agreement with this fundamental approach. This is also called the Taylor series approximation.

Now if a quantity q is measured using some input variables x, y and z which are measured with uncertainties δx, δy and δz, respectively. If we assume that all the measured uncertainties are small, then δq can also be find out using the Taylor series approximation mentioned above,

$$\begin{aligned} \delta q^2 &= \left[\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta z\right)^2 + \left(\frac{\partial q}{\partial z}\delta z\right)^2\right] \\ &+ \left[\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}\delta x\delta y + \frac{\partial q}{\partial y}\frac{\partial q}{\partial z}\delta y\delta z + \frac{\partial q}{\partial z}\frac{\partial q}{\partial x}\delta z\delta x\right]. \end{aligned} \tag{29}$$

The above expression is conveniently referred as the 'law of uncertainty propagation'. The square terms are always positive and never cancel each other. However, the cross terms may cancels out due to the fact that each term may be positive or negative. Hence the exact formula for uncertainty is,

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta z\right)^2 + \left(\frac{\partial q}{\partial z}\delta z\right)^2}. \tag{30}$$

Equation (30) shows a direct relationship between multiple variables and their standard uncertainties.

References

[1] John R. Taylor, "*An introduction to Error analysis*", University Science Books, pp. 181-199, (1997).

[2] G. L. Squires, "*Practical Physics*", Cambridge University Press, pp. 12-23, (1999).

[3] Andy Buffler, Saalih Allie, Fred Lubben and Bob Campbell "*An introduction to measurement in the Physics Laboratory*", University Science Books, pp. 127-132, (2002).

[4] Les kirkup and Bob Frenkel, "*An introduction to uncertainty in measurement*", Cambrigde University Press, pp. 43-47, (2007).