

# Thermodynamics of the Heat Engine

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The experiment provides an introduction to thermodynamics. Our principle objective in this experiment is to understand and experimentally validate concepts such as pressure, force, work done, energy, thermal efficiency and mechanical efficiency. We will verify gas laws and make a heat engine using glass syringes. A highly interesting background resource for this experiment is selected material from Bueche's book [1]. The experimental idea is extracted from [2].

## KEYWORDS

Temperature · Pressure · Force · Work · Internal Energy · Thermal Efficiency · Mechanical Efficiency · Data Acquisition · Heat Engine · Carnot Cycle · Transducer · Second law of thermodynamics

**APPROXIMATE PERFORMANCE TIME** 6 hours

## 1 Conceptual Objectives

In this experiment, we will,

1. learn about the first and second laws of thermodynamics,
2. learn the practical demonstration of gas laws,
3. learn how to calculate the work done on and by the system,
4. correlate proportionalities between pressure, volume and temperature by equations and graphs,
5. make a heat engine using glass syringes, and
6. make a comparison between thermodynamic and mechanical efficiencies.

## 2 Experimental Objectives

The experimental objectives include,

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1. a verification of Boyle's and Charles's laws,
2. use of the second law of thermodynamics to calculate the internal energy of a gas which is impossible to directly measure,
3. making a heat engine,
4. using transducers for measuring pressure, temperature and volume, and
5. tracing thermodynamical cycles.

## 3 Theoretical introduction

### 3.1 First law of thermodynamics

We know from our lessons in science that energy is always conserved. Applying the same principle to thermodynamics, we can write an equation that relates different variables such as heat entering a system, work done on the system and the system's internal energy. Heat is defined as the energy transferred from a warm or to a cooler body as a result of the temperature difference between the two.

Consider a gas contained in a cylinder. We call the gas our 'system'. If we heat it, this energy may appear in two forms. First, it can increase the internal energy  $U$  of the system. The change in the internal energy is given by,

$$U_B - U_A = \Delta U \quad (1)$$

where  $U_B$  and  $U_A$  represent the final and initial internal energies. Second, the heat input to the system, can cause the gas to expand. This increases the volume and work is done by the system on its surroundings, *reducing* the internal energy content of the system itself. Therefore heat entering the system  $Q$  manifests as an increase in internal energy and work done by the system on its surroundings. Expressing this mathematically, we obtain,

$$Q = \Delta U - W, \quad (2)$$

or

$$\Delta U = Q + W, \quad (3)$$

We will revisit the concept in section 3.3. Note that the work done by the system is conventionally taken to be negative  $-W$ .

Equation (3), which is simply rearrangement of (2) is another way of saying that the internal energy can increase in two ways; first by heat entering the system and second, by work done *on* the system.

Equations (2) and (3) are called the first law of thermodynamics relating heat, internal energy and work. If a small amount of heat  $dQ$  results in a tiny change in the internal energy  $dU$  as well as some infinitesimal work  $dW$ , then Equation (2) can be also written in its differential form,

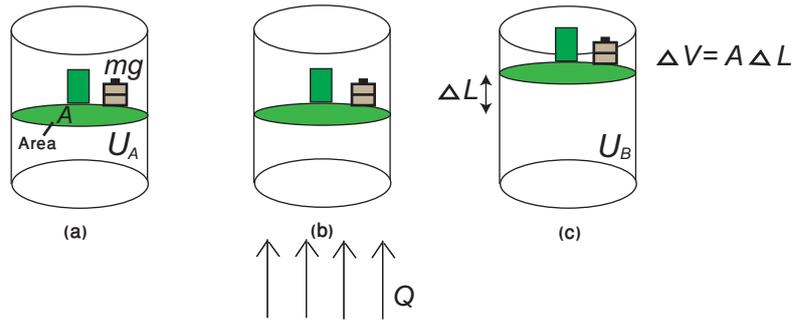


Figure 1: (a) System with internal energy  $U_A$ . (b) Heat  $Q$  enters the system. (c) Work done by the system as the gas expands.  $W = -P\Delta V$ , is negative by convention. The internal energy also increases to  $U_B$ . The first law states  $\Delta U = Q + W = Q - P\Delta V$ .

$$dQ = dU - dW. \quad (4)$$

In summary, heating a system has two effects, first, on the internal energy, and second, on the mechanical work done by the system.

### 3.2 Effect of heat on internal energy

Suppose we have  $n$  moles of gas inside a container with rigid and fixed boundaries implying that the gas cannot contract or expand, hence  $W = 0$ . The volume of the gas in such a container remains constant. Now if we supply heat, the thermal energy input will only increase the internal energy of the system. Since, there is no change in volume,  $dV = 0$ ,  $dW = 0$ , we can write Equation (4) as,

$$dQ = dU, \quad (5)$$

meaning that all the heat added brings a change in the internal energy of the system. The heat required to raise the temperature of  $n$  moles of gas at constant volume is given by,

$$dQ = nC_v dT, \quad (6)$$

where  $C_v$  is the specific heat capacity of the gas at constant volume, and  $dT$  is the small change in temperature. A constant-volume process is also called an isochoric process.

### 3.3 Work done as a result of heating

Now suppose that the gas at atmospheric pressure is instead contained in a cylinder fitted with a moveable piston. Since the piston can freely move, the volume of the gas is variable but the pressure remains constant. If we now heat the container, the

internal energy of the gas molecules increases. The gas molecules perform work on the piston by pushing it upwards and bring about an increase in the volume  $dV$ . By our definition, the work done on the system,  $W$ , is negative. This is called an isobaric (constant-pressure) process. The heat absorbed by  $n$  moles of gas at constant pressure is given by,

$$dQ = nC_p dT. \quad (7)$$

where  $C_p$  is the specific heat capacity of gas at constant pressure.

Thermodynamic processes are commonly drawn on  $PV$  diagrams, where the pressure is plotted on the  $y$ -axis and volume on the  $x$ -axis. The  $PV$  diagrams for some representative processes are shown in Figure 2.

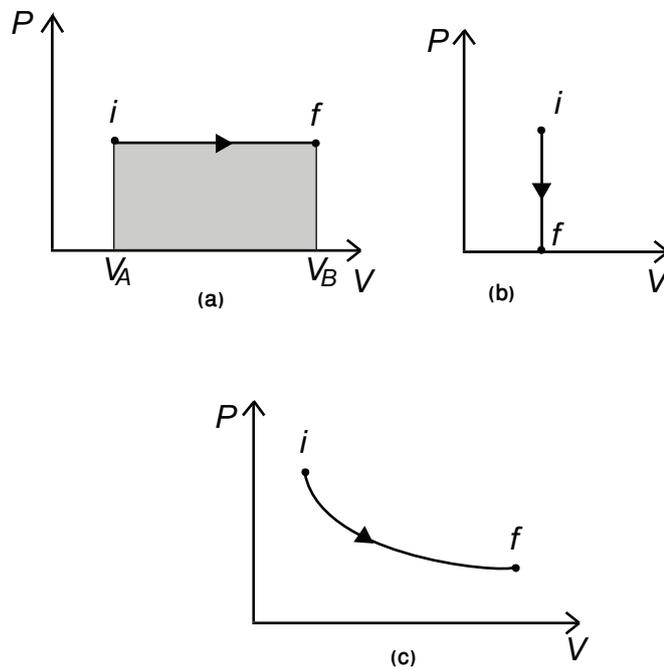


Figure 2: Typical  $PV$  curves for (a) isobaric, (b) isochoric and (c) isothermal processes,  $i$  and  $f$  represent initial and final states.

The calculation for the work done is easy. For example, consider Figure 2 (a) which shows an expansion process at constant pressure. The initial volume is  $V_A$ . Expansion results in an increase to a new volume  $V_B$ . Since, work is given by Force  $\times$  distance = Pressure  $\times$  area  $\times$  distance = Pressure  $\times$  change in volume, we have the relationship,

$$W = -P\Delta V = -P(V_B - V_A). \quad (8)$$

Here the negative sign is introduced to ensure compliance with our convention. ( $\Delta V$  is positive, and work done on the system  $W$  is negative). Now  $P(V_B - V_A)$

is just the area of the shaded rectangle shown, or the area under the  $PV$  curve. This rule is in general true, work done can be calculated by measuring the area under the  $PV$  curve. Let's illustrate this through a numerical example.

### Example

Suppose a gas contracts in the way shown by the  $PV$  diagram in Figure 3. We are asked to find the work done by the gas in going from the situation represented by point A, through point B, to point C.

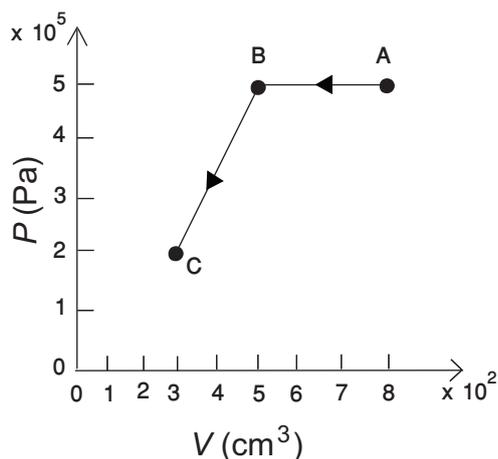


Figure 3: An example for calculating the area under the curve when a system traverses the path  $A \rightarrow B \rightarrow C$ .

As expected, we must compute the area under the curve. Notice that this irregular shape consists of three simple shapes: two rectangles and one triangle. We can calculate these three simple areas and add them to get the total area we need. The area under the portion AB is given by,

$$(5.0 \times 10^5 \text{Pa})[(800 - 500) \times 10^{-6} \text{ m}^3] = 150 \text{ J.} \quad (9)$$

Similarly, the area under the curve from B to C is,

$$(2.0 \times 10^5 \text{Pa})(200 \times 10^{-6} \text{ m}^3) + \frac{1}{2}(3.0 \times 10^5 \text{Pa})(200 \times 10^{-6} \text{ m}^3) = 70 \text{ J} \quad (10)$$

where we have used the fact that the area of a triangle is one-half the base times the height. Therefore, the total area under  $PV$  curve =  $150 \text{ J} + 70 \text{ J} = 220 \text{ J}$ . Since, the process we are considering involves a decrease in volume, the work done on the system  $-P\Delta V$  is positive.

**Q 1.** Substitute Equations (7) and (8) into Equation (4) and show that,

$$nC_p dT = dU + PdV. \quad (11)$$

In an ideal gas, the internal energy  $U$  depends only on the temperature. Therefore, if the temperature change at constant pressure has the same value as the temperature change at constant volume, the increase  $dU$  must be the same between an isobaric and isochoric process. Hence, substituting Equation (6) into Equation (11) yields,

$$nC_p dT = nC_v dT + PdV. \quad (12)$$

**Q 2.** Using the ideal gas law equation  $PV = nRT$  and Equation (12), show that,

$$C_p = C_v + R, \quad (13)$$

where  $R$  is called the molar gas constant. It has the value  $8.31 \text{ J kg}^{-1} \text{ K}^{-1}$ . Equation (13) relates the heat capacities of a gas at constant pressure and constant volume.

### 3.4 Adiabatic processes

Suppose the gas is now contained in a cylinder that is fitted with a moveable piston but in addition, the cylinder is perfectly insulated from the surroundings. No exchange of heat is possible between the gas and the surroundings,  $dQ = 0$ . The idealized process in which no heat is absorbed or released is called an adiabatic process. Consider Figure 4 which shows an isolated system with internal energy  $U_A$ . A weight is rapidly dropped onto the piston, compressing the gas and increasing its internal energy to  $U_B$ . The work done on the system is now positive and given by,

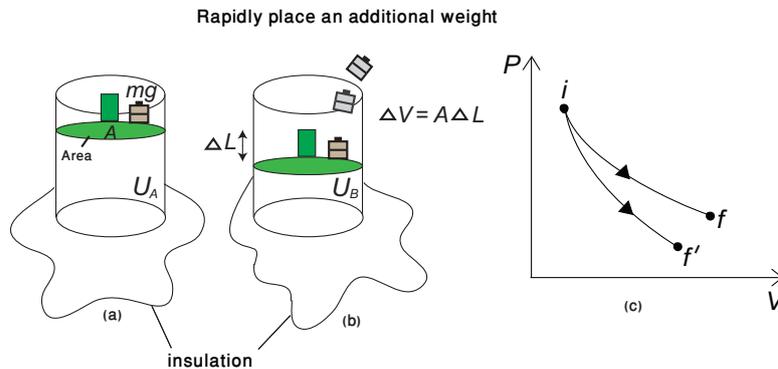


Figure 4: (a) An insulated cylinder with gas inside. (b) Rapid placement of the weight makes the process adiabatic. (c) The  $PV$  diagrams for isothermal and adiabatic curves, shown respectively by  $i \rightarrow f$  and  $i \rightarrow f'$ .

$$dW = -PdV. \quad (14)$$

Remember the negative sign: it ensure  $dW$  is positive when  $dV$  is negative! Since no heat enters or leaves the system we have,

$$dQ = 0, \quad (15)$$

and using Equation (12) we obtain,

$$nC_v dT + PdV = 0. \quad (16)$$

Rearranging the terms and using  $PV = nRT$ ,

$$nC_v dT + nRT \frac{dV}{V} = 0. \quad (17)$$

The solution of the above equation can be found in [1], and in your higher classes you will learn how to solve such equations, but let's suffice for the time being to stating just the result. The solution is,

$$PV^\gamma = \text{constant}. \quad (18)$$

Here,  $\gamma = \frac{C_p}{C_v}$  is the ratio of the specific heats at constant pressure and volume.

Figure 5 shows the values of  $\gamma$  for difference materials. Figure 4(c) shows how an adiabatic process looks like on a  $PV$  diagram. For a comparison, an isothermal process for which  $PV^1 = \text{constant}$  is also shown. Notice the steeper slope for the adiabatic process.

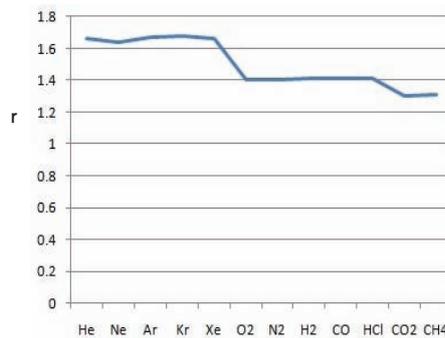


Figure 5:  $\gamma = C_p/C_v$  for various gases.

**Q 3.** Derive the solution (17) or show that Equation (18) is indeed a solution.

## 4 Introduction to the Apparatus

**A photograph of the experimental assembly is presented in Figure (6).**

1. **Beakers and Hot plate** for heating water. The heating is achieved by the provided hot plate. The maximum temperature of the hot plate is around 400 °C.

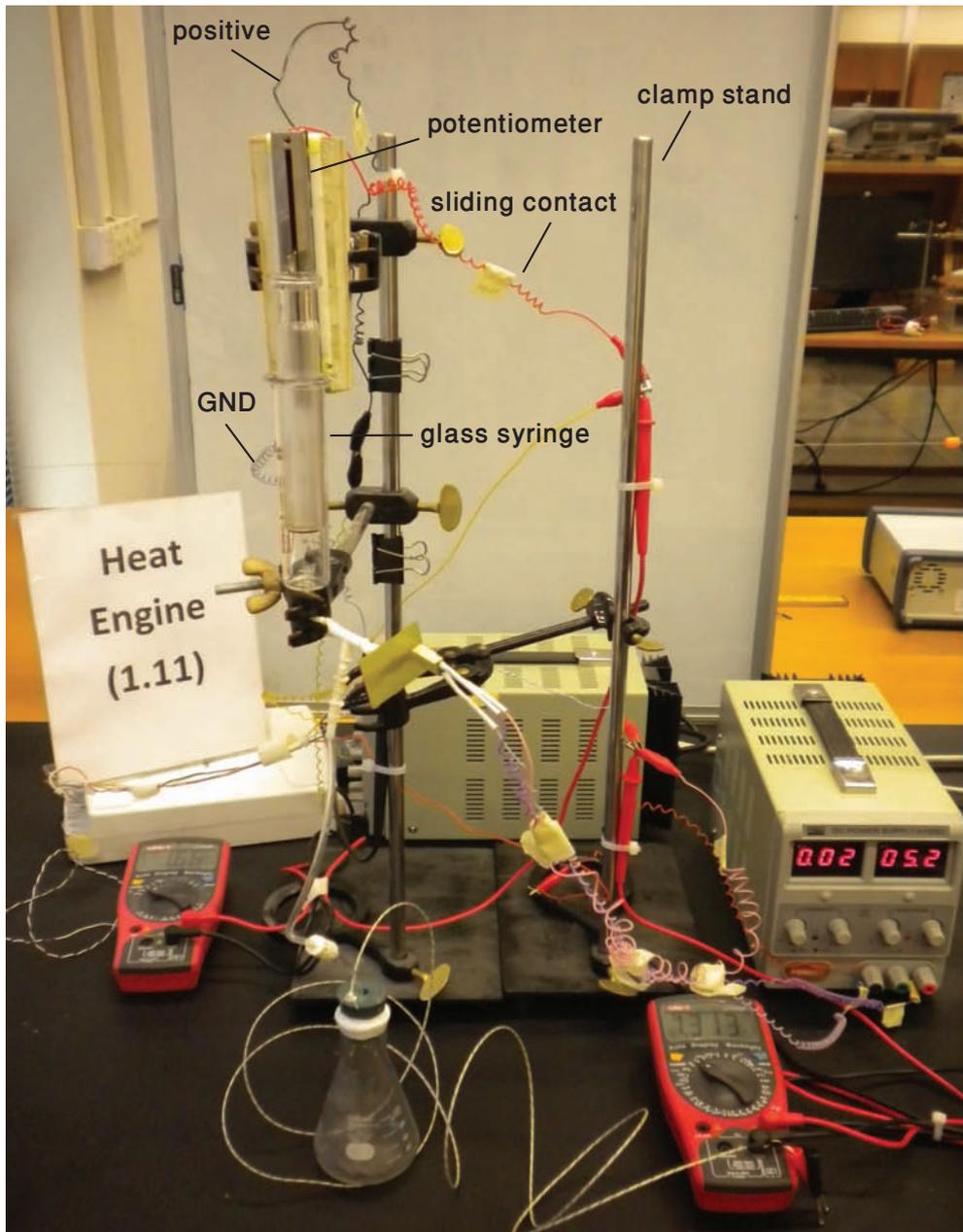


Figure 6: Photograph of the experimentally assembly.

2. **Conical flasks** We will use 37 and 50 mL conical flasks. The small volume of the flasks provides the essential pressure required to lift the piston of the syringe.

3. **Piezoresistive pressure transducer**

A pressure sensor (MPXH6400, Freescale Semiconductor) [3] is used to monitor the pressure variation inside the conical flask. The sensor is connected to a glass syringe through luer connectors (Harvard Apparatus). Three wires labeled  $V_{IN}$ ,  $GND$  and  $V_{OUT}$  are attached to the terminals 2, 3 and 4. The numbering starts from the end with a notch. A fixed DC

voltage of +5 volts is applied at pin 2 by a 30 Vdc power supply and the output is read at pin number 4 using a digital multi-meter. The pin number 4 is also connected to the data acquisition system (DAQ) to record the data electronically. Here are some important characteristics of the pressure transducer.

Characteristics	Symbol	Min	Typ	Max	Unit
Pressure range	$P_{OP}$	20	-	400	kPa
Supply voltage	$V_S$	4.64	5.0	5.36	Vdc

#### 4. Thermocouple (J type)

We have used two thermocouples in this experiment. One is connected to a digital multi-meter and placed inside the beaker containing water to observe the temperature of the hot or cold reservoir. The second thermocouple is suspended in air inside the conical flask and records the temperature variation of the air through the DAQ.

#### 5. Syringes

We have used 10 and 20 mL glass syringes (Cole Palmer). The choice of the syringe depends upon the size of the flask. The best option is to use the 37 mL flask with a 20 mL syringe for achieving a reasonable span of variation in pressure and volume. **The glass syringes are breakable, therefore they must be handled with care.** If you feel friction in the piston, place your finger at the tip and press the piston back and forth. This will make the piston move smoothly. One 60 mL plastic syringe is also used in the experiment.

#### 6. Hot and cold reservoirs

The hot reservoir is made by heating on the hot plate. We place a beaker containing water on the hot plate and immerse the flask with air inside the water bath. For the cold reservoir, we can use normal tap water or ice chips crushed inside water.

#### 7. Potentiometer

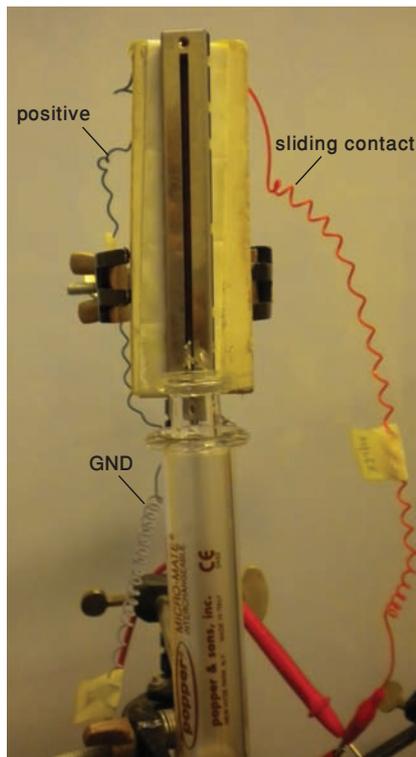
A linear potentiometer, shown in Figure 7 (a) is used to measure the volume changes inside the glass syringe. It has three terminals, two on one side and one on the opposite. From the two terminals at one side, the top one is connected to +5 volts and the bottom one is connected to the ground. On the other side we have the sliding contact directly measuring the extension of the potentiometer. For calibration, the potentiometer is placed horizontally on the table. While for the remainder of the experiment, it will be used in the vertical position.

#### 8. Digital Vernier Calliper

A digital Vernier calliper is used to measure the change in length precisely while performing the calibration of the potentiometer. The initial reading of the calliper is zero.

#### 9. Digital multi-meter

We have used two digital multi-meters in our experiment. One is connected to the pressure sensor to observe the voltage signal generated for the pressure changes. The second multi-meter is connected to the terminals of the thermocouple to monitor the temperature changes inside the flask.



(a)



(b)

Figure 7: (a) Connections to the potentiometer. (b) Fixation of the digital Vernier calliper to measure the increase in length

10. **Power supply for potentiometer and pressure transducer** One 30 V DC power supply is used to provide a fixed +5 V to the pressure transducer. The second is connected to the potentiometer and provides a +5 V fixed DC voltage.

## 5 The Experiment

### 5.1 Calibration of the potentiometer

A potentiometer is a three-terminal device with one sliding contact that acts as a voltage divider. The other two terminals are connected to voltage and ground through a power supply. In our experiment, we will use a linear potentiometer to measure the volume change of air in a glass syringe. The change in volume of air in a syringe is recorded in terms of voltage that is recorded electronically using DAQ and we can convert it into volume using a calibration equation of the potentiometer. The first step is this calibration.

**Q 4.** Place the potentiometer horizontally on the table and fix it with the help of scotch tape such that it does not move. Connect a multi-meter to the potentiometer and using the provided support and tape, fix the calliper as shown

in Figure 7 (b). Try to keep the Vernier calliper and the potentiometer at the same level and fix the sliding bar of the calliper to the potentiometer using tape. Make sure that the calliper does not move but its length can be changed.

**Q 5.** Make connections to the power supply.

**Q 6.** Note down the initial reading of the voltage when the displacement of the sliding contact is zero.

**Q 7.** Now start increasing the displacement and measure the corresponding voltage. Try to increase the length in small increments to get a large number of readings. The reading on the calliper directly measures the length and the voltmeter shows the corresponding voltage.

**Q 8.** Plot the voltage versus displacement and using the curve fitting tool in MATLAB, obtain the calibration equation for the potentiometer. Find the slope and intercept of this curve, paying close attention to the units. You will use the slope ( $V/mL$ ) and intercept ( $mL$ ) in the LabView codes.

## 5.2 Verifying Charles's law

In the present section, we aim at verifying the Charles's law and predicting the absolute zero of temperature. If you are not sure, what the law is, look up [1].

**Q 9.** Write a mathematical equation for the Charles's law.

**Q 10.** Connect a 10 mL glass syringe to a 37 or 50 mL flask with the help of plastic tubing as shown in Figure 8 (a). Make sure there is no droplet of water inside the flask. This may produce extra pressure due to the evaporation of the water molecule and not due to the expansion of air. Guided by the figure, complete the assembly.

**Q 11.** To make the flask sealed properly, use silicone or teflon tape. Place the flask in a beaker containing water and heat it on a hot plate.

**Q 12.** Open the LabView file **temperatureversusvolume.vi** and click **Run**. The data will be saved in a file **charleslaw.lvm** in the Z drive containing the folder **heat\_engine**.

**Q 13.** As the water is heated, you will notice a change in the volume of the gas, making the piston of the syringe to move upwards. This change in volume is recorded using the linear potentiometer whose output is a variable voltage connected to the DAQ.

**Q 14.** The change in temperature is monitored using the thermocouple placed in the water and connected to the DAQ.

**Q 15.** Open the saved file in Matlab and using the recorded data, plot the curve between volume and temperature and calculate the value of the absolute zero by extrapolation. The predicted value is  $-273^{\circ}\text{C}$ . For this part use the calibration equation achieved in **Q8**.

**Q 16.** Repeat the same experiment by directly heating the flask on a hot plate

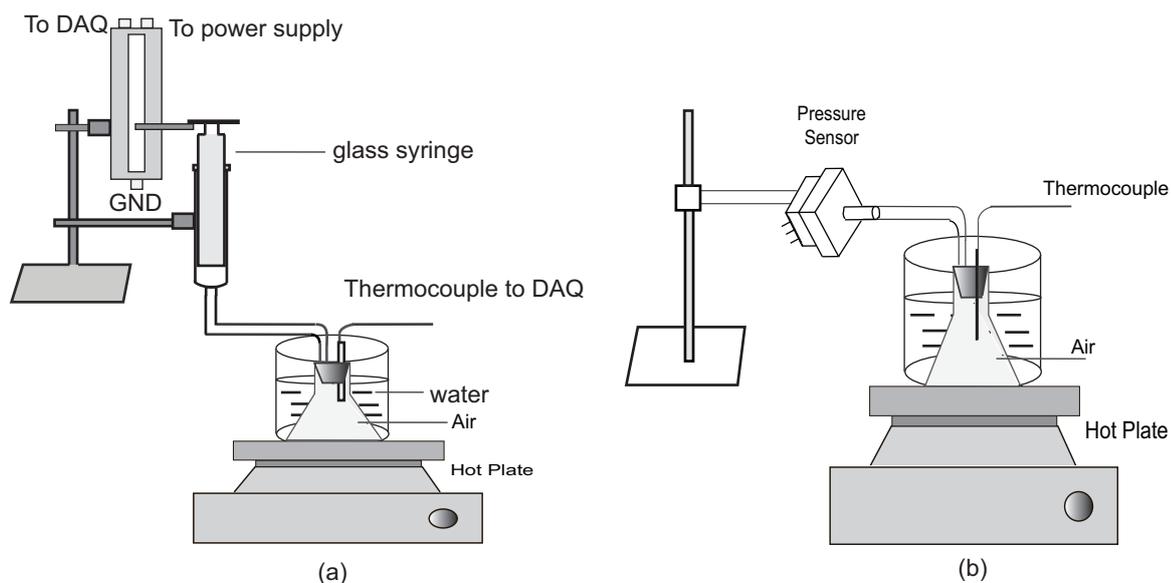


Figure 8: (a) Experimental scheme for the verification of Charles's law. (b) Setup to see the behavior of pressure versus temperature when the volume is fixed.

and calculate the absolute zero temperature.

**Q 17.** Explain the difference between the two methods. Which method provides more reliable readings?

### 5.3 Verifying the change in pressure versus temperature at constant volume

**Q 18.** Connect a 37 mL flask using plastic tubing directly to the pressure sensor as shown in Figure 8 (b). Prefer using the smaller volume flask to have sufficient expansion of the syringe. Make the connections tight using silicone or teflon tape to avoid any leakage.

**Q 19.** Place the flask in water. Open the LabView file named as **pressure-versustemperature.vi** and click **Run** after setting the hot plate on for heating. The values for temperature and pressure will be recorded in the file **pressureversustemperature.lvm** in the same folder as described in section 5.2. Plot a curve for pressure versus temperature.

**Q 20.** Estimate the value of absolute zero temperature.

**Note:** Do not heat the flask directly for a long time. The maximum temperature should not exceed 90°C.

## 6 The heat engine

The industrial revolution in the late eighteenth century catalyzed the formation of a machine that could efficiently convert thermal energy into mechanical work [1]. Initially, these were low efficiency machines but with time a sufficient increase in efficiency was made possible. A device that converts thermal energy to mechanical work is called as a heat engine.

In principle, heat is taken in from a hot reservoir and used to do work. There is also some on flow of energy to a cold body. We know from our daily experiences that the conversion of work to heat is easy but it is more challenging to obtain work from thermal energy. Kelvin formulated this perception in his second law of thermodynamics.

**No process is possible whose sole result is the complete conversion of heat into work.**

An alternative statement of the second law is, **it is impossible experimentally to convert thermal energy into work with out any loss of energy to a cold environment.**

Now let's try to understand the above statement using heat engines, such as steam and petrol engines. Steam and automobile engines convert heat into mechanical energy. The steam engine obtains heat from the combustion of coal in a boiler whereas the automobile engine obtains heat from the combustion of petrol in its cylinders. A substantial amount of heat is lost to surroundings through the condenser of the steam engine and the radiator of an automobile engine.

Consider the flowchart of a simple heat engine shown in Figure 9. An amount of heat  $Q_H$  is given to the engine by the hot reservoir held at the temperature  $T_H$ . This amount of input energy will be utilized in doing some work  $W$  and the remaining energy will be transferred to the cold reservoir  $Q_C$  at temperature  $T_C$ , in line with the energy conservation principle. Using the first law of thermodynamics we can write,

$$Q_{net} = Q_H - Q_C = W, \quad (19)$$

where  $W$  is the mechanical work done by the engine in one complete cycle, implying  $\Delta U = 0$ .

Now we define the thermodynamic efficiency, the ratio of work done to the input thermal energy, given by,

$$\eta = \frac{W}{Q_{net}}. \quad (20)$$

Substituting Equation ( 19) in the above equation we obtain,

$$\eta = 1 - \frac{Q_C}{Q_H}, \quad (21)$$

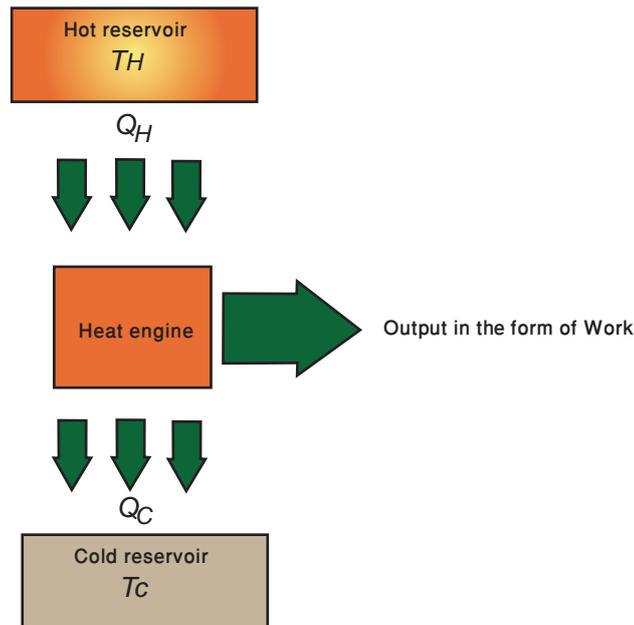


Figure 9: Working principle of a heat engine.

which can also be written in terms of absolute temperatures as,

$$\eta = 1 - \frac{T_C}{T_H}. \quad (22)$$

The above equation shows that the efficiency of any heat engine is defined by the difference in temperatures of the cold and hot reservoir.

**Q 21.** Derive Equation (21) and Equation (22).

**Q 22.** Under what conditions is the thermodynamic efficiency  $\eta = 100\%$ ?

Before moving to the experimental part of making a simple heat engine using a glass syringe, we discuss an ideal engine working in a cyclic process.

## 6.1 The Carnot cycle

Sadi Carnot in 1824 proposed an ideal heat engine capable of converting thermal energy into work. When a system performs work with the application of the thermal energy, the variables such as pressure and temperature, defining the state of the system change. If at the end of the processes, the initial state is restored, it is called a thermodynamical cycle.

To understand how the Carnot cycle works, consider an example of a gas contained in a cylinder that is fitted with a piston as shown in Figure 10.

If we place the cylinder in contact with a high temperature thermal reservoir at

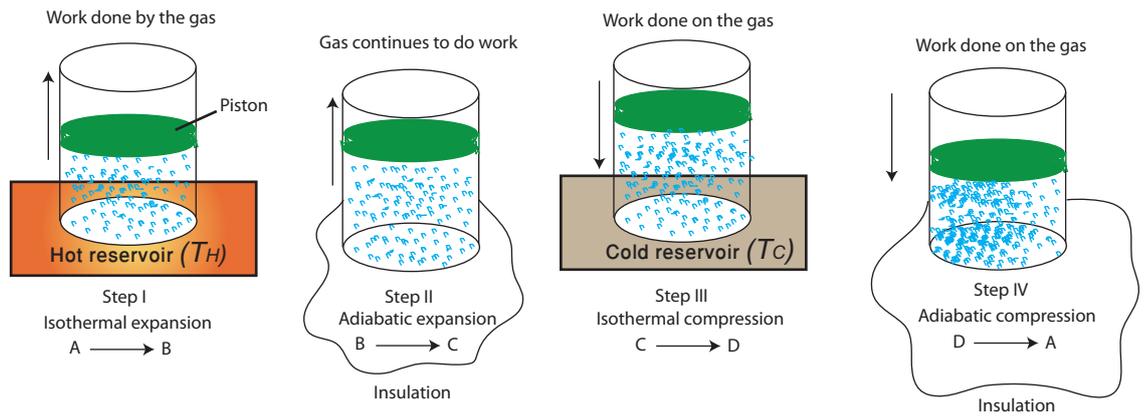


Figure 10: Demonstration of different phases of the Carnot cycle.

temperature  $T_H$ , the gas expands and does some work on the piston by moving it to a new position as shown in step I. The temperature of the gas does not change in this process. This process in which the engine absorbs heat and performs work at a constant temperature is known as the isothermal process and leads to the trajectory,  $A \rightarrow B$  as shown in Figure 11.

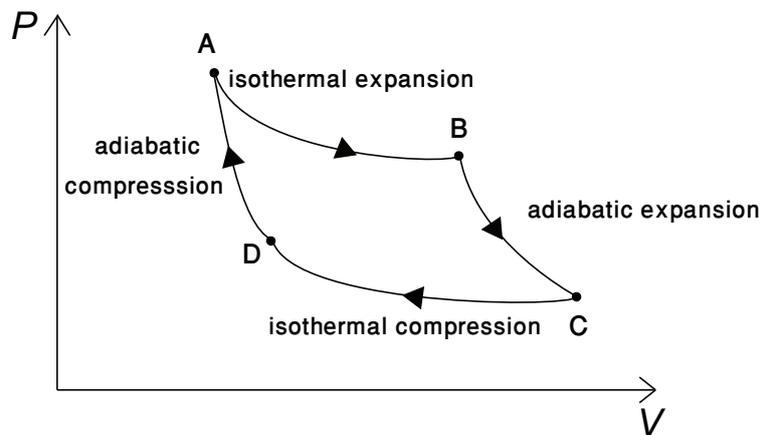


Figure 11:  $PV$  diagram for the Carnot cycle.

If we now remove the cylinder from the heat reservoir, the gas will continue to expand. This is an adiabatic process in which the gas does work while being insulated from its surroundings. As the gas is expanding, its temperature will continue to decrease. The adiabatic curve representing step II is shown in Figure 11 by the path  $B \rightarrow C$ .

Now we place the cylinder in contact with a low temperature reservoir at temperature  $T_C$ . The heat now flows from the gas to the reservoir thus decreasing its energy. As the temperature remains constant, therefore, this is also an isothermal

process and is shown in Figure 11 from C → D.

Now, if we sever the contact between the low temperature reservoir and the cylinder, the gas will continue to reduce its volume. Since no heat is added or taken out of the system, this is adiabatic compression. The original volume has been restored and initial pressure value is achieved. The path D → A in Figure 11 depicts this process and completes the Carnot cycle.

In the following experiment, we will employ the above principles and construct a simple heat engine that operates in a cyclic process between two temperatures  $T_H$  and  $T_C$ . You will be able to calculate the thermodynamic and mechanical and efficiencies and see how they differ from each other. The mechanical efficiency is defined later.

## 6.2 Practical demonstration of the heat engine

**Q 23.** Connect a 20 mL glass syringe to the pressure sensor, 37 ml conical flask and a linear potentiometer as shown in Figure 12. The position of the potentiometer should be vertical allowing the free movement of the sliding contact.

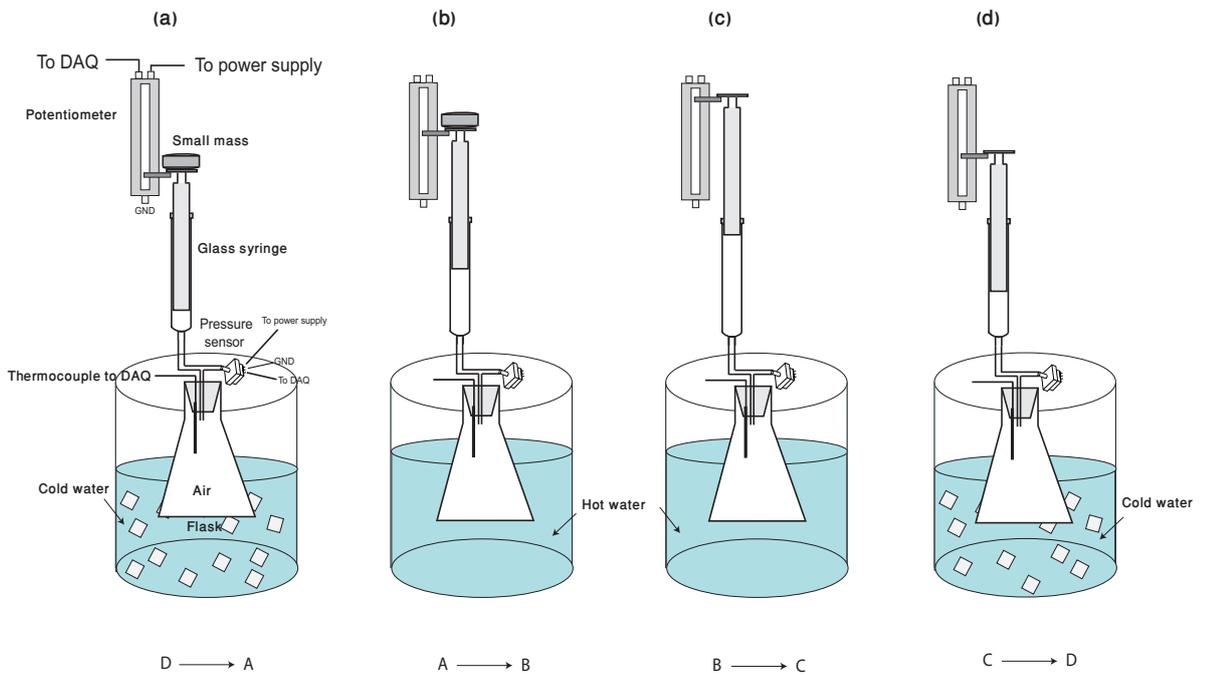


Figure 12: Steps depicting the implementation of the Carnot cycle.

**Q 24.** Place the flask in the cold reservoir whose temperature is monitored using the thermocouple. Open the LabView file **heatengine.vi** and click **Run**. The values of pressure and volume will be stored in a file **heatengine.lvm** in the **heat\_engine** folder in the Z drive. Now place a small mass (e.g. 50 or 100 g) at the top of the syringe slowly, without displacing the assembly, see Figure 12 (a). The gas inside the flask contracts adiabatically and moves the potentiometer. The relevant change in the position is monitored using the DAQ system. Now move

the flask to a hot reservoir with known temperature, as shown in Figure 12 (b) and observe the changes. Next, remove the mass slowly from the piston, making the piston to expand more, Figure 12 (c) representing the expansion. Last, place the flask back in the cold reservoir as shown in Figure 12 (d). The gas achieves its original volume. Stop the vi file.

**Q 25.** Open the saved file in Matlab and plot the pressure versus volume curve.

**Q 26.** Discuss your result with the demonstrator and give reasoning why is it different from the ideal curve?

**Q 27.** Calculate the total work done and heat added during the process.

**Q 28.** Calculate the thermodynamic efficiency.

**Q 29.** Calculate the mechanical efficiency defined as,

$$\eta_{mech} = \frac{\text{Work output}}{\text{Work input}}. \quad (23)$$

**Q 30.** Make a semi-logarithmic plot for the data obtained in the adiabatic process of the heat engine cycle and calculate the value for  $\gamma$  for air. Compare your value with the published value of 1.31 [4].

**Q 31.** Calculate the uncertainty in  $\gamma$ ?

## References

- [1] Fredrick J. Bueche and David A. Jerde, *Principles of Physics*, (McGraw Hill, 1995), pp. 352-391.
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