The Magnetic Pendulum

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Nonlinearity is a profound concept in the study of physical systems. The characteristics of seemingly very simple systems may turn out to be extremely intricate due to non-linearity. The study of chaos also begins with the study of such simple systems. The magnetic pendulum can be one such system.

A pendulum is one of the simplest and diverse systems in terms of its mathematical basis and the range of fields of science that it can relate to. Without doubt, it is a gift of reflective simplicity for reductionist science. With slight modifications, it can exhibit even exotically insightful phenomena, chaos being one of them. In this experiment, we will explore the notion of nonlinear and chaotic dynamics using a "magnetic pendulum".

KEYWORDS

Determinism · Chaos · Supersensitivity · Phase Portrait · Poincare Map · Attractor · Resonance · Rotary motion sensor.

PREREQUISITE EXPERIMENT: Chasing Chaos with an RL-Diode Circuit

Contents

1 Objectives

In this experiment, we will discover:

1. how apparently simple systems can be highly non-linear and exhibit a complex behavior under certain conditions,
2. the physical structure of dynamical systems, and
3. the conditions and consequences of the notion of super-sensitivity and its relationship with chaos through simulation and experiment.

References

[1] Yaakov Kraftmakher, "Experiments with a magnetically controlled pendulum", Eur. J. Phys. 28, 1007-1020, (2007)

[2] A. Siahmakoun, V. A. French, J. Patterson, "Nonlinear dynamics of a sinusoidally driven pendulum in a repulsive magnetic field", Am. J. Phys. 65, (1997).

[3] Priscilla W. Laws, "A unit of oscillations, determinism and chaos for introductory physics students", Am. J. Phys. 72, (2003).

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2 Introduction

Almost all of the known physical systems are essentially nonlinear. Yet, for simplicity, they can be treated as linear systems within some operating constraints. The magnetic pendulum is one such system. It is essentially a nonlinear system. So, it can help us to look into nonlinear phenomena such as chaos.

3 Apparatus

The schematic of our homebuilt apparatus is shown in Figure 1 (a) while (b) is a photograph.

Basically an AC induction motor has been used as the driving device. We use it for its readily available power supply and for its ease in speed control, which is simply achieved by turning the varying the input voltage.

The flywheel and connecting rod assembly converts the circular motion of motor's shaft to an approximately linear harmonic motion of the bearing-rod assembly to which the pendulum and rotary motion sensor are attached.

Think: What is the advantage of using a flywheel?

The Vernier Rotary Motion Sensor (RMS) encodes the angular information of the shaft into a digital stream and sends it to your LabVIEW program through LabPro. This information can then be used in Matlab for further processing.

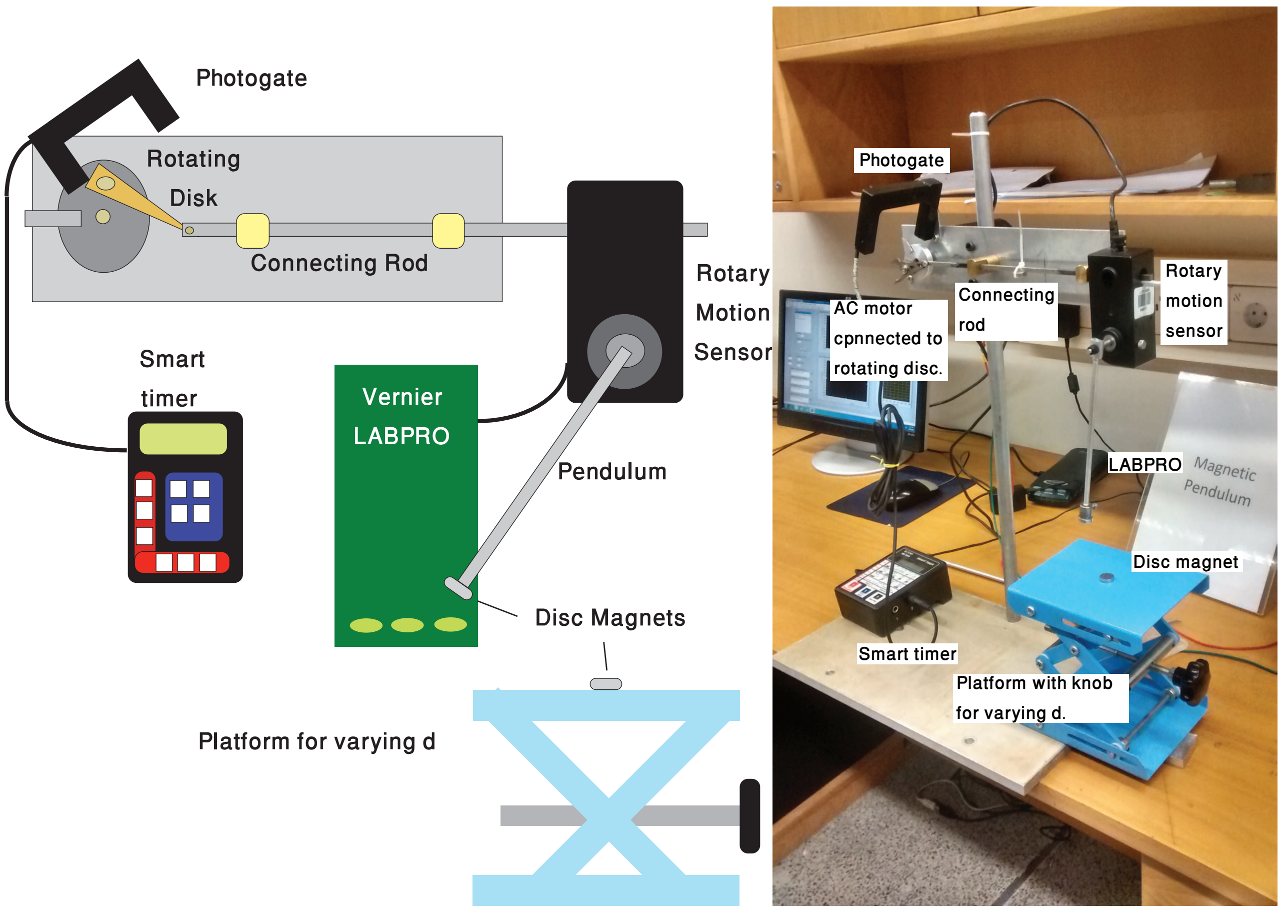
A small idential disc magnet provides the magnetic field to interact with that of the small magnet at the end of pendulum. In this way, we can control the magnitude and nature of the restoring torque and hence the nonlinearity of pendulum.

Both the magnets are kept with their same magnetic poles facing each other so that both of them repel each other.

Design Idea: Can you design a better setup to meet the same qualitative requirements?

Note that the magnetic pendulum capable of exhibiting chaotic dynamics is required to fulfill the following demands:

(a) variable amplitude and frequency of the driving force, and



(a) (b)

Figure 1: (a) A schematic illustration of the magnetic pendulum and (b) photograph of the apparatus.

(b) variable damping, such asdue to repulsive force between two magnets, and

(c) a way to display the graphical data obtained from the system (such as time series, phase plots, Poincare Map etc).

4 The Experiment

The current experiment is divided into three parts:

1. Exploring nonlinearities,
2. varying experimental parameters i.e., distance between two magnets and varying initial ampli-

tudes to study the sensitive behaviour of the pendulum. The effects will be studied both through simulation and through practical experiment, and

3. plotting a graph of maximum amplitude against angular frequency of the driving rod and observing the phenomena of hysteresis and bistability.

4.1 Exploring Non-Linearities

4.1.1 Simulating the Simple Pendulum

We start off this experiment through a pre-lab exercise exploring nonlinearity. Of course, the purpose is to see how a simple pendulum can become nonlinear. Tn this regime, the assumption that the time period is independent of the initial pendulum breaks down.

If we ignore friction a simple pendulum's equation of motion can be written as:

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{1}$$

If we make the small angle approximation i.e. $\sin(\theta) \approx \theta$ the we get:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{2}$$

Here, θ is the angular displacement of the pendulum of the length, l, and g is the acceleration due to gravity.

Question What factors control the time period of the pendulum in the case of small angles? Does time period depend on the initial amplitude from which the pendulum is displaced? Derive Eq. (1) from Newton's force equation.

For the large amplitude case we can rearrange and integrate Equation (2) to give us the following formula for time period of a pendulum:

$$T(\theta_o) = 4\sqrt{\frac{l}{g}}\,K(k) \tag{3}$$

where,

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \tag{4}$$

here $k = \sin\frac{\theta_o}{2}$ and $\sin\frac{\theta}{2} = k\sin\phi$. It can also be observed that the time period, T, is a function of θ_o which is the initial amplitude or initial point of release of the pendulum. Furthermore $K(k)$ is called the complete the complete elliptic integral of the first kind and can be easily tabulated through Matlab. If you type in the following set of commands in Matlab[1]:

```
>> theta_o=(20/180)*pi;
>> k=sin(theta_o/2);
>>  a=pi/2;
>> int('1/sqrt(1-k*k*sin(x)*sin(x))',0,a)
```

Here $theta_o$ is the initial amplitude of the pendulum defined in radians.This command will return you **EllipticK(k^2)** which is actually the answer of the integral. In order to calculate the exact value you will of the integral type in the following command:

```
>> Y=mfun('EllipticK',k^2);
```

Here Y would be your value of the integral for that particular θ_o. Substitute the value of the integral back in the time period formula and the value of the time period can be calculated for that specific θ_o. For further clarity please refer to [5]. The reference is also available on the experiment webpage.

Exercise Plot a graph time period versus θ_o. Vary values of θ_o from 0^o to 90^o. Take the value of l to be 0.1338 m. At what point would you say the transition from linearity to non-linearity occurs?

Question With the help of [5], derive Eq. (4).

4.1.2 Experiment

Now you will actually try to see the transition from linearity to non-linearity by measuring time periods for varying initial amplitudes of the pendulum. You will use a stop watch to measure the time period and ou will use your own judgment call to estimate the value for θ_o or the point of release of the pendulum.

The pendulum will be allowed to oscillate freely and the magnet below the pendulum will be removed for this part of the experiment.

Exercise Measure the time period of the pendulum for varying initial amplitudes. How do your experimental and simulation results compare?

[1]These commands require Matlab's Symbolic Math Toolbox. Type '**ver**' in your Matlab command window to check and see if you have this toolbox.

4.2 Varying the Experimental Parameters

4.2.1 Varying the distance, d, between the magnets

Precaution: Never run the motor at high speeds. It may damage the apparatus. Start from very low speeds and gradually increase when needed. If the motor instantaneously gains speed, switch it off immediately.

In this part of the experiment you will steadily decrease the distance, d, between the two magnets and observe the changes that occur in the phase portrait of the pendulum for each distance where d is the minimum distance between the two magnets. The effort will be basically focused at trying to find out that how and when the system gets trapped in a lopsided attractor and how does it jump between the two equilibrium positions (or attractors). One of the aims would also be to figure out that at what point does the pendulum exhibit chaotic behavior. In this section, all the phase portraits will be plotted using Matlab.

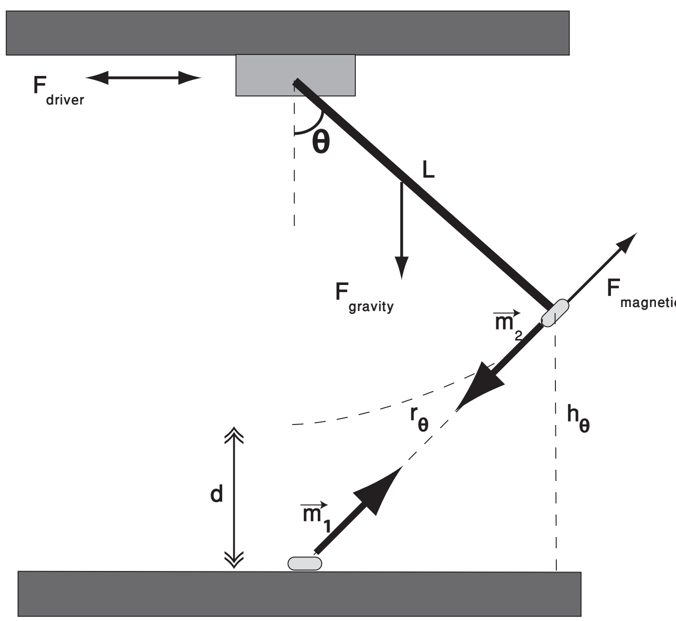


Figure 2: Schematic diagram of the physical pendulum and the forces acting on it. The distance d is between magnets when the pendulum is in the resting position. The thick arrows show the magnetic moment vectors. The angle θ is measured with the rotary motion sensor.

4.2.1.1 Simulating the Magnetic Pendulum

Once again, we will gain insight by simulating the motion of our pendulum in Matlab. For that we will need to know the equation of motion of the pendulum that will be numerically solved. The motion exhibited by our pendulum is of the rotational kind so if we can write an equation describing all the torques acting on a pendulum at any point we can derive its equation of motion. Figure 2 shows the various torques acting on the pendulum, where upon the torques can be derived.

Using Newton's Second Law we can write:

$$I\frac{d^2\theta}{dt^2} = \Sigma\tau_i \tag{5}$$

where I is the moment of inertia of the pendulum and $\Sigma\tau_i$ is the vector sum of all the torques acting on the pendulum and is given by:

$$\Sigma\tau_i = \tau_{gravity} + \tau_{driver} + \tau_{damping} + \tau_{magnetic} \tag{6}$$

Therefore we are now able to arrive at the following set of differential equations:

$$\frac{ML^2}{3}\,\dot{\omega} = -\frac{L}{2}Mg\ \sin\theta + T_{driver}\sin(\Omega t) - \gamma\omega + \frac{|\theta|}{\theta}L\frac{\mu_o}{4\pi}\frac{m_1 m_2}{r_\theta^2} \times \cos\left(|\theta| + \arctan\left(-\left|\frac{h_\theta}{L\sin\theta}\right|\right)\right) \tag{7}$$

$$\dot{\theta} = \omega \tag{8}$$

Where,

$$F_{magnetic} = \frac{\mu_o}{4\pi}\frac{m_1 m_2}{r_\theta^2}\hat{r} \tag{9}$$

Equation (9) assumes a Coulomb-like inverse square law between two magnetic moments m_1 and m_2 separated by a a distance r_θ. The unit vector $\hat{r}$ points radially away from the line joining the two magnets and indicates the direction of the magnetic force at some distance.

Here $r_\theta = \sqrt{(L\sin\theta)^2 + h_\theta^2}$ and $h_\theta = d + L(1-\cos\theta)$. The variables r_θ and h_θ are shown in Figure 2.

In this case μ_o is the permeability of vacuum, L is the length of the pendulum, M is the mass of the pendulum, g is the gravitational acceleration, T_{driver} is the maximum value of the periodic torque produced by the horizontal displacement of the pivot, Ω is the angular frequency of the driver and γ is the damping constant.

Exercise Derive each term of Equation (7). Provide an explanation where possible.

Equation (7) can be simplified and rewritten as:

$$\dot{\omega} = -A\sin\theta + B\sin(\Omega t) - C\omega + \frac{|\theta|}{\theta}\frac{E}{r_\theta^2} \times \cos\left(|\theta| + \arctan\left(-\left|\frac{h_\theta}{L\sin\theta}\right|\right)\right) \tag{10}$$

where A, B, C and E are constant coefficients that depend only the physical constants associated with the system. We are going values of these constants given for the pendulum system in [2].

The following parameter values and initial conditions will be used for the Matlab simulations: $A = 110\ \mathrm{s}^{-2}$, $B = 0.01\ \mathrm{s}^{-2}$, $C = 0.001\ \mathrm{s}^{-1}$, $E = 0.2\ \mathrm{m}^2\mathrm{s}^{-2}$, $\theta(0) = 0.2$ rad and $\omega(0) = 0$. All these values and conditions will be kept constant throughout all simulation runs.

Now to start work on the simulation download the **pendode1.m** file from the web page of the experiment. Open the m-file and you will see the following:

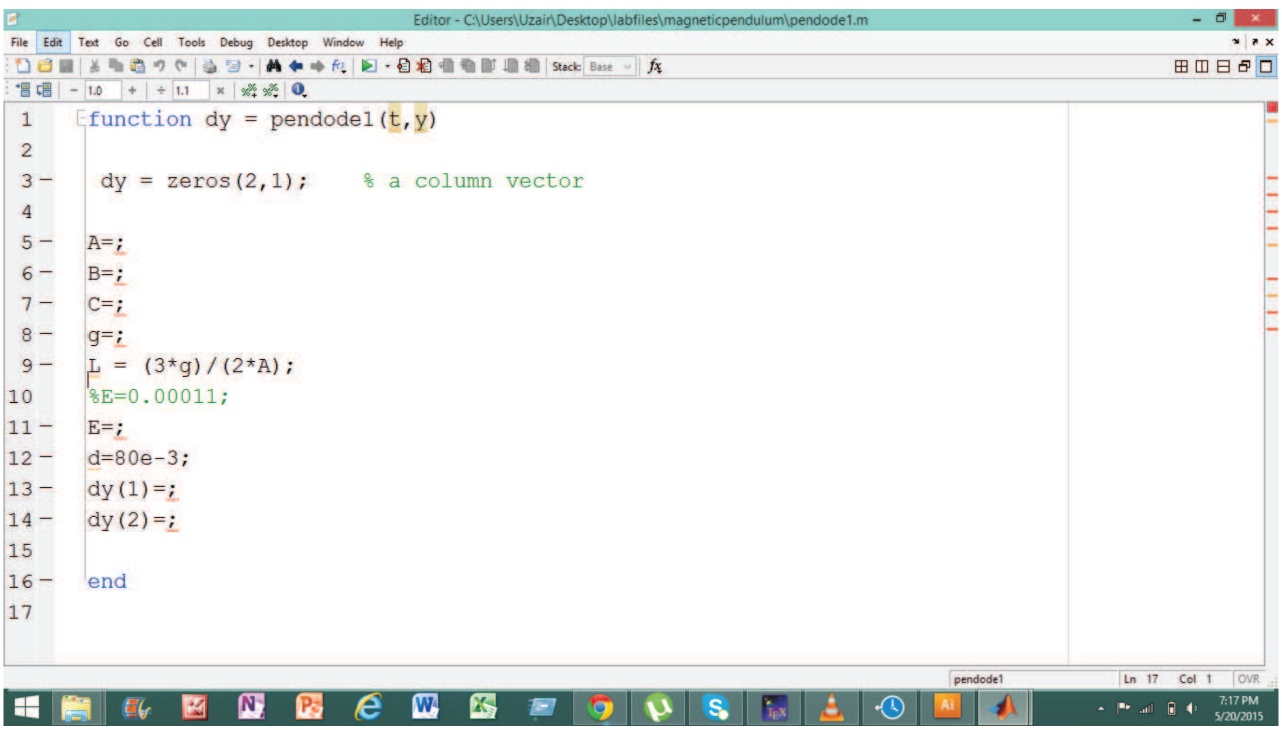


Figure 3: The **pendode1.m** file.

1. Write your values for parameters A, B, C and E. You may calculate values of these parameters for your own pendulum system and use them.

2. Write the two Equations (8) and (7), respectively in front of **dy(1)=;** and **dy(2)=;**. In our simulation the defined variables are $y(1) \equiv \omega$, $y(2) \equiv \theta$, $dy(1) \equiv \dot{\omega}$ and $dy(2) \equiv \dot{\theta}$. For example for writing $\dot{\theta} = 10\omega - 2$ you will write **dy(2)=10y(1)-2;**

3. Set the value of d for which want to run your simulation. After the parameter values have been set and the equations have been defined you will now initiate the simulation by typing in the following commands in Matlab command window:

   NOTE: The **pendode1.m** file should be saved in the current directory of Matlab because only then will you be able to call that file using Matlab's command window.

   >> **options = odeset('RelTol',2.22045e-14,'AbsTol',[1e-14 1e-14]);**
   >> **[T,Y] = ode113(@pendode1,[0 1000],[0 0.2],options);**

   Here $ode113$ is being called on to solve the system of differential equations which you have defined in **pendode1.m**. The function $ode113$ is one of the many differential equation solvers that come with Matlab. The interval [0 1000] indicates the time in seconds (or the values of time vector, T) for which the simulation will be run. In real time, the $ode113$ would take about 10-20 s to solve the equation for the specified time range of 1000 s. The vector [0 0.2] indicates the initial conditions of our two variables, $y(1) \equiv \omega$ and $y(2) \equiv \theta$.

   The $options$ command sets the relative and absolute tolerance levels of the ode solver for our two parameters: ω, θ. As we are looking at a very sensitive pendulum system therefore the tolerance levels have been set extremely low to give us a high degree of accuracy in our results. These tolerance levels have been adjusted after trials.

   After finishing off its processing the ode solver will return to you two vectors in the Matlab Variable Workspace : a time vector T and one vector Y which would have two columns each corresponding to our two variables, ω and θ.

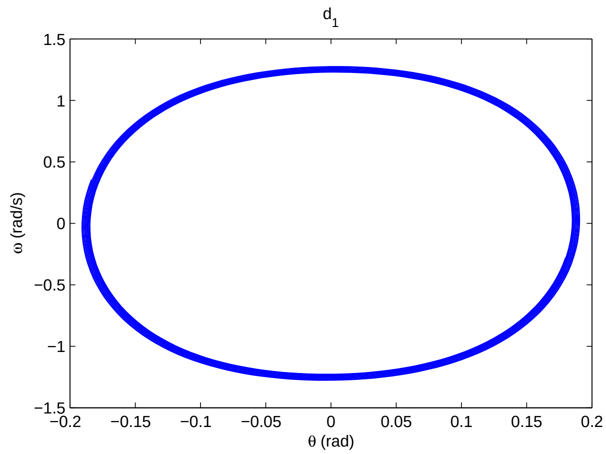
4. Plot the waveform time series of θ.

   NOTE: In our phase portrait and time series plots we will only be plotting that part of our data where the pendulum has reached a steady state. So it will have to be ensured that for any set of data of θ being used to plot there are no transient states present. It will be your judgment call in each case to tell when the simulation reaches the steady state.

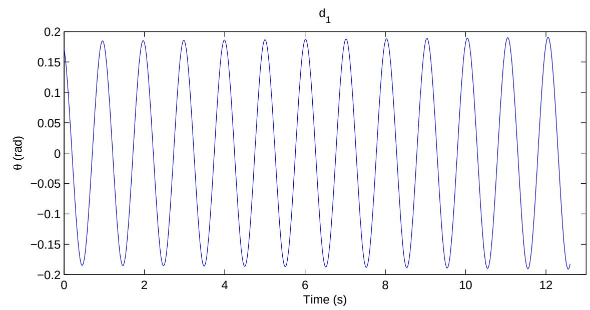
5. Plot the phase portrait using the same set of data of θ which you used to make time series.

   HINT: To plot the phase portrait you will need to plot ω vs θ. Here ω is basically $\frac{\Delta\theta}{\Delta T}$. So $\Delta\theta$ would be simply obtained by using the **diff()** command. Moreover, ΔT would have to be inferred in the same way from your corresponding T vector data set.

6. A set of sample results for varying distances are shown in the following pages. Here $d_1 > d_2 > d_3 > d_4 > d_5 > d_6$. The step size between each of these distances is not necessarily equal. You will need to reproduce these results and also find out a value of d for which the pendulum exhibits chaotic behavior. All the times in the in the time series graphs have been 'zeroed', starting from zero to maintain consistency.

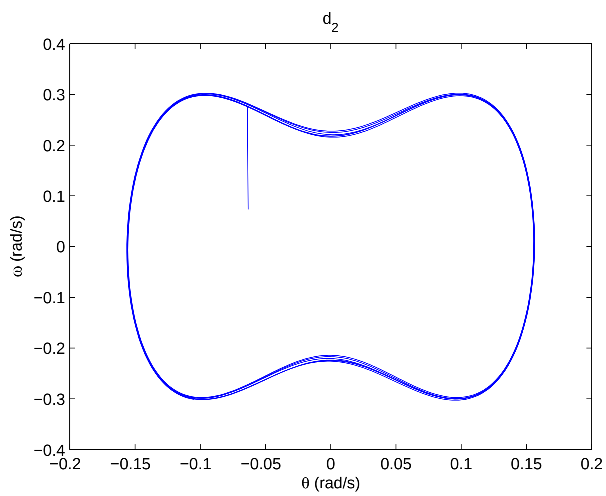


(a)



(b)

Figure 4: (a) Phase plot and (b) time series showing periodic motion for d_1. For relatively large d, the motion is approximately similar to a simple pendulum.



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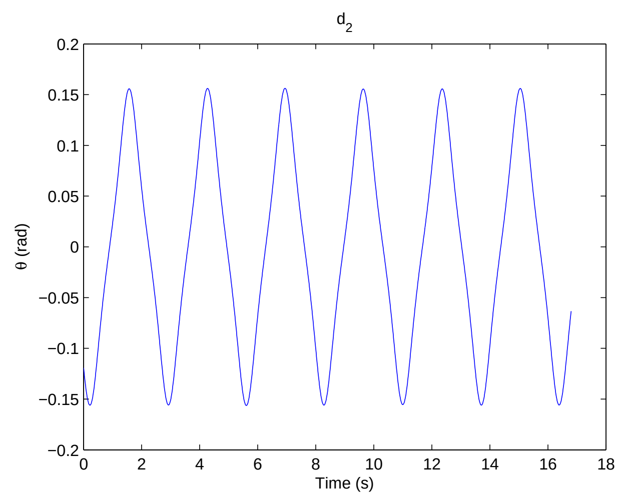
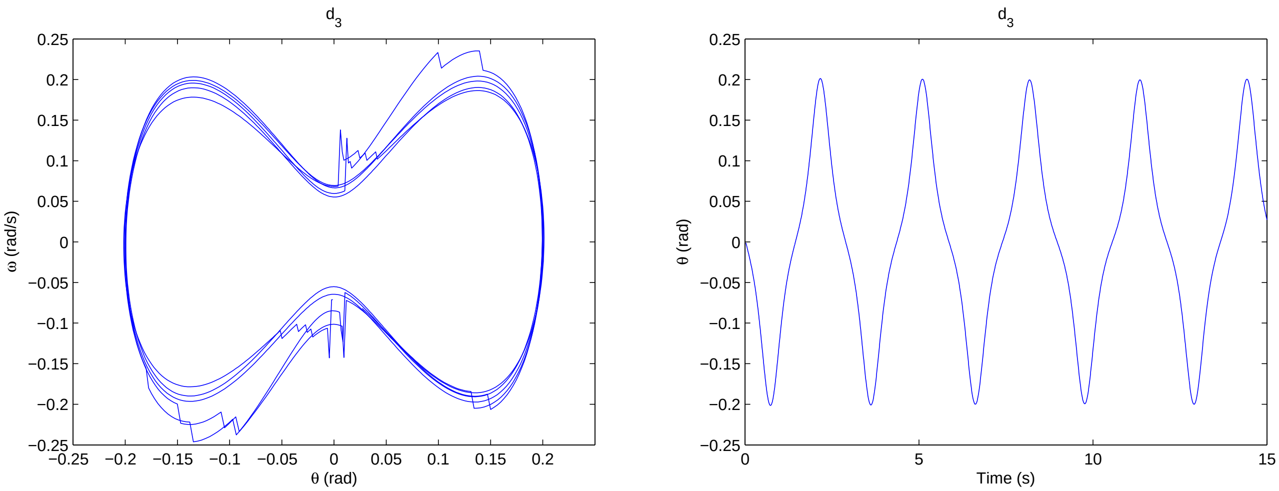


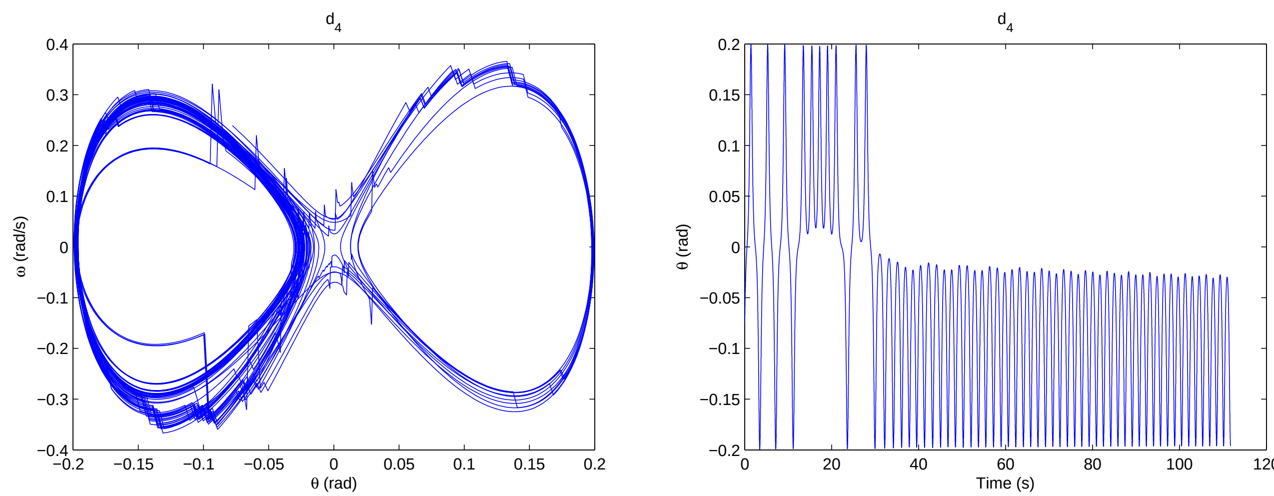
(b)

Figure 5: (a) Phase plot and (b) time series showing motion for d_2. As the distance is decreased the periodic orbits around the two attractors become more complicated.



(a) (b)

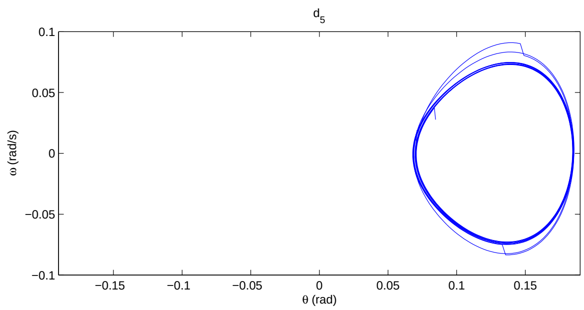
Figure 6: (a) Phase plot and (b) time series showing motion for d_3. The repulsive force starts slowing the magnet down when it passes between the two attractors. This effect is clearly evident in the 'pinching' of the phase portraits near $\theta \approx 0$ rad



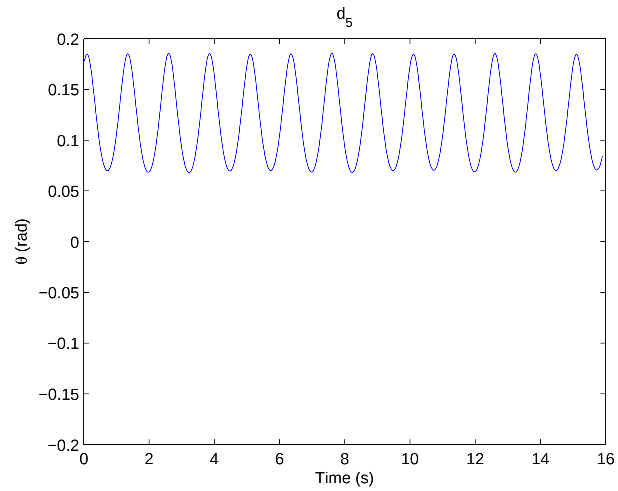
(a) (b)

Figure 7: (a) Phase plot and (b) time series showing periodic motion for d_4. It can be seen in the time series the pendulum exhibits chaotic motion in the beginning and then gets stuck in the attractor on the right.

Exercise Select a value of d where the simulation exhibits chaotic motion and plot its phase portrait.

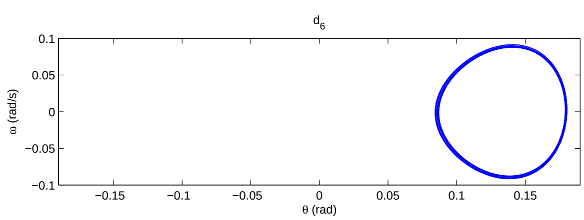


(a)

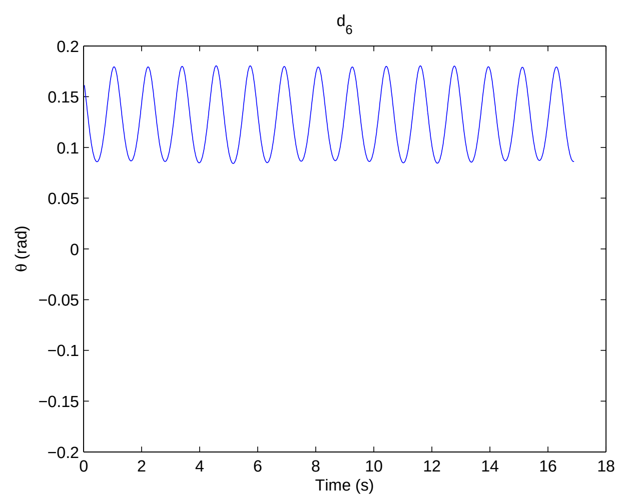


(b)

Figure 8: (a) Phase plot and (b) time series showing periodic motion for d_5. The repulsive force becomes really strong at this point and after only a few handful cycles the pendulum gets stuck in the right attractor.



(a)



(b)

Figure 9: (a) Phase plot and (b) time series showing periodic motion for d_6. The pendulum is unable to overcome the repulsive force and remains in the right attractor since the start as its initial condition is 0.2 rad.

Question: What are your values for d_1, d_2, d_3, d_4, d_5 and d_6?

4.2.1.2 The Experiment

In this section, we will practically verify the effect of successively decreasign d on ht behaviour of the magnetic pendulum.. For that first you will need to find out the resonant frequency of our pendulum. When the resonant frequency is known you will try to replicate the phase portraits at varying distances that you obtained from the simulation. For all your experimental runs the driving frequency will be kept constant and slightly lower than the resonant frequency.

1. Download the VI file of the experiment located on the experiment web page named **RotaryMotion2.vi**.

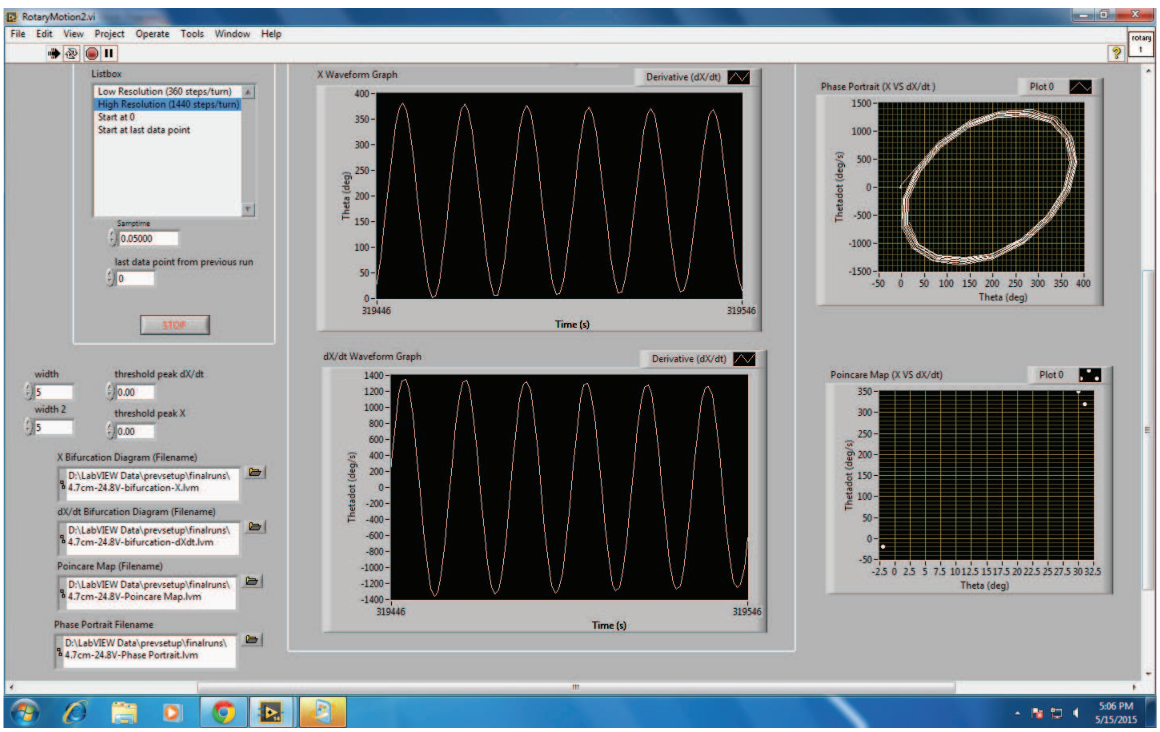


Figure 10: A screenshot of the **RotaryMotion2.vi** file.

2. Turn on the LABPRO and the Smart Timer.
3. In the list box select *High Resolution(1440 steps/turn)*.
4. Set the Sampling time to 0.05 s. Set each of **width** and **width 2** to 5 and **threshold peak X** and **threshold peak dX/dt** to -500.
5. Define paths to save your data. The phase portrait data file will save data from the Phase Portrait graph in the file.

6. The first step will be the to find the resonant frequency of your pendulum system. Increase the frequency of the motor from the power supply by changing voltage in steps of 0.2 V from 22.4 V to 28.6 V.

   NOTE: Before taking a reading make sure that the pendulum is exhibiting periodic motion.

   NOTE: At each step you will have to wait for the transient states (which will be in the form of beats) to die down. In our system it would take about roughly 5 min for the transients to die down. Throughout this process you will be constantly recording your data using your VI. Once you have started the motor just run the VI and it will start recording. The data points of the two waveforms of **Theta** and **Thetadot** signals will be recorded as two separate columns in the Phase portrait **.lvm** file.

   In this section of the experiment the distance d would be kept constant at around 6.0 cm. You will use a wooden meter rule to measure this distance d between the two magnets. Also note the corresponding driving frequency for each voltage step of the power supply using the Smart timer. Calculate the max amplitude of the **Theta** signal at each step.

   HINT: The amplitude will keep increasing till you reach the resonant frequency and after that it will start to decrease.

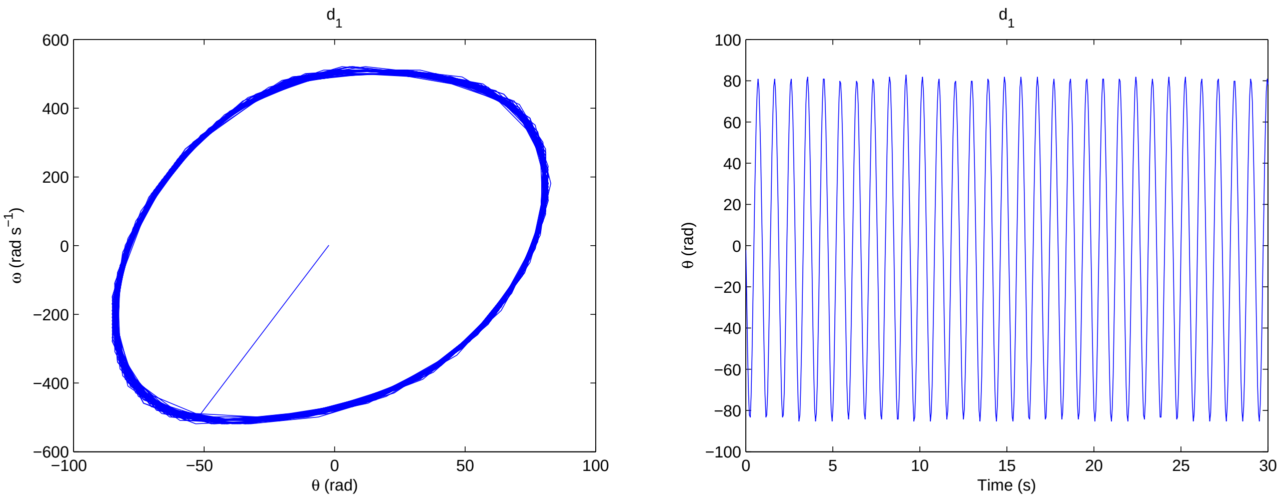
7. Once you know the resonant frequency you will now keep the frequency constant and change d to replicate the results of the simulation keep the driving frequency slightly less, (say $10 - 20$ % less) than the resonant frequency. This is because at the resonant frequency the pendulum is able to overcome the magnetic repulsion in most cases which we do not want.

8. After fixing the driving frequency you will now vary d in the same way as you did for the simulation. In this case as the smallest unit on the ruler will be 0.1 cm therefore we would not be able to gain the same of amount accuracy as we had in the simulation. However our pendulum will behave in the same way as the simulation predicted. The distance will be varied by turning the knob of the jack on which the lower magnet is fixed.

   IMPORTANT: Each time before starting any experimental run with the pendulum make sure that your initial conditions are the same for all runs. For example, your driver should start from right above the magnet fixed below and the pendulum should not be moving when you start driving it.

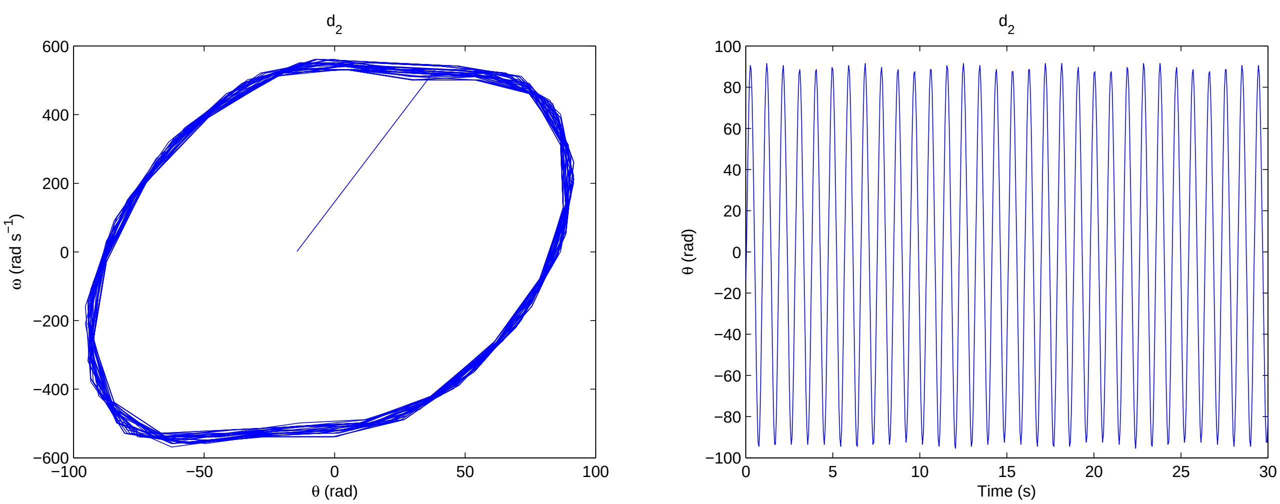
   NOTE: Before taking a reading make sure that the pendulum is exhibiting periodic motion and not is chaotic or aperiodic.

9. A set of sample experimental results is shown on the next pages. Here $d_1 > d_2 > d_3 > d_4 > d_5 > d_6 > d_7 > d_8$. The step size between each of these distances is not necessarily equal. You will need to reproduce similar results and also find out a value of d for which the pendulum exhibits chaotic behavior



(a) (b)

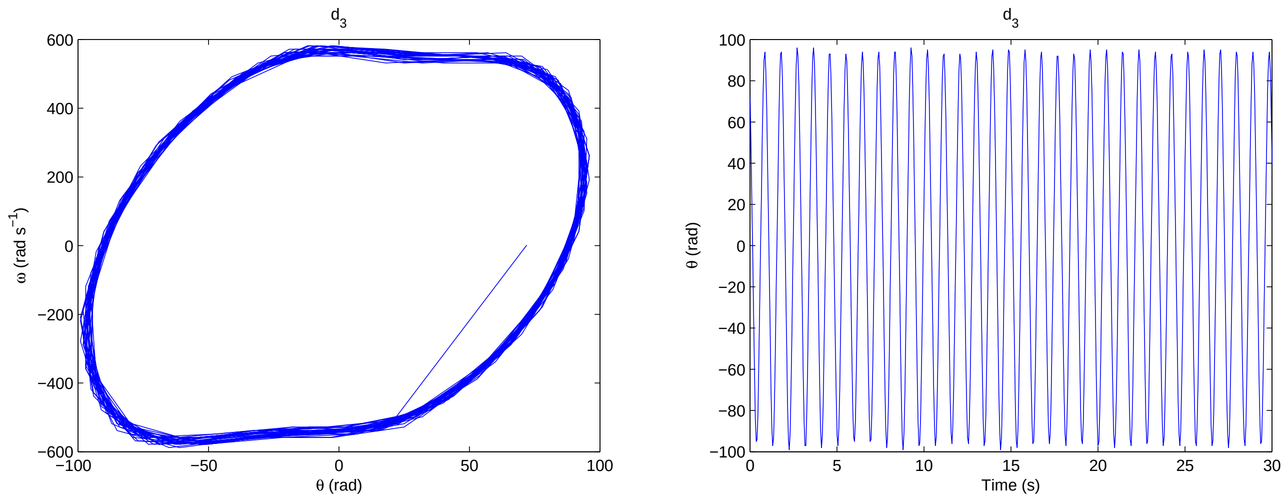
Figure 11: (a) Phase plot and (b) time series showing periodic motion for d_1. The pendulum oscillates perfectly in sync with the driving motor.



(a) (b)

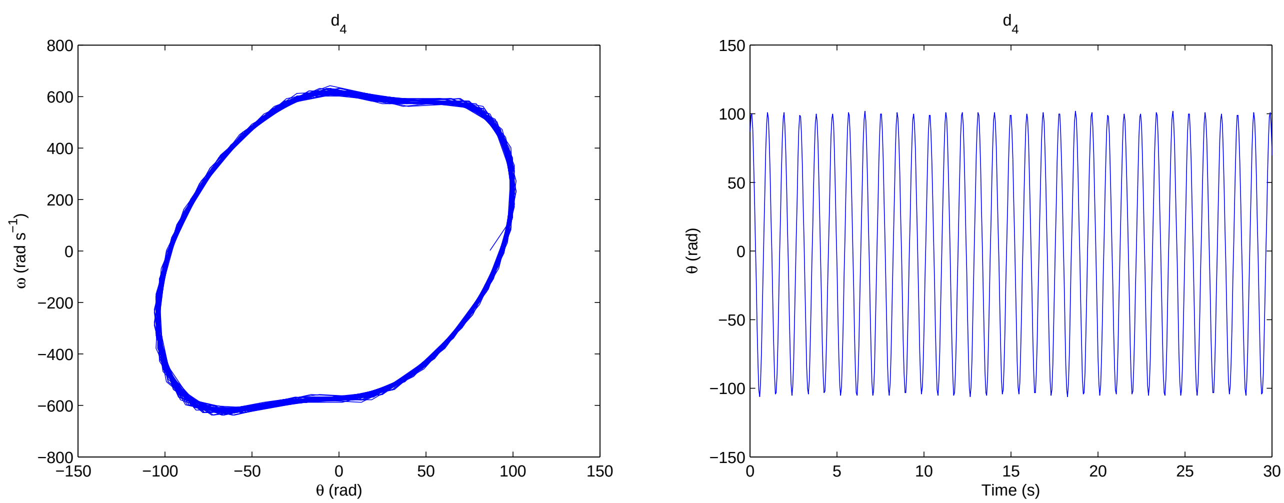
Figure 12: (a) Phase plot and (b) time series showing periodic motion for d_2. As the distance is decreased the periodic orbits around the two attractors become more complicated and we start to observe a large bulge in the center.

Question: What should a chaotic phase portrait should look like? Select a value of d or driving voltage for which the system shows chaotic and plot its phase portrait in Matlab.



(a) (b)

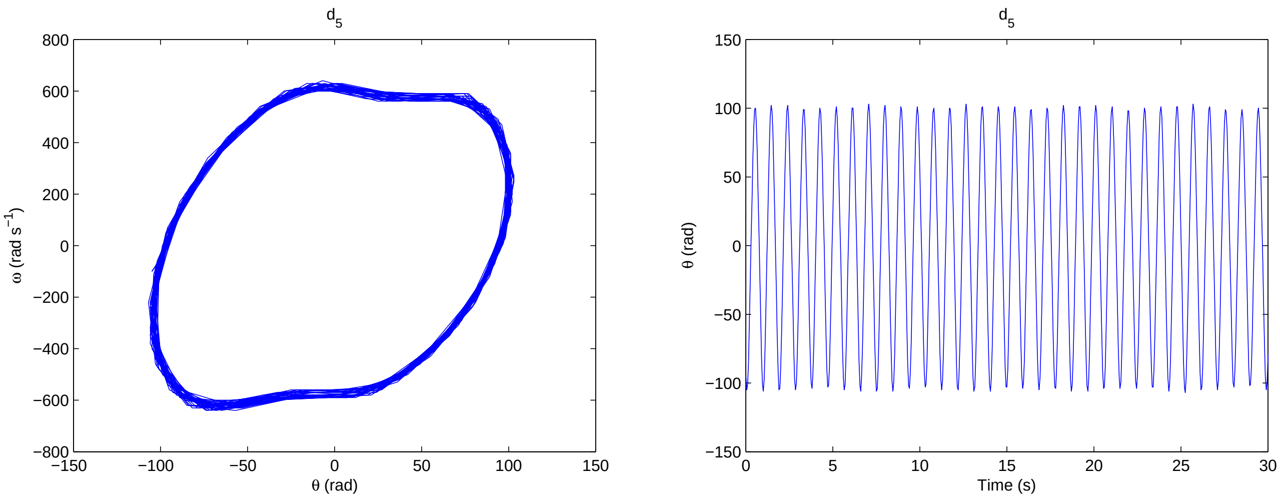
Figure 13: (a) Phase plot and (b) time series showing periodic motion for d_3.



(a) (b)

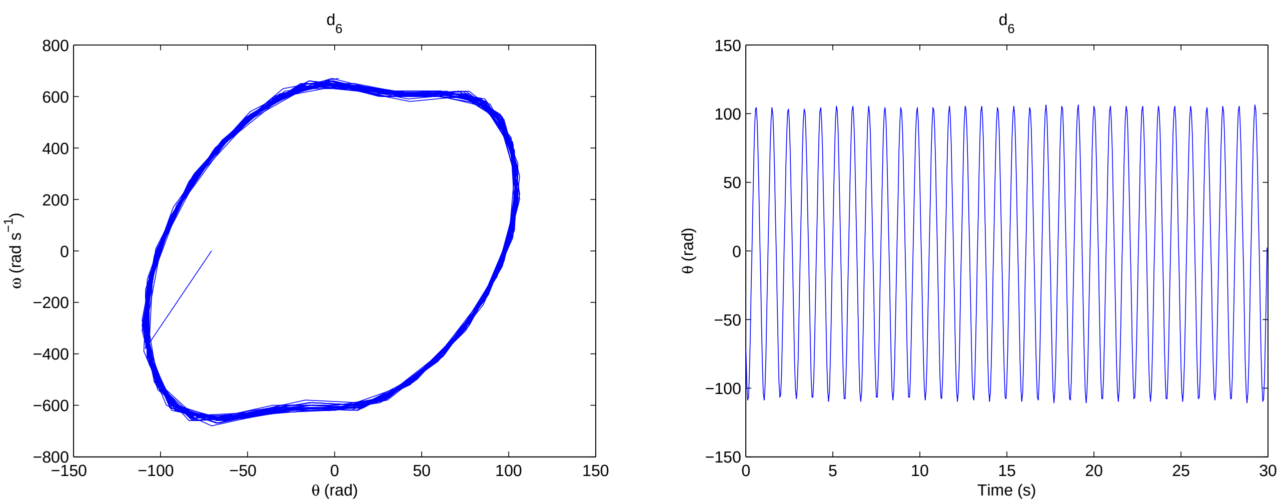
Figure 14: (a) Phase plot and (b) time series showing periodic motion for d_4. As the distance is decreased the bulges in the center become more noticeable and the pendulum seems to be favoring the right attractor in this case as it has a shorter orbit for the right attractor as compared to the left one.

Question: Explain the similarities and differences between your experimental and simulation results?



(a) (b)

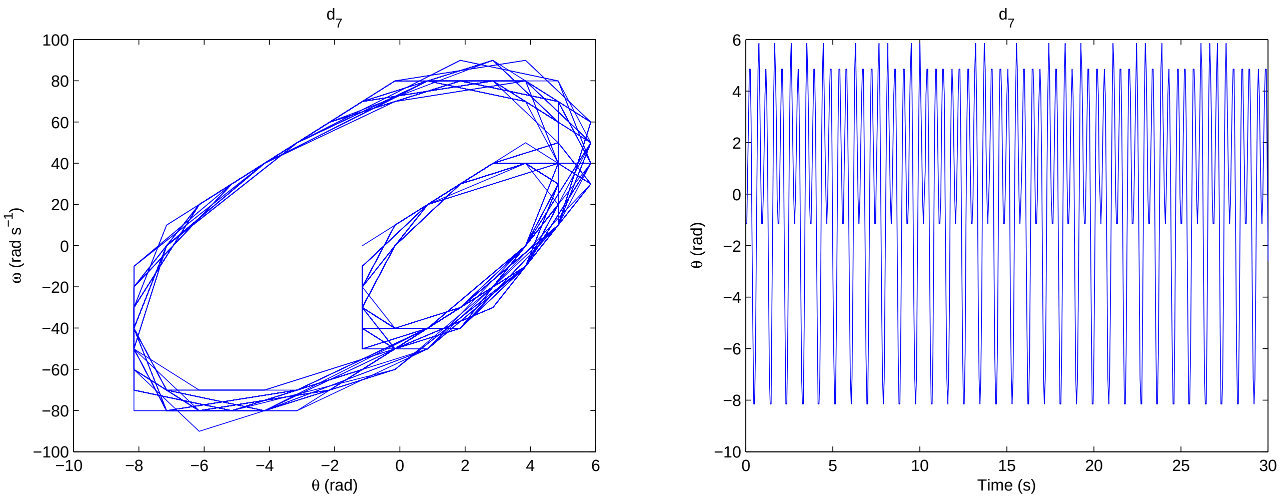
Figure 15: (a) Phase plot and (b) time series showing periodic motion for d_5. As the distance is decreased the periodic orbits around the two attractors become more complicated.



(a) (b)

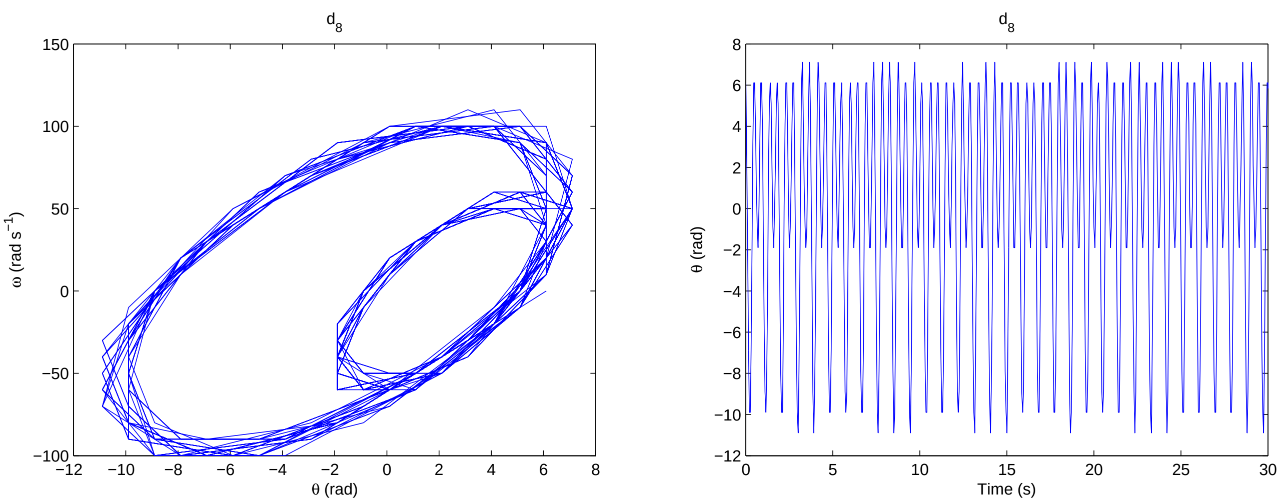
Figure 16: (a) Phase plot and (b) time series showing periodic motion for d_6. The inward bulges have now started to become larger in size.

Question: What are your values for d_1, d_2, d_3, d_4, d_5, d_6, d_7 and d_8 ?



(a) (b)

Figure 17: (a) Phase plot and (b) time series showing motion for d_7. At this point the pendulum gets stuck in the right attractor and remains there for the rest of the time. It also exhibits a period doubling bifurcation.



(a) (b)

Figure 18: (a) Phase plot and (b) time series showing motion for d_8. As the distance is decreased further the pendulum remains stuck in the right attractor and exhibits the same behavior.

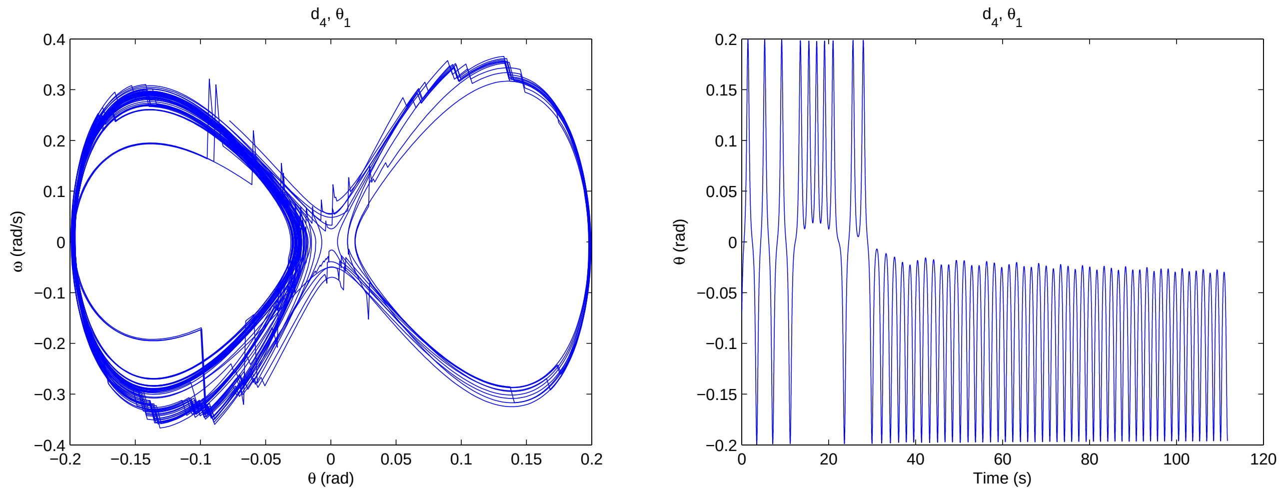
4.2.2 Obserivng the Sensitivity of the System by Varying the Initial Conditions

As you might have noticed earlier the pendulum system is extremely sensitive to its initial conditions. Even if practically you keep everything the same and drive the pendulum, the trajectories on the phase portrait will never match. This highlights our lack of control over the nonlinear system and its hyper sensitivity to initial conditions.

To reaffirm our ideas in this regard we will now use our simulation to vary the initial amplitude very slightly and observe the change in the behavior of the pendulum.

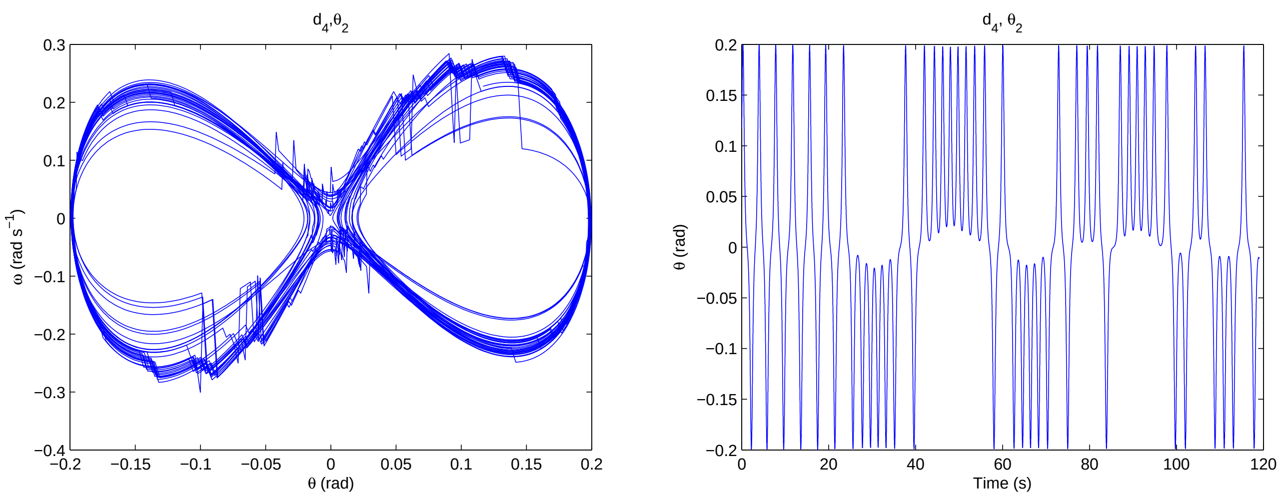
1. Once again, open the **pendode1.m** file and select a specific d. For example, in our case we selected the d_4 of our simulation results shown previously.
2. The simulation will now be run twice with slightly incongruent initial values. In each case the d will be kept constant and the initial amplitude will be varied. For example, the amplitudes can be set at 0.2000 rad and 0.2002 rad in the two runs.
3. Plot the waveform time series and phase portraits for both cases. A set sample results are shown below.

Question: Do you results agree to our initial hypothesis that the system is extremely sensitive to the initial conditions?



(a) (b)

Figure 19



(a) (b)

Figure 20: Figure 19 (a) Phase plot and (b) time series showing periodic motion for d_4 and θ_1. After exhibiting the pendulum settles into the right attractor. Figure 20 (a) Phase plot and (b) time series showing periodic motion for d_4 and θ_2 with a slight variation in initial amplitude. The pendulum exhibits completely chaotic motion throughout. You can see that in the case the pendulum is unable to decide which attractor it should settle in.

4.3 Observing Amplitude Hysteresis in Pendulum's Behavior

In this last section of the experiment you will carry out a simple exercise. The driving frequency of the rod will be varied by, as usual, changing the voltage of the power supply. The voltage will be changed from 22.4 to 28.6 V in steps of 0.2 V. At each step you will calculate and record the maximum amplitude of the signal, obtained in the LABVIEW window. The value of *d* would be kept constant at, say 60 cm.

The process will first be carried out for increasing voltage (or drive frequency) and then for decreasing voltage (or drive frequency).

NOTE: Only record the amplitude once the pendulum has reached its steady state at each step.

HINT: You have already noted amplitudes for the increasing frequency part in Section 4.2.1.2. You only have to note the amplitudes for decreasing frequency.

HINT: Use **min()** and **max()** functions in Matlab to calculate the minimum and maximum of the signal.

Your results would look similar to what is shown in Figure 21. This is a plot between signal amplitude and driving voltage. This clearly shows hysteresis, a hallmark of nonlinear systems.

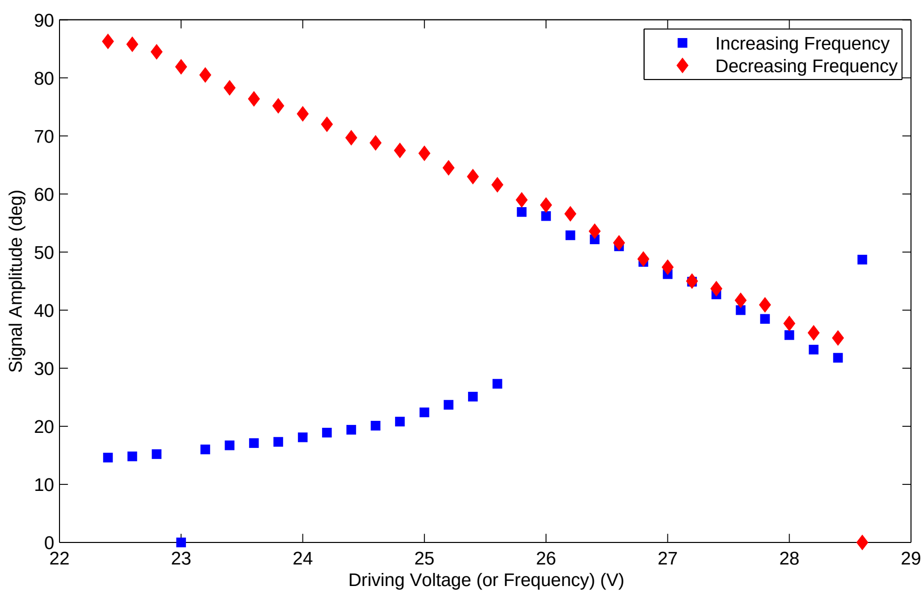


Figure 21: The graph shows the pendulum's varying behaviour with increasing and decreasing driving frequencies (or voltages). In the decreasing part the pendulum just kept increasing amplitude as we decreased the frequency. It did so until 19 V (not shown here) at which point our pendulum loosened up from the pivot and had to be stopped. In the increasing behavior it shows a clear point for maximum amplitude at our system's resonant frequency.

Question: What is the reason the pendulum system exhibits hysteresis? Does this have anything to do with initial conditions?