Dynamics of water discharge from a cylinder*

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This experiment demonstrates the dynamics of water flowing out of a tank. Using the rate change of mass for an emptying cylinder, we investigate the application of Bernoulli's equation and the resulting Torricelli's theorem. We also observe the effects of constriction on the parcel of water flowing out of the tank. Students will investigate fluid dynamics, pressure and will relish how a phenomena as simple as water flowing out from a tank can lead to rich dynamics that can be explored mathematically.

KEYWORDS

Pressure · viscosity · Bernoulli's equation · Torricelli's law · Laminar Flow · Continuity Equation

1 Conceptual Objectives

In this experiment, we will,

1. understand and apply Bernoulli's equation,
2. understand Torricelli's law,
3. understand the continuity equation,
4. learn how to numerically differentiate data, and
5. make plots of variables derived from directly measured quantities.

2 Theoretical background

A fluid is a collection of molecules held together by weak cohesive forces. Usually liquids and gases are termed as fluids because they deform in response to external forces. Some general properties of fluid flow are summarized here.

1. **Steady or non-steady:** The flow of a fluid is described by pressure, density and flow velocity at every point of the fluid. If these variables are constant in time then the flow is steady.

2. **Compressible or incompressible:** If the density of a fluid remains constant and does not depend on x, y, z and t, then the flow is incompressible.

3. **Viscous or non-viscous:** Viscosity is the resistance towards flow. When a fluid flows such that there is no energy dissipation, then it is non-viscous flow. Such a flow is really an idealization.

4. **Rotational or irrotational:** If any element of the fluid does not rotate about an axis through the center of mass of the element, then the flow is irrotational.

2.1 Pressure inside a fluid

Consider a small segment of the fluid of density ρ at a distance y above some reference level as shown in Figure 1(a). This segment is a thin disk with thickness dy and area A, as illustrated in part (b) of the diagram. The mass of the element is $dm = \rho dV = \rho A\, dy$ and its weight $W = (dm)g = \rho g A\, dy$. Since there is no acceleration the net vertical force is zero,

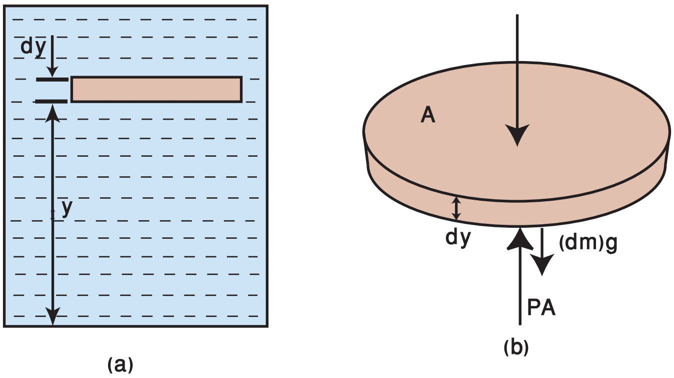


Figure 1: A static fluid. (a) Small element at rest inside the fluid, (b) forces acting on a small element.

$$\Sigma F_y = PA - (P + dP)A - \rho g A\, dy = 0, \tag{1}$$

yielding,

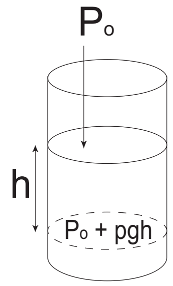
$$\frac{dP}{dy} = -\rho g. \tag{2}$$

This equation describes the variation of pressure with elevation above some reference level.

As the height increases (dy positive), the pressure decreases (dP negative). For an incompressible and homogeneous liquid with difference in height, the pressure difference is found by integrating the Equation (2)

$$P_2 - P_1 = -\rho g(y_2 - y_1), \tag{3}$$

and if the liquid has a free surface exposed to the atmospheric pressure P_o, then,



$$\begin{aligned} P_o - P &= -\rho g(y_2 - y_1), \\ P &= P_o + \rho g h, \end{aligned} \tag{4}$$

where, $y_2 - y_1 = h$. This shows that the pressure in a liquid increases with depth but would be same at all those points that are on the same level.

2.2 Bernoulli's equation

In our discussion on pressure, we have seen how pressure depends on the weight of the fluid above a level. However, pressure will also change with speed and elevation. You must have noticed how an object could be pulled into the wake of a fast-moving train; or by narrowing the hose of a water pipe, the stream of water can go further.

Q 1. Why do hordes of birds fly in a characteristic V-shaped pattern?

When a fluid moves through a region in which either the speed of the fluid or elevation above the earth's surface changes, the impact is that the pressure in the fluid changes. The relationship between fluid speed, pressure and elevation was first derived by Daniel Bernoulli in 1738. Bernoulli's equation, a fundamental relation in fluid mechanics is derivable from basic laws of Newtonian mechanics, as well as from the work-energy principle which stems from the conservation of energy.

Consider a steady, incompressible and nonviscous flow of a fluid through a pipeline from the position shown in Figure 2(a) to (b). The portion at the left has a cross sectional area A_1 and at an elevation y_1 from some reference level. A mass of fluid Δm gradually rises and after time Δt, it moves to the right end with cross sectional A_2, at an elevation y_2.

According to the work-energy theorem, the work done by the resultant force that acts on a system is equal to the change in kinetic energy. Assuming that there is no viscous force, the only forces that do work on the system are the pressure forces and the force of gravity. The net work done on the system by all the forces is,

$$W = P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - (\Delta m)\, g(y_2 - y_1). \tag{5}$$

This is the work done as the mass Δm displaces from (a) to (b). The pressure force $P_2 A_2 \Delta l_2$ bears a negative sign because its direction is opposite to the horizontal displacement Δl_2. The gravitational force is also negative because it acts in a direction opposite to the vertical displacement. As $A_1 \Delta l_1 = A_2 \Delta l_2$ is the volume of the fluid (ΔV) displaced, we can replace this $\Delta m/\rho$. The change in kinetic energy, therefore is,

$$\begin{aligned} \Delta K &= \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 \\ &= \frac{1}{2}\Delta m(v_2^2 - v_1^2) \\ &= P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - (\Delta m) g(y_2 - y_1) \end{aligned} \tag{6}$$

This can be rearranged to give,

$$\frac{1}{2}\Delta m(v_2^2) + P_2 A_2 \Delta l_2 + (\Delta m) g y_2 = \frac{1}{2}\Delta m(v_1^2) + P_1 A_1 \Delta l_1 + (\Delta m) g y_2 \tag{7}$$

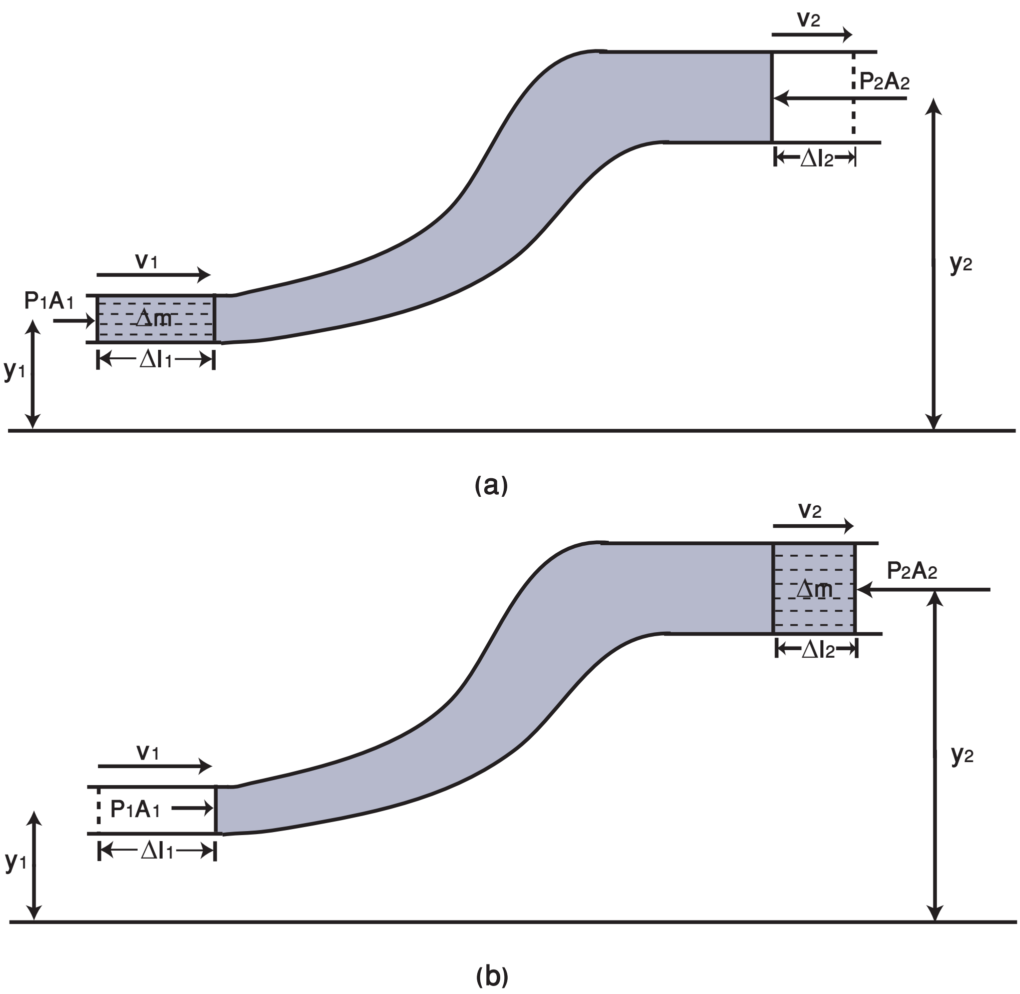


Figure 2: A fluid is flowing through a pipe from position (a) to (b). The net effect is the transfer of the element (Δm) from the left to the right end. We calculate the work done in this transfer process.

Dividing each side by the respective volume of the element $A_2(\Delta l_2) = A_1(\Delta l_1)$,

$$P_2 + \frac{1}{2}\,\rho\, v_2^2 + \rho g y_2 \;\; = \;\; P_1 + \frac{1}{2}\,\rho\, v_1^2 + \rho g y_1\,. \tag{8}$$

The above equation is often expressed as,

$$P + \frac{1}{2}\,\rho\, v^2 + \rho g y \;\; = \;\; \text{constant}. \tag{9}$$

This is a statement of Bernoulli's equation.

The relation in Equation (8) can be modified in many different ways depending upon the situation. This leads to interesting corollaries. If the fluid is at rest i.e. $v_2 = v_1 = 0$ then,

$$P_1 + \rho g h_1 = P_2 + \rho g h_2, \tag{10}$$

where the term $(P + \rho g y)$ is called the *static pressure*.

Likewise, if both ends of the pipe are placed at same height then Equation (9) can be re-written as,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2, \tag{11}$$

showing that high speeds corresponds to low pressures. The term $\frac{1}{2}\rho v^2$ has dimensions of pressure and is called *dynamic pressure*.

2.3 Continuity equation

The equation of continuity for incompressible fluids states that,

$$\rho A v = \text{constant} \tag{12}$$

where v is the velocity and ρ is density of the fluid. This relation is easy to understand. Consider Figure (3) which shows a tapered horizontal pipe. The area at the left end is A_1 and at the right is $A_2 < A_1$. In a unit time Δt, a mass of liquid Δm is transported between the ends. Since the fluid cannot be compressed, we must conserve the mass of fluid transferred, otherwise the liquid will turn denser in some regions and rarer in others. Therefore,

$$\rho A_1 \Delta l_1 = \rho A_2 \Delta l_2, \tag{13}$$

which dividing by Δt yields the equation of continuity, (12).

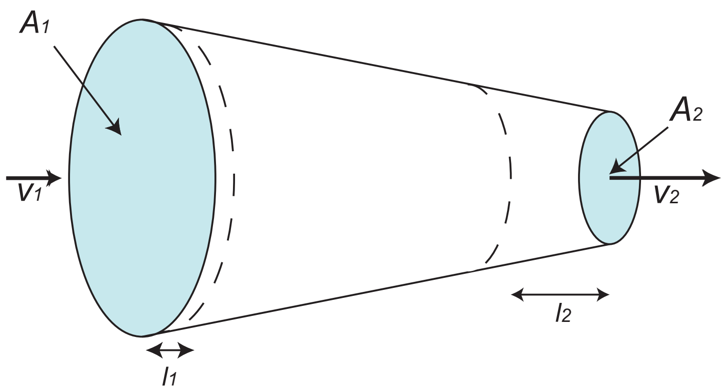
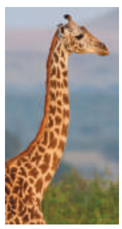


Figure 3: A tapered horizontal pipe. The horizontal velocity vectors are depicted by arrows.



Q 2. A giraffe needs a strong heart because of its long neck. Suppose the difference of height between the aortic valve (the place where the arterial blood comes out of the heart) and the head of a giraffe is 2.50 m, and the artery leading from near the aortic valve to the head has constant cross section all the way to the head. Blood is an incompressible fluid with density 1.0 g/cm^3. Assume the pressure at the head is zero.

(a) What is the minimum required pressure at the aortic valve? Compare this pressure to the peak output pressure of the human heart (1.6×10^4 Pa)?

(b) What would be the effect on the giraffe if the artery diameter narrowed down as it approached the brain?

2.4 Water discharge from a cylinder

A cylinder contains water which flows out from a narrow circular orifice at a fixed height y_2 from the base. [right]right The orifice has a small area A_2 compared to the cross sectional area A_1 of the cylinder. As time progresses, the level of the water $y_2(t)$ in the cylinder descends and water issues out with a speed $v_2(t)$. Let's apply Bernoulli's law to points **1** and **2**. Note that at the orifice, the jet of water is also exposed to the atmospheric pressure P_o. From Bernoulli's principle,

Consider Fig (4)

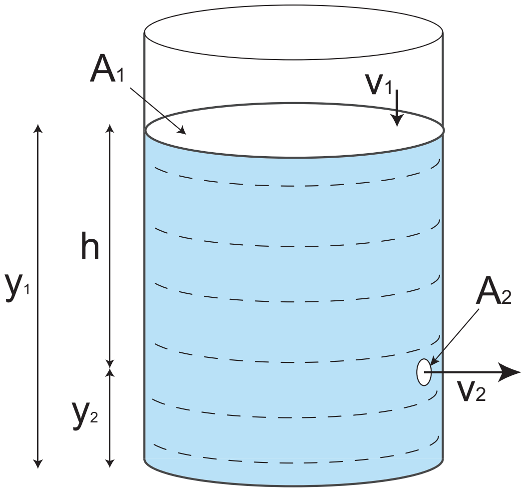


Figure 4: A cylinder with water flowing out from a narrow orifice at a fixed height y_2 from the base.

$$\cancel{P_o} + \frac{1}{2}\rho v_1^2 + \rho g y_1 = \cancel{P_o} + \frac{1}{2}\rho v_2^2 + \rho g y_2 \tag{14}$$

Since $A_1 \gg A_2$, $v_1 \approx 0$, leading to

$$\begin{aligned} \tfrac{1}{2}\rho v_2^2 &= \rho g(y_1 - y_2) \\ v_2^2 &= 2gh, \end{aligned} \tag{15}$$

which shows the relationship between the speed $v_2(t)$ and the instantaneous head $h(t)$ of water *above* the orifice.

2.5 Torricelli's Law

Torricellis Law describes the relationship between the velocity of fluid leaving the cylinder v_2 and the height h of the fluid. This relationship is given in Equation (15). In its simplest form, the speed, v, of a liquid flowing under the force of gravity out of an opening in a tank is proportional to the square root of the vertical distance, h. The speed of efflux is independent of the direction of flow. The theorem is named after Evangelista Torricelli, who formulated it in 1643. Notice that this speed is identical to the speed acquired by a mass falling under gravity through a height h.

In the experiment, you will observe if a linear relationship between v_2^2 and h exists. Furthermore, v_2^2 will in fact be observed to be smaller than $2gh$. The discrepancy will be accounted for by water's viscosity, and the effective narrowing of the orifice.

3 The Experiment

3.1 Preparation

You are provided a graduated cylinder with an orifice at a fixed height y_2 from the base. Place it on the provided electronic mass balance and set it to zero by pressing the `TARE` button on the front panel. This subtracts the mass of cylinder from the subsequent data points. Place the provided plastic box in the line of orifice to collect the discharging water. Complete the assembly as shown in Figure 5. Before you begin make sure that the mass balance and the LabView program have been configured according to sections 3.2 and 3.3 respectively.

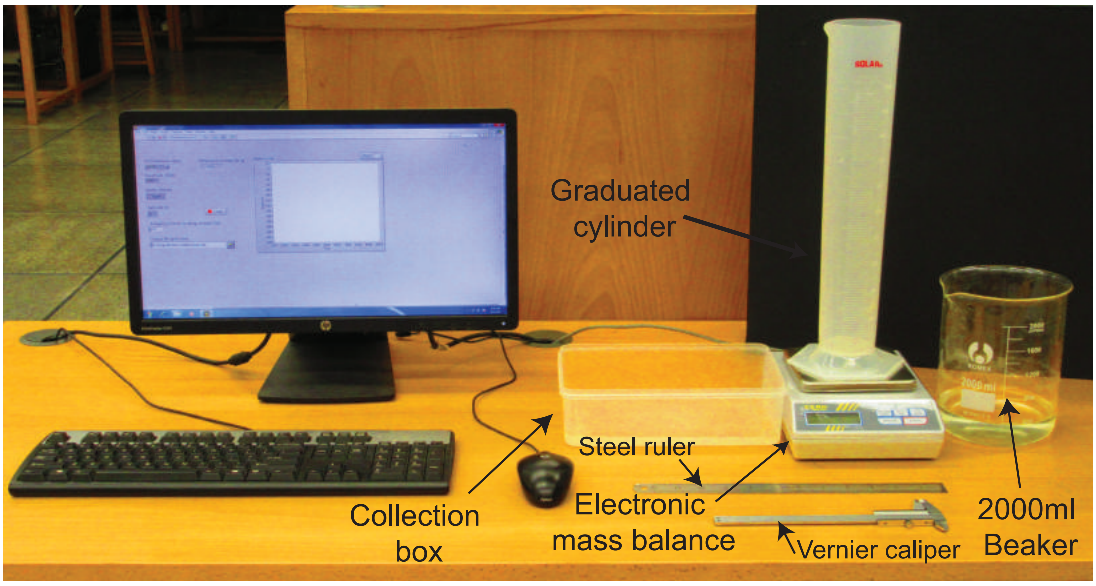


Figure 5: The experimental assembly, for observing the rate of discharge from a graduated cylinder.

3.2 Mass Balance Setup

A digital mass balance Kern 440-47N is connected to the serial port COM1 on the computer using an RS-232 interface connection. The balance will send mass (g) readings to the computer where a LabView code will be used to collect data into an output file. The balance needs to be set up on `AU PC` mode before it can transmit data to the computer. For this purpose one needs to follow the steps given in Table 1.

Also, turn off the `Dosing and Zero-tracking` function of the mass balance. If active, this function is used to tare small variations in weight automatically which can cause improper weighing results. So, it is advisable to switch this function off. Follow the steps given in Table 2 for this purpose.

Setting of Data Transfer Mode	Balance Display
1. Keep the Print key pressed until Unit is displayed.	Unit
2. Press Mode key till Pr appears.	Pr
3. Press the Set key to change the settings of the balance.	
4. Keep pressing the Mode key until the balance displays AU PC.	Au PC
5. Press the Set key to confirm this change in settings.	
6. The balance returns to the weighing mode.	0.0 g

Table 1: Steps to setup the mass balance for data transfer mode.

Activate/deactivate Zero-Tracking	Balance Display
1. Keep the Print key pressed until Unit is displayed.	Unit
2. Press the Mode key several times until tr is displayed.	tr
3. Press the Set key to activate the function.	tr on
4. By pressing once more the Mode key, the function is deactivated.	tr off
5. The changed setting takes over by pressing the Set key.	
6. The balance returns to the weighing mode.	0.0 g

Table 2: Steps to activate/deactivate zero-tracking.

3.3 Using the LabView Application

To collect the data of mass (g) and time (t) you will be using a **LabView** code which is available for download from the experiment's website. Download and run the file from the website and present it with an output path for your data collection. Some other settings relevant for the data acquisition are given in Table 3.

Functional Title	Value
Visa Resource Name	COM1
Baud Rate	9600
Bits	8
Parity	None
Stop bit	1
Flow Control	None

Table 3: Settings for the LabView interface.

3.4 Experimental procedure and analysis

In this experiment, you are required to verify a linear relationship between $v_2^2(t)$ and $h(t)$. You have access to vernier calipers that will be used to measure the diameter of the cylinder and the orifice. Of course, the balance will return the rate of water flowing out from the cylinder. Use available data to verify the Torricelli's theorem. Write a computer script that converts mass flow rate to speed.

Does your experimental data support Toricelli's theorem? Ideally the slope of the v_2^2

verses h graph should have a slope $2g$. Is the slope of your data smaller or greater than 2g? How do you account for the difference?

References

[1] Raymond A. Serway, John W. Jewett, Jr. *Physics for Scientists and Engineers with modern Physics*, pp. 465-483, (2010).