Chapter 13

A backyard swing provides an example of oscillatory motion. Such motion occurs everywhere in the physical world, from vibrations in molecules to oscillations in the shape of the Sun.

# Oscillatory Motion

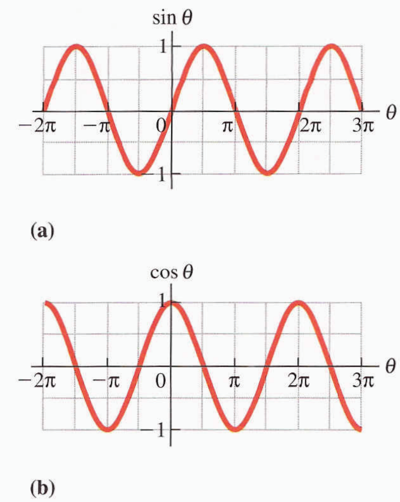
Rhythmic motion—also known as periodic motion—is a common occurrence in the physical world. The very concept of time arises from the observation that certain motions, such as the human heartbeat and the cycling of the seasons, repeat themselves in a reliable and regular way. An important class of periodic motions involves what are known as restoring forces, forces that act to bring an object back to an equilibrium point. As we have already seen in Chapter 7, such restoring forces have potential energy functions with minima at the equilibrium point. Objects in this sort of motion oscillate, and oscillatory motion is the central subject of this chapter. The most basic type of oscillatory motion is omnipresent in nature: **simple harmonic motion**. This motion occurs when the strength of the restoring force is directly proportional to the object's displacement from the equilibrium point. Everyday examples are the motion of a mass on the end of a spring and the motion of a pendulum. The position of an object in simple harmonic motion varies with time as a sine or a cosine. While the spring force is an example that we will use repeatedly, simple harmonic motion is of universal importance because virtually any small oscillatory motion about a stable equilibrium point is simple harmonic motion.

We'll also see the effects of dissipative forces in this chapter—which not surprisingly cause the motion to progressively die out—and the effects of an oscillatory driving force. The presence of the driving force illustrates the remarkable feature known as resonance, in which the motion can become catastrophically large if the frequency of the driving force is just right.

## 13–1 The Kinematics of Simple Harmonic Motion

**Simple harmonic motion**, which describes the small repeating motion followed by a mass on the end of a spring or a pendulum, is a simple form of **oscillatory** motion. The word "harmonic," signifying agreement and accord, reveals that humankind have always seen beauty in this motion. In the back-and-forth of simple harmonic motion, the position $x(t)$ of an object is of the form $\sin(\omega t)$ or $\cos(\omega t)$, where the coefficient $\omega$ is the **angular frequency**. Both sines and cosines repeat themselves periodically as time $t$ passes. The trigonometric functions are functions of a dimensionless argument, an angle measured in radians (or, sometimes, degrees). Thus the coefficient of the time must have the dimensions $[T^{-1}]$. We'll see later that the angular frequency $\omega$ is a fundamental property of the motion, determined by the inertia of the moving objects and the restoring force acting on them.

How do we figure out whether the motion of a mass on the end of a spring is described by a sine or by a cosine? Let's look at a graph of $\sin\theta$ versus $\theta$ next to a graph of $\cos\theta$ versus $\theta$ (Fig. 13–1). Both functions repeat every time the angle $\theta$ changes by $2\pi$ rad. When $\theta = 0$, the sine function is zero, whereas the cosine function is $+1$, but this is only a matter of placing the axis. Indeed, the functions are *identical* if the origin of the $\theta$ axis is shifted. We can specify such a shift of $\theta$ by an angle we call the **phase**, $\delta$. By what angle $\delta$ would $\theta$ have to be shifted so that the $\sin\theta$ curve in Fig. 13–1a is coincident with the $\cos\theta$ curve of Fig. 13-1b? If $\delta$ is chosen properly, the function $\sin(\omega t + \delta)$ can represent $\sin(\omega t)$, $\cos(\omega t)$, or anything in between. The phase simply makes explicit the "starting" point for harmonic motion. Both sine and cosine have the same shape, but displaced, and the phase sets the amount of displacement.




▲ **FIGURE 13–1** Plots of (a) $\sin\theta$ and (b) $\cos\theta$, both as a function of $\theta$.

Another quantity that characterizes oscillatory motion is how far the moving object gets from the equilibrium position before it turns around. In the case of a point mass in simple harmonic motion in, say, the $x$-direction, the motion is symmetric from one side to the other, and the maximum distance of displacement to the right of the equilibrium point equals the maximum distance of displacement to the left. We call this distance the **amplitude**, $A$. It is by definition positive. The sine function is dimensionless and varies between $-1$ and $+1$. But $x(t)$ has dimensions of length. To express $x(t)$, we therefore have to multiply the harmonic sine (or cosine) function by a constant with dimensions of length, and this constant is the amplitude $A$ described above. The resulting expression for the position of an object in simple harmonic motion is

$$x(t) = A\sin(\omega t + \delta), \qquad (13\text{–}1a)$$

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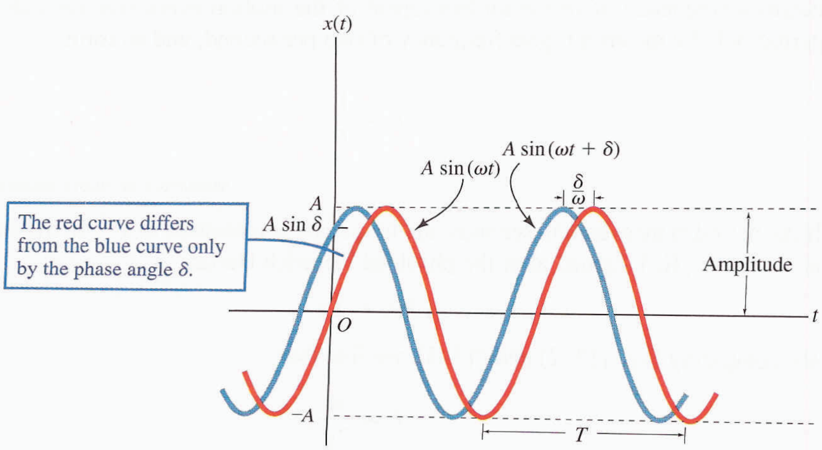
and we can immediately confirm that $A$ describes the magnitude of the maximum excursion away from the point of zero displacement (Fig. 13–2). An alternative form of this expression turns out to be very useful. We can use the basic trigonometry rule $\sin(x + y) = \sin x\cos y + \cos x\sin y$ to rewrite $\sin(\omega t + \delta)$ as

$$x(t) = [A\cos\delta]\sin(\omega t) + [A\sin\delta]\cos(\omega t).$$




◀ **FIGURE 13–2** In simple harmonic motion the phase, $\delta$, corresponds to a sliding of the curve of displacement versus time to earlier or later times. The amplitude and period of the motion are also shown.

The original quantities $A$ and $\delta$ are two constants that characterize the motion, and we can think of the two quantities in square brackets as combinations of these two constants that equally well characterize the motion. In other words, simple harmonic motion can alternatively be expressed as

$$x(t) = a_1 \sin(\omega t) + a_2 \cos(\omega t). \tag{13–1b}$$

Comparing the intermediate step with Eq. (13–1a), we find relations that can be used to connect the constants $a_1$ and $a_2$ to the constants $A$ and $\delta$,

$$a_1 = A \cos \delta,\ a_2 = A \sin \delta.$$

Inverting, we can express $A$ and $\delta$ in terms of $a_1$ and $a_2$:

$$A^2 = a_1^2 + a_2^2,\ \tan \delta = a_2/a_1.$$

Which of these two forms, Eq. (13–1a or b), is more convenient depends on the circumstances, and we'll sometimes use one and sometimes the other.

## Properties of Simple Harmonic Motion

Three independent parameters appear in simple harmonic motion and describe the motion: the amplitude $A$, the phase $\delta$, and the angular frequency $\omega$. The amplitude and the phase are determined by specifying the position $x(t)$ at $t = 0$ and the maximum magnitude of $x(t)$. It follows in this case from Eq. (13–1a) that $x(0) = A \sin \delta$, while $|x_{max}| = A$. These two equations give $A$ and $\delta$ in terms of $x(0)$ and $|x_{max}|$. Or one may know the position $x(t)$ and velocity $v(t) = dx/dt$ at an initial time $t = 0$. In this case we say that $A$ and $\delta$ are determined by the **initial conditions** for the motion. It follows from Eq. (13–1a) that $x(0) = A \sin \delta$. We can also use Eq. (13–1a) to find the velocity $v(t) = dx/dt = A\omega \cos(\omega t + \delta)$ [see Eq. (13–7)], so that $v(0) = A\omega \cos \delta$. The two expressions $x(0)$ and $v(0)$ are enough to specify both amplitude and phase provided that $\omega$ is known. A similar analysis can be done for Eq. (13–1b), in which the two constants $a_1$ and $a_2$ are determined by initial conditions. The fact that $\omega$ needs to be known here suggests that we should turn to that constant next.

The angular frequency $\omega$ is a measure of the repetition time for the motion, i.e., the time for one full cycle of the motion. We call this time the **period** $T$. The sine function repeats itself either when the angle increases by $2\pi$ rad (see Fig. 13–1) or, because $\delta$ is a constant, when $\omega t$ increases by $2\pi$. Thus the period satisfies $\omega T = 2\pi$. We can solve for the period:

$$T = \frac{2\pi}{\omega}. \tag{13–2}$$

*PERIOD OF SIMPLE HARMONIC MOTION*

Thus the value of the angular frequency $\omega$ determines the period. In Chapter 3, where we described uniform circular motion, we defined the **frequency**, $f$, as the number of full oscillations per unit time, or equivalently the inverse of the period. A period of 5 s means a frequency of one complete repeat of the motion every five seconds, while a period of 0.5 s means a repeat frequency of two per second, and so forth:

$$f = \frac{1}{T}. \tag{13–3}$$

*FREQUENCY OF SIMPLE HARMONIC MOTION*

If the period is measured in seconds, the frequency is measured in $s^{-1}$. In SI, the unit $s^{-1}$ is the **hertz** (Hz), named after the physicist Heinrich Hertz:

$$1\ \text{Hz} = 1\ \text{s}^{-1}. \tag{13–4}$$

By comparing Eqs. (13–2) and (13–3), we find that

$$f = \frac{\omega}{2\pi}. \tag{13–5}$$

**CONCEPTUAL EXAMPLE 13–1** Your classmate states that if the acceleration of a mass acted on by a spring is proportional to the displacement from the equilibrium point of the mass, then the farther the mass gets from the equilibrium, the larger the acceleration, and the mass will soon be accelerating so much that it will be in the next county in a few minutes. Is he right? How would you correct him?

**Answer** It is indeed true that the acceleration is proportional to the displacement, but as a look at Eq. (13–8) verifies, *there is a crucial minus sign* in the relation. This sign keeps the motion within bounds. If the displacement is to the right of the equilibrium point, the acceleration is to the left, tending to send the mass back to the left; if the displacement is to the left, the acceleration is to the right, tending to send the mass back to the right. Without the minus sign, your classmate is correct. In that case, the position is an exponential function of time rather than oscillatory.

We will see in Section 13–2 that the angular frequency $\omega$ can be identified with the angular speed, a quantity we have already defined and used in Sections 3–5 and 9–1 in connection with circular motion. Inversion of Eq. (13–2) or (13–5) gives

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{13–6}$$

When the position is specified as a function of time, the velocity and the acceleration are determined by taking successive derivatives. As a consequence of Eq. (13–1a), we have (see Appendix IV–7)

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}[A\sin(\omega t + \delta)] = \omega A\cos(\omega t + \delta). \tag{13–7}$$

One further derivative gives the acceleration as a function of time:

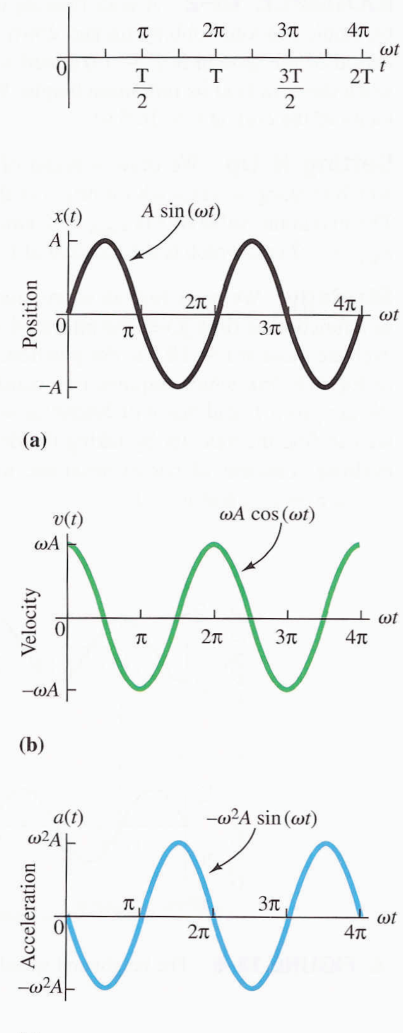
$$a(t) = \frac{dv}{dt} = -\omega^2 A\sin(\omega t + \delta) = -\omega^2 x(t). \tag{13–8}$$

*The acceleration is proportional to the displacement.* Since we will argue that virtually all stable equilibrium situations, from the back and forth of a rocking chair to the oscillation of a spider on his web in the breeze, are associated with simple harmonic motion; thus, the proportionality of the acceleration and the displacement is a universal property of motion near equilibrium.

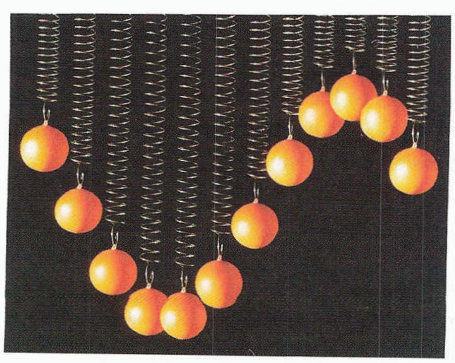
## Relations Among Position, Velocity, and Acceleration in Simple Harmonic Motion

In Fig. 13–3 we plot the position, velocity, and acceleration of an object in simple harmonic motion over two full periods, starting with $x(t) = A\sin(\omega t)$. (For convenience, the phase has been taken to be zero. The relations discussed here are not affected by the phase.)

The photo in Fig. 13–4 represents the up-and-down motion of a ball on a spring, presented so you can follow the ball's vertical position as a function of time. This motion matches the motion described in Fig. 13–3. In Fig. 13–3a the object is at the origin at $t = 0$. As we see in Fig. 13–3b, the velocity at $t = 0$ is maximum in magnitude and is positive, while Fig. 13–3c shows that at this time the acceleration is zero, so that the velocity is not changing. After one-quarter of the period ($\omega t = \pi/2$), the object has moved to the right-hand extreme of its motion and is ready to turn around. The velocity is zero at this turnaround point, but the acceleration has actually reached a maximum in magnitude and is negative, indicating that the velocity will be turning to the left and will become




▲ **FIGURE 13–3** Starting with (a) the graph of position $x(t) = A\sin(\omega t)$, a single derivative gives (b) the velocity $v(t) = \omega A\cos(\omega t)$. One further time derivative gives (c) the acceleration $a(t) = -\omega^2 A\sin(\omega t)$. We have in each case plotted two cycles, or periods, for zero phase; the curves repeat after every period. Note also, as the upper scale indicates, the correspondence between the number of periods and $\omega t$ as a multiple of $2\pi$.



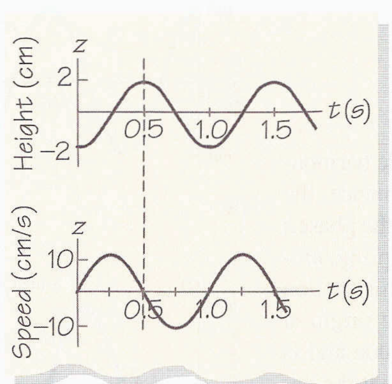
◀ **FIGURE 13–4** A photograph of the simple harmonic motion of the mass on the end of a spring.

negative. (Think of a ball thrown in the air; at the maximum height—the turnaround point—the velocity is zero even if the acceleration is nonzero and directed toward Earth.) After one-half the period $(\omega t = \pi)$, the object once again passes through the origin, this time moving to the left. The acceleration is again zero. The three-quarter mark $(\omega t = 3\pi/2)$ is at another turnaround, characterized by a maximum negative value of $x$—the object is at its left-hand extreme—and zero velocity. The acceleration is maximum and positive, meaning that the velocity is becoming positive, and the object will subsequently move back to the right. Finally, after one full period $(\omega t = 2\pi)$, the object has come back to its starting point, moving to the right through the origin with its largest positive velocity and zero acceleration. The situation at $t = 2\pi/\omega$ is identical to what it was at $t = 0$.

**EXAMPLE 13–2** A cork floating on a pond moves in simple harmonic motion, bobbing up and down over a range of 4 cm. The period of the motion is $T = 1.0$ s, and a clock is started at $t = 0$ s when the cork is at its minimum height. What are the height and velocity of the cork at $t = 10.5$ s?

**Setting It Up** We draw a graph of the motion in Fig. 13–5, which is along a $z$-axis whose origin is the midpoint of the motion. The maximum value of $z$ is $z_{max} = 2$ cm, and the minimum value is $z_{min} = -2$ cm, which is the location at $t = 0$.

**Strategy** We must find an expression for position and velocity as a function of time given the information in the problem, and then evaluate these at $t = 10.5$ s. For position, we'll use the general form of Eq. (13–1b), which requires two constants and knowledge of $\omega$. We are given $T$, and that will determine $\omega$ directly. With Eq. (13–1b) we can find the velocity by taking the derivative of the position. To evaluate constants of our expressions, we can use the facts that at $t = 0$, $z = z_{min}$ and $v = 0$.




▲ **FIGURE 13–5** The height and speed of a bobbing cork in a pond.

**Working It Out** We know the period, $T$, and from Eq. (13–6), $\omega = 2\pi/T$. The motion (position) takes the general form

$$z(t) = a_1 \sin(\omega t) + a_2 \cos(\omega t).$$

With a single derivative, we also get the velocity:

$$v(t) = a_1\omega \cos(\omega t) - a_2\omega \sin(\omega t).$$

To find the constants $a_1$ and $a_2$, we use the initial conditions. As stated above, these read, $z = z_{min}$ and $v = 0$ at $t = 0$. The second equation above is simple to apply: $v$ can only be 0 at $t = 0$ if the constant $a_1 = 0$. Applying this, the condition that $z = z_{min}$ at $t = 0$ then gives immediately $a_2 = z_{min}$. Finally, $\omega = 2\pi/T = 2\pi/(1\text{ s}) = 2\pi$ rad/s. In summary,

$$z(t) = a_2 \cos(\omega t)$$

and

$$v(t) = -a_2\omega \sin(\omega t)$$

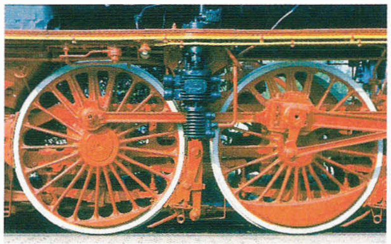
with $a_2 = -2$ cm and $\omega = 2\pi$ rad/s.

The second part of Fig. 13–5 shows the velocity of the cork.

It is straightforward to plug $t = 10.5$ s into these expressions. We can also employ some simple reasoning to make a shortcut to the numerical answer. Since both $z$ and $v$ repeat themselves every period, the values of $z$ and $v$ at 10.5 s are the same as at 0.5 s (0.5 period). Moreover, after half a period, the cork moves from the bottom of the motion to the top, i.e., $z$ will move from $z_{min}$ to $z_{max}$ and the velocity will once again be zero as the motion of the cork turns around. Thus

$$\text{for } t = 10.5\text{ s}, z = z_{max} = +2\text{ cm} \quad \text{and} \quad v = 0\text{ m/s}.$$

**What Do You Think?** For what value(s) of $z$ does the acceleration of the cork have maximum magnitude? For what value(s) of $z$ does the acceleration have minimum magnitude? *Answers to What Do You Think? questions are given in the back of the book.*



▲ **FIGURE 13–6** The relation between uniform circular motion and simple harmonic motion is evident in the piston-linkage connection on the train wheel and the resulting motion.

## 13–2 A Connection to Circular Motion

In Chapter 3 we discussed another kind of periodic motion: uniform circular motion. The photograph in Fig. 13–6 of the wheels and driving piston of a steam engine suggests that circular motion has a simple connection to harmonic motion, and we next demonstrate this connection in more detail. Figure 13–7 shows uniform circular motion for a point moving in the $xy$-plane a constant distance $R$ from the origin. The motion is described by an angle $\theta$, measured from the $x$-axis, that varies linearly with time:

$$\theta = \omega t + \delta. \tag{13–9}$$

The phase, $\delta$, is just the value of $\theta$ at time $t = 0$.

If we were to look at a side view of the uniform circular motion of a pin stuck on a rotating turntable, we would see the pin oscillate in simple harmonic motion. Figure 13–7

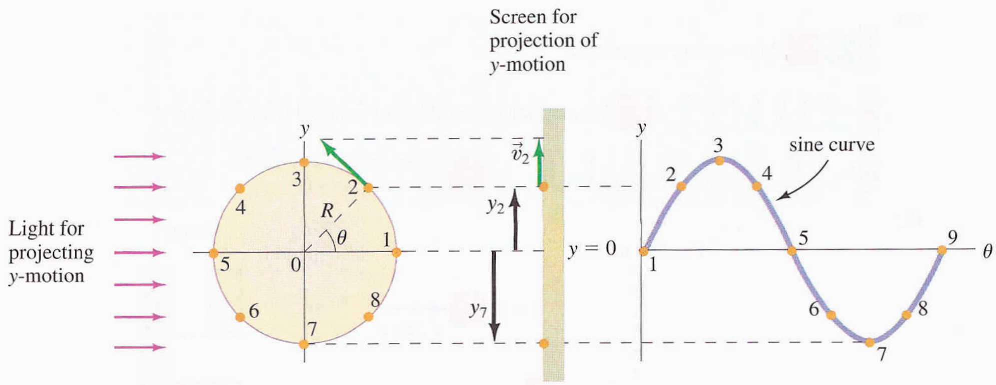



**FIGURE 13–7** Uniform circular motion in the $xy$-plane, and its projection onto the $y$-axis. The projection represents simple harmonic motion, easily visible in a plot of $y$ versus $\theta = \omega t + \delta$, as on the right.

indicates the *projection* of the circular motion on the $y$-axis, but you could easily project onto both $x$ and $y$. Simple trigonometry gives us these projections:

$$x = R\cos\theta = R\cos(\omega t + \delta); \tag{13–10}$$

$$y = R\sin\theta = R\sin(\omega t + \delta). \tag{13–11}$$

Thus uniform circular motion corresponds to simple harmonic motion in both the $x$- and $y$-directions. A cosine rather than sine appears in $x$, but as we discussed above, this is just the standard form with a different phase. We can use the trigonometric identity $\sin[\theta + (\pi/2)] = \sin\theta\cos(\pi/2) + \cos\theta\sin(\pi/2) = \cos\theta$ to replace the cosine in Eq. (13–10) with a sine function, and we thereby obtain

$$x = R\sin\left(\omega t + \delta + \frac{\pi}{2}\right). \tag{13–12}$$

Both the $x$- and $y$-motions are now in the standard form of Eq. (13–1a). The two motions have a phase that differs by exactly $\pi/2$ (90°), and the sign of this phase difference specifies the direction—clockwise or counterclockwise—of the corresponding uniform circular motion (see Problem 22).

## 13–3 Springs and Simple Harmonic Motion

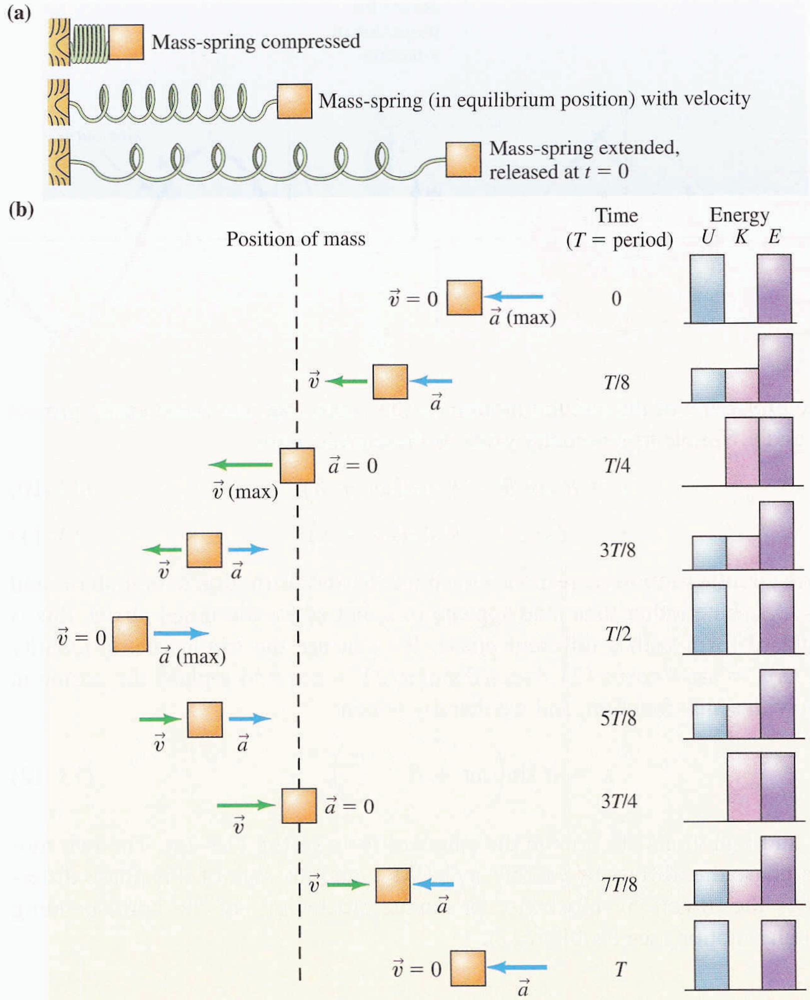
Having described simple harmonic motion—the kinematics—and armed with our knowledge of Newton's second law, we now can turn to the cause of the motion. Springs give rise to simple harmonic motion. Let's restrict ourselves to one-dimensional motion and dispense with vector notation. The spring force on a mass displaced by $x$ from the equilibrium position of the spring is a restoring force linearly dependent on $x$, the form known as *Hooke's law*:

$$F = -kx. \tag{13–13}$$

This form is valid provided the spring is not overly stretched or compressed, in which case it loses its "springiness" and distorts—this is why we have spoken about "small" motions about the equilibrium point. Here $k$ is the *spring constant*. It is the minus sign in Eq. (13–13) that indicates that the force is a *restoring force*. A displacement in the $+x$-direction gives rise to a force that acts in the $-x$-direction and vice versa. Figure 13–8a shows a series of possible starting points for the motion. Let us choose the third one, where the mass is released at $t = 0$ from an extended position. The resulting motion is shown in Figure 13–8b over a complete period of the motion. Newton's second law provides us with the connection between the force and the acceleration; namely, $F = -kx = ma$. Thus the acceleration of a mass on the end of a spring is proportional to its displacement, with a minus sign:

$$a = -\frac{k}{m}x. \tag{13–14}$$

An acceleration proportional to the position, with a minus sign, is just the kinematic characteristic that we found in Section 13–1 for simple harmonic motion. Comparison of




▶ **FIGURE 13–8** (a) Some possible starting points for the motion of a mass on the end of a spring. (b) The simple harmonic motion of the mass when it is released from the stretched position. The speed is lowest (and the acceleration is highest) when the displacement from equilibrium is a maximum, and the speed is highest (and the acceleration is lowest) when the displacement is a minimum. We can also see the play between kinetic and potential energy; one is large where the other is small.

Eqs. (13–8) and (13–14) yields the important result that the angular frequency is determined by the mass and the spring constant:

$$\omega^2 = \frac{k}{m}; \tag{13–15}$$

$$\omega = \sqrt{\frac{k}{m}}. \tag{13–16}$$

*ANGULAR FREQUENCY FOR MASS ON A SPRING*

In turn, Eqs. (13–2) and (13–3) give the period and the frequency of the oscillations:

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{13–17}$$

Remarkably, *the period of the motion is independent of the amplitude*. The same is then true for the frequency.

The spring is the prototype of dynamical systems moving back and forth about a stable equilibrium—virtually all such systems exhibit simple harmonic motion. All these systems reduce to a mass on the end of a spring, in that the *form* of the force is the same as that of the spring, a restoring force linear in some variable.

**CONCEPTUAL EXAMPLE 13–3** The spring constant $k$ of a mass-spring system is doubled. By what factor does $m$ have to change so that (a) the acceleration at $x = 0$ is unchanged; (b) the acceleration at $x = A$ [$A$ is the original amplitude] is unchanged; (c) the velocity at $x = A$ is unchanged; (d) the period of the motion is unchanged?

**Answer** (a) The acceleration is proportional to $x$; at $x = 0$, the acceleration remains zero, regardless of the values of $k$ or $m$.

(b) From Eq. (13–14), the original acceleration at $x = A$ is $a = -(k/m)A$. If $k$ is doubled, doubling $m$ will leave $a$ unchanged.

(c) The velocity at the extremes of the motion—i.e., $x = A$—is zero, and this is independent of the values of $k$ or $m$.

(d) The period is inversely proportional to the angular frequency, which is in turn a function of $k/m$. So double $m$ to leave the period unchanged.

**EXAMPLE 13–4** A mass $m = 0.50$ kg moves along the $x$-direction under the influence of a spring with spring constant $k = 2.0$ N/m. The origin of the $x$-axis is at the equilibrium point of the mass. At $t = 0$ s, the mass is at the origin and moving with a speed of 0.50 m/s in the $+x$-direction. (a) At what time $t_1$ does the mass first arrive at its maximum extension? (b) What is this maximum extension?

**Setting It Up** We note specifically that we are given initial conditions, in this case the position and velocity at $t = 0$ s.

**Strategy** We can give a description of the position at all times, then substitute specific times. The motion as a function of time is given by either of the two Eqs. (13–1)—we'll use Eq. (13–1a) here. With $k$ and $m$ known, we can find the angular frequency, $\omega$. And the initial conditions will be sufficient to find the two remaining parameters $A$ and $\delta$ in Eq. (13–1a). In part (a) we want the time to go from the origin to the maximum extension, and this is just a quarter period—for that we need only $\omega$. For part (b) the parameter $A$ *is* the maximum extension.

**Working It Out** From Eq. (13–16), the angular frequency, $\omega$, is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.0\ \text{N/m}}{0.50\ \text{kg}}} = \sqrt{4.0\ \text{s}^{-2}} = 2.0\ \text{rad/s}.$$

(a) The time to go from the equilibrium position to the maximum extension is $T/4$:

$$t_1 = \frac{1}{4}T = \frac{1}{4}\frac{2\pi}{\omega} = \frac{1}{4}\frac{2\pi\ \text{rad}}{2.0\ \text{rad/s}} = \frac{1}{4}3.1\ \text{s} = 0.78\ \text{s}.$$

(b) We use the information about $x$ and $v$ at $t = 0$ s to find the amplitude. Writing $x(t) = A\sin(\omega t + \delta)$, we have $x(t = 0) = A\sin\delta = 0$. This implies that $\delta = 0$. We use this, in turn, for the value of $v$ at $t = 0$, $v(t = 0) = A\omega\cos(0) = A\omega$. (The argument of the cosine is zero because both $t$ and $\delta$ are zero.) Thus

$$A = \frac{v(t = 0)}{\omega} = \frac{0.50\ \text{m/s}}{2.0\ \text{rad/s}} = 0.25\ \text{m},$$

which is the maximum excursion of the mass from the origin.

**What Do You Think?** If the speed at $t = 0$ were doubled, then the time to reach the maximum extension would be (a) doubled (b) the same (c) halved.

## Additional Constant Forces

Suppose we start with a spring force and we add a constant force to it that acts along the same line. How different is the motion of an object under the influence of both these forces from the motion with the spring force alone? The answer is, remarkably little. The only thing that changes is the equilibrium point. As we have seen, the original (one dimensional) spring force always takes the form $F_{\text{spring}} = -k(x - x_0)$ (the sign takes into account the vector nature in one dimension), here aligned with the $x$-axis. The quantity $x - x_0$ is the displacement of the mass from its equilibrium point at $x = x_0$. The period of this spring, or indeed any spring, is independent of the equilibrium point. Now imagine adding (also acting along the $x$-axis) a constant force $F_c$. We can *always* write $F_c$ in the form

$$F_c = kx_1,$$

where $k$ is the same spring constant as for the original spring and $x_1 \equiv F_c/k$. That means the net force takes the form

$$F_{\text{net}} = F_{\text{spring}} + F_c = -k(x - x_0) + kx_1 = -k(x - [x_0 + x_1]).$$

This is again a spring force, with the same spring constant as the original spring force. Thus the motion will have the same frequency, but a shifted equilibrium point, $x_0 + x_1$ instead of $x_0$.

This behavior is exhibited by a mass hanging vertically from a spring. The supplementary constant force is that of gravity. The frequency of the simple harmonic motion will be the same whether the spring is hanging vertically or not. For the hanging case, and assuming the spring itself is much less massive than the mass attached to its end, the equilibrium position will be lowered by an amount $\Delta y$ proportional to the additional weight of the mass, as in Fig. 13–9. More precisely, we have

$$mg = k\,\Delta y, \quad \text{or} \quad \Delta y = mg/k.$$

The harmonic motion is measured from the new equilibrium position.

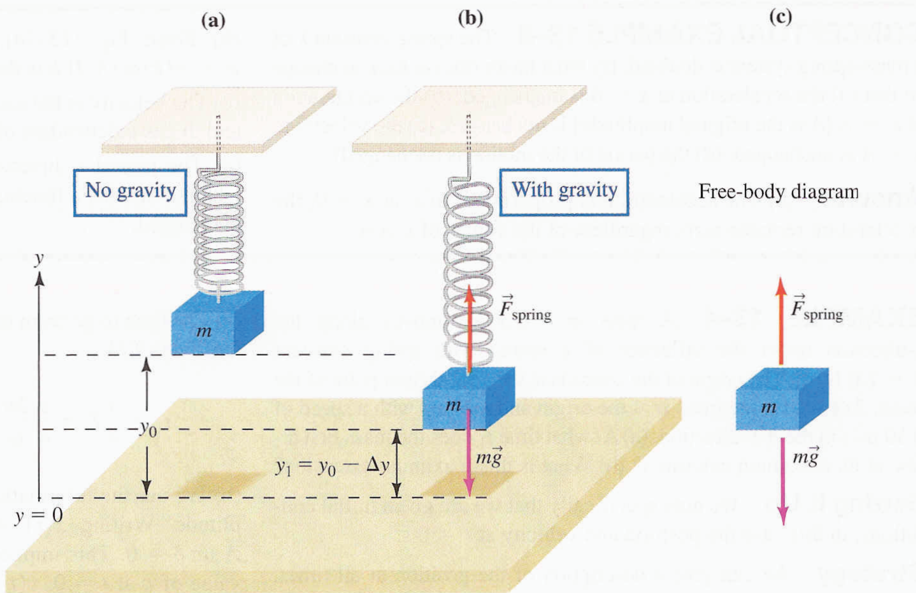



**FIGURE 13–9** A mass on the end of a spring is suspended vertically. (a) If its equilibrium length would place it at height $y_0$ in the absence of gravity, then (b) it will be stretched an additional amount, $\Delta y$, to a new equilibrium position, $y_1$, under the influence of gravity. (c) Free-body diagram for the mass.

## 13–4 Energy and Simple Harmonic Motion

We examined energy considerations for the spring force in Chapter 7, where we found that the work done by a spring force in moving a mass from one position to another is independent of the path taken by the mass. That means that the spring force is conservative and has a potential energy function $U(x)$ associated with it. The total energy $E$ (the sum of kinetic energy, $K$, and potential energy) is *conserved* throughout any motion.

In Section 7–1 we computed the potential energy $U(x)$ of an object attached to a spring and found

$$U(x) = \frac{1}{2}kx^2. \tag{13–18}$$

*POTENTIAL ENERGY FOR MASS ON A SPRING*

In Eq. (13–18) zero potential energy has been chosen at the equilibrium position of the spring, $x = 0$. The kinetic energy is simply

$$K = \frac{1}{2}mv^2. \tag{13–19}$$

Because both $x$ and $v$ are known for simple harmonic motion from Eqs. (13–1) and (13–7), the variation in time of $U$ and $K$ can be plotted. If we write the argument $\omega t + \delta$ as $\theta$, we have

$$U = \frac{1}{2}kA^2 \sin^2 \theta, \tag{13–20}$$

and using $\omega^2 = k/m$ [Eq. (13–15)],

$$K = \frac{1}{2}mA^2 \omega^2 \cos^2 \theta = \frac{1}{2}kA^2 \cos^2 \theta. \tag{13–21}$$

Figure 13–10 is a plot of the potential and kinetic energy functions as $\theta$ varies between 0 and $2\pi$, which corresponds to a complete cycle. Both $\sin^2 \theta$ and $\cos^2 \theta$ vary between 0 and 1; when $\sin^2 \theta$ is a minimum, $\cos^2 \theta$ is a maximum and vice versa. Thus $U$ and $K$ each vary between 0 and $kA^2/2$. Suppose that an object attached to a spring starts at the origin and moves to the right, motion you can follow on the graphs of Fig. 13–10. At the origin the potential energy is zero and $K$ is a maximum. As the mass moves to the right, it slows until it has reached its turnaround point at one-quarter cycle, where the velocity and hence $K$ are zero. Because $x$ is at its maximum here, $U$ is also a maximum. The mass now moves to the left, gaining speed until the speed is a maximum as it passes

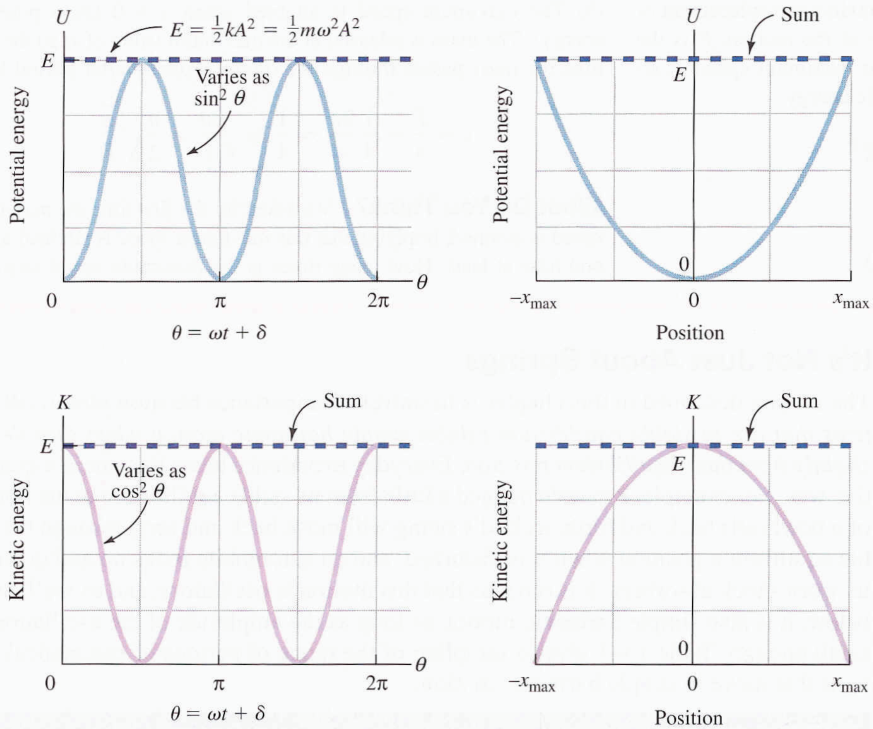



**FIGURE 13–10** The potential energy and the kinetic energy of a mass in simple harmonic motion plotted over one cycle (a) as a function of $\theta$, with the origin at the equilibrium point, and (b) as a function of displacement $x$. When one is a maximum, the other is a minimum, and their sum, the total energy, is conserved.

through the origin once more. Here, after one-half cycle, $K$ is a maximum and $U$ is a minimum. Finally, at the left-hand turnaround point, $K$ is a minimum and $U$ is a maximum. *The energy flows back and forth between U and K.*

## The Total Energy

The total energy, $E = U + K$, must be constant. We have (again, $\theta = \omega t + \delta$)

$$E = \frac{1}{2}kA^2 \sin^2\theta + \frac{1}{2}kA^2\cos^2\theta$$
$$= \frac{1}{2}kA^2[\sin^2\theta + \cos^2\theta]. \qquad (13\text{–}22)$$

Because the sum of $\sin^2\theta$ and $\cos^2\theta$ is unity for any $\theta$, $E$ is indeed constant in time:

$$E = \frac{1}{2}kA^2. \qquad (13\text{–}23)$$

*TOTAL ENERGY OF MASS ON A SPRING*

The dependence of energy on the square of the amplitude is typical of simple harmonic motion.

**EXAMPLE 13–5** A mass $m$ attached to a spring of spring constant $k$ is stretched a length $X$ from its equilibrium position and released with no initial motion. (a) What is the maximum speed attained by the mass in the subsequent motion? (b) At what time is this speed first attained?

**Strategy** For part (a) the conservation of energy is a useful tool. Initially all the energy is potential, and the maximum speed occurs later, when all the potential energy is converted to kinetic energy. Once we know the maximum kinetic energy, we also know the maximum speed. For part (b) we are asked about time, and we need more information than energy alone can supply. However, we can use our knowledge that in spring motion the potential energy is zero when the mass passes through the origin, and that time is one-quarter period later than the time it is at a maximum extension, which in this case is the starting point of the motion.

**Working It Out** (a) Just before the mass is released from rest at a position $x = X$, all of its energy is potential energy; that is, the total energy is

$$E = \frac{1}{2}kX^2.$$

*(continues on next page)*

This agrees with Eq. (13–23) because the maximum displacement of the motion is, by definition, the amplitude of the motion. $E$ is the value of the energy at all times. When the maximum speed is attained, all the energy is in the form of kinetic energy:

$$\frac{1}{2}mv_{\max}^2 = E = \frac{1}{2}kX^2.$$

We solve for $v_{\max}$:

$$v_{\max} = \sqrt{\frac{k}{m}}X = \omega X.$$

(b) The maximum speed is attained when $x = 0$ (zero potential energy). The mass is released at the maximum value of $x$, so the first time the mass passes through the origin is one-quarter period later:

$$t = \frac{T}{4} = \frac{1}{4}\frac{2\pi}{\omega} = \frac{1}{4}2\pi\sqrt{\frac{m}{k}} = \frac{\pi}{2}\sqrt{\frac{m}{k}}.$$

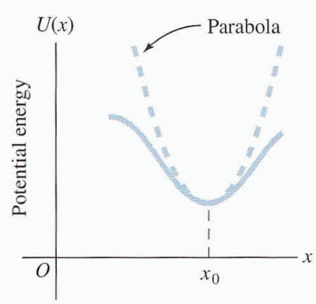
**What Do You Think?** We asked for the first time the maximum speed is attained, implying that this maximum speed is attained a second time at least. How many times is the maximum speed attained?

## It's Not Just About Springs

The motion described in this chapter is of universal importance because *almost all systems that are in stable equilibrium exhibit simple harmonic motion when they depart slightly from their equilibrium position.* Everyday experience bears this out in a qualitative way. For example, a marble nudged a little from its stable equilibrium at the bottom of a bowl rolls back and forth, a child's swing will move back and forth through the stable equilibrium position when it is disturbed, and an automobile rocks up and down on its worn shock absorbers. It is obvious that this motion is oscillatory, and as we'll argue below, it is also simple harmonic motion as long as the amplitude of the oscillations is small enough. Table 13–1 gives a sampling of the range of periods of mechanical systems that move in simple harmonic motion.

**TABLE 13–1 • Periods of Mechanical Systems in Simple Harmonic Motion**

| Mechanical System | Period (s) |
|---|---|
| Sloshing of water in a tidal basin or large lake | $10^2$ to $10^4$ |
| Large structures (bridges, buildings) | $> 1$ |
| Strings or air columns of musical instruments | $5 \times 10^{-2}$ to $10^{-4}$ |
| Piezoelectric crystals, ultrasound generators | $10^{-5}$ to $5 \times 10^{-1}$ |
| Vibrations in molecules | $10^{-14}$ |




▲ **FIGURE 13–11** A potential well, in which potential energy has a minimum at $x = x_0$. This point is a point of stable equilibrium. The dashed line is a parabola that matches the minimum of the well.

The discussion of energy in this chapter tells us why simple harmonic motion occurs in these situations. For a spring, and indeed for every case of stable equilibrium, a mass is confined to a *potential energy well* (Fig. 13–11). A potential energy well has a *minimum* on a graph of potential energy versus a position variable. For a spring the position variable $x$ is the stretch of the spring, and the minimum potential energy occurs at zero stretch, the position of stable equilibrium. In this case the potential energy function is parabolic in $x$; it is proportional to $x^2$. The reason that almost any oscillation about a stable equilibrium point is simple harmonic motion is that in most cases any minimum in a potential energy–versus–position curve is a parabola close enough to the minimum point, at least if the amplitude of the motion is not too large.

The Taylor expansion (Appendix IV–8) is a very general mathematical result that allows us to see why the minimum of a potential energy well forms a parabola and explains why simple harmonic motion is universal near equilibrium. Suppose we apply the Taylor expansion to a potential energy function near a minimum. Let's label the position of the minimum as the origin, $x = 0$. Then the Taylor expansion says that

$$U(x) = \{U(0)\} + x\{U'(x)|_{x=0}\} + (x^2/2)\{U''(x)|_{x=0}\} + \cdots$$

where we have labeled differentiation with respect to $x$ with a prime. The quantities in curly brackets in this expression are constants, and the variable $x$ no longer appears in them. The constant $U(0)$ plays no physical role, and as we know, we can always replace it by 0. (This is implicit in the expression $U = \frac{1}{2}kx^2$ that applies for the spring itself.) The first derivative of $U$ at $x = 0$ is zero because that is a minimum point. Thus, if we keep the first nonzero term in the Taylor expansion—and this is a good approximation if $x$ remains small, so our result refers to small oscillations—we find

$$U(x) \cong (x^2/2)U''(0). \tag{13–24}$$

[Here we have used the notation $\{U''(x)|_{x=0}\} = U''(0)$]. This is indeed in the form of a spring force, with $U''(0)$ playing the role of the spring constant. Thus the force takes the general form near the equilibrium point:

$$F(x) = -\frac{dU}{dx} \cong -U''(0)x. \tag{13–25}$$

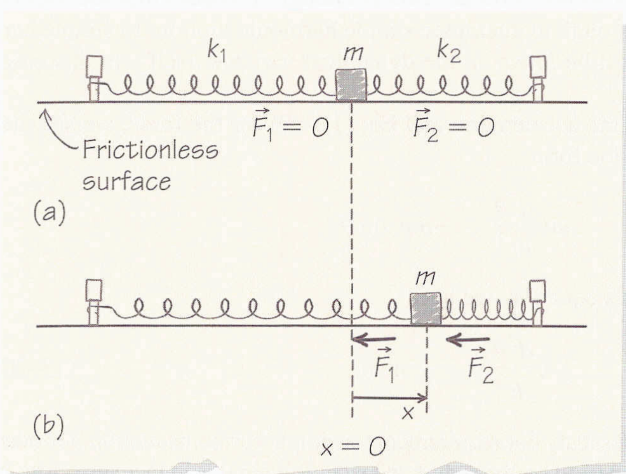
The force is proportional to the displacement and in a direction opposite to the displacement. It is the familiar linear restoring force of the spring.

We can conclude that almost all stable equilibrium behavior is simple harmonic motion close to the equilibrium point. (The "almost" is present as it is conceivable that a force might have a potential for which the term $U''(0)$ is zero. This requires, however, a restoring force of a very special form, and for these cases you would have to go to the term of order $(x^3)$ in $U$ to find the leading term. Figure 13–11 shows how a minimum on a potential energy curve can be approximated by a parabola, the dashed curve in Fig. 13–11.

**EXAMPLE 13–6** A mass $m$ on a frictionless table is attached to two pegs by springs with spring constants $k_1$ and $k_2$, respectively. The mass can move along the straight line between the pegs. The separation between the pegs has been arranged so that each spring is in its relaxed position, neither stretched or compressed, when the mass is placed at an equilibrium position. What is the motion of the mass when it is displaced from this position? In particular, assuming the motion is periodic, what is the period?

**Setting It Up** In Fig. 13–12 we show in part (a) the mass at equilibrium, at the point $x = 0$ where there is no force on the mass from either spring; in part (b) the mass is displaced to position $x \neq 0$ as indicated. We measure $x$ positive to the right. The physical situation implies a point of equilibrium at $x = 0$. When the mass moves away from $x = 0$, the forces tend to send it back to that point, so it is stable in this position. We want to show that when the mass is displaced from $x = 0$, the net force is a linear restoring force, and then find the period of the harmonic motion.

**Strategy** The motion is one-dimensional, along the line between the pegs. We find the net force acting on the mass, which is a force composed of the forces from the two springs. From the general discussion of stable equilibrium, we expect that the net force will be proportional to the displacement $x$, and the coefficient will give us the *net, or effective, spring constant*. From this we can deduce the period of the motion. To calculate the net force, we simply add the two forces, taking into account their signs.




▲ **FIGURE 13–12** In (a) the mass is at its equilibrium position (no net force acts on it). In (b) the mass is no longer in the equilibrium position, and it feels a force from both springs.

**Working It Out** We let positive values of force be to the right, which takes care of the vector aspect of this problem. From Fig. 13–11 we see that for the displacement shown, the force from the left-hand spring is $F_1 = -k_1x$, while the force from the right-hand spring is similarly $F_2 = -k_2x$. [You can check that the signs are correct: With $x$ positive (to the right), the left-hand spring is stretched and its force is to the left, while the right-hand spring is compressed and its force is also to the left.] Adding, the net force on the mass is

$$F_{\text{net}} = F_1 + F_2 = -(k_1 + k_2)x.$$

Thus the two springs together act as a single spring with effective spring constant

$$k_{\text{eff}} = k_1 + k_2.$$

The motion is simple harmonic motion, with period

$$T = 2\pi\sqrt{\frac{k_{\text{eff}}}{m}} = 2\pi\sqrt{\frac{k_1 + k_2}{m}}.$$

**Alternative Strategy** A different strategy utilizes the potential energy in the two springs. For the displacement of the figure, the potential energy in springs 1 and 2 are

$$U_1 = \frac{1}{2}k_1x^2 \quad \text{and} \quad U_2 = \frac{1}{2}k_2x^2,$$

respectively—we have chosen the zero of potential energy at $x = 0$ for each spring. The total potential energy is then

$$U = U_1 + U_2 = \frac{1}{2}(k_1 + k_2)x^2.$$

This is a harmonic oscillator potential energy for a spring with spring constant $k_1 + k_2$, so we get a period corresponding to spring constant $(k_1 + k_2)$, the same result we obtained using forces.

**What Do You Think?** Suppose that the initial separation between the pegs were larger than that of the problem. Would there still be harmonic motion for movement on the line between the pegs?