Determination of Verdet constant from combined ac and dc measurements

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Two methods of determination of the Verdet constant (V) for an optically transparent medium are described. These methods use the same apparatus in which the intensity of a monochromatic beam of light is modulated by passing it, successively, through a polarizer, the medium under test (contained within a solenoid which carries ac of frequency f in its windings), and an analyzer. The emergent beam is incident on a detector which produces a steady output voltage U_o for the special conditions of the optical system set for maximum transmittance and, simultaneously, zero modulation. Methods I and II, which were tested by determining V for water for the wavelength 643.3 nm, are based, respectively, on the ratios u_1/U_o and u_2/U_o, where u_1 is the rms value of detector output voltage varying at frequency f and u_2 is the corresponding output varying at frequency $2f$. The three mean values of V from method I, method II, and a literature survey agree to within 1%. The standard deviation in the values of V by method I (eight determinations) is 0.8% of the mean and by method II (16 determinations) is 1.4%, leading to respective standard errors of the mean (attributable to random errors) of 0.3% and 0.4%. By either method there is, additionally, 0.4% systematic uncertainty in V arising from calibration accuracy of the instrumentation. The total range of Faraday rotation angles studied spanned the range 2×10^{-4}–0.24 deg rms.

I. INTRODUCTION

The Faraday effect is the phenomenon of induced optical activity in a material by a magnetic field which has a component along the propagation direction of the light. For passage of linearly polarized light through an element of length dl of the material, the plane of polarization is rotated through the angle $VH\cdot dl$, where H is the magnetic field at the element and V, which is a characteristic property of the material for a given wavelength and temperature, is the Verdet constant.

The standard dc method of determination of V, exemplified by Refs. 1–5, involves passing monochromatic light successively through a polarizer, the material under test (mounted axially within a solenoid which carries a steady current in its windings), an analyzer, and then on to a detector. A similar geometrical arrangement, but with alternating current in the windings of the solenoid, was used by Duffy and Netterfield (DN)[6] and by Brevet-Philibert and Monin (B-PM)[7] to determine the amplitude of Faraday rotation per unit amplitude of solenoid current. The apparatus as used by B-PM is more automated than that used by DN but, essentially, the methods are the same[8] and only trivial additional experimental data are needed to convert from amplitude of Faraday rotation per unit solenoid current to Verdet constant.

An ac method was used by Williams *et al.*[9] to investigate the Faraday effect in glasses although the emphasis was on the fractional change of V per unit temperature increase rather than on the absolute values of V. Several instruments, based on the Faraday effect in optical fibers, have been developed as novel ammeters for ac and pulsed current but, usually, the same instruments have not been used to determine V for the fiber material.[10–16]

In the method described by DN and B-PM the output signal from the detector, on which light from the analyzer is incident, contains components at the frequencies f and $2f$, where f is the frequency of the ac in the solenoid windings. The rms values, u_1 and u_2, respectively, of these voltage components are measured and V is found from the ratio u_2/u_1. In the present work, using apparatus similar to that used by DN and, also like them, using ac for the solenoid current, separate determinations of V are made from the ratios u_1/U_o and u_2/U_o, where U_o is the steady output from the detector under conditions of zero solenoid current and the analyzer set for maximum transmittance. This pair of related techniques for measuring V was tested using water as the Faraday medium. An advantage of the proposed methods is that, for the small additional effort of measuring U_o, the u_1 and u_2 data yield independent results for V and this forms a useful check. A further advantage is that, by the choice of analyzer settings used, the value obtained for V is less sensitive to uncertainty in these settings. This allows one to achieve similar accuracy in the value of V to that attainable by the DN/B-PM method but with the use of a cheaper analyzer (in the present work the analyzer was a polaroid film device whereas DN and B-PM each used a Glan–Thompson prism). Another advantage is that the result for V, when measuring rotation angles less than about 0.1 deg rms, is less influenced by accidental second-harmonic current component through the solenoid windings. It is conceded that the greater degree of automation and the compensation for drift in the output intensity of the source are important advantages of the procedure used by B-PM. In the present work the ranges of Faraday rotation angle used to measure V for water were, respectively, 0.24–0.01 and 0.24–0.03 deg rms for the methods based on the ratios u_1/U_o and u_2/U_o, although rotations down to 2×10^{-4} deg. rms were detectable by the

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former method. These rotations are smaller than those measured for glass by DN (minimum rotation of 0.5 deg. rms) or for water by B-PM (minimum rotation of 5.1 deg. rms).

II. THEORY UNDERLYING THE METHODS

Modulation of the Faraday rotation in the form $\hat{\theta}\cos\omega t$ for a medium placed between a pair of polarizers differing in azimuthal setting by ϕ leads to transmitted power W through the system given by

$$W=\frac{W_0}{2}\{1+\cos[2(\phi-\hat{\theta}\cos\omega t)]\}, \qquad (1)$$

where W_0 is the transmitted power for $\phi=\hat{\theta}=0$. Expansion of the right-hand side of Eq. (1) into sine and cosine terms of the angles 2ϕ and $2\hat{\theta}\cos\omega t$, followed by use of standard properties[17] of Bessel functions enables development of the expression into its Fourier components with the result

$$W=\frac{W_0}{2}\left\{1+\left[J_0(2\hat{\theta})+2\sum_{m=1}^{\infty}(-1)^m J_{2m}(2\hat{\theta})\cos 2m\omega t\right]\right.$$
$$\times\cos 2\phi+2\left[\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(2\hat{\theta})\right.$$
$$\left.\left.\times\cos(2m+1)\omega t\right]\sin 2\phi\right\}, \qquad (2)$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν for the argument x. Extraction from Eq. (2) of the first-harmonic (angular frequency ω) and second-harmonic (angular frequency 2ω) terms, together with series expansion[17] of the Bessel functions, leads to

(amplitude of first harmonic in W)

$$=W_0\left[\frac{\hat{\theta}}{0!1!}-\frac{\hat{\theta}^3}{1!2!}+......\right]|\sin 2\phi| \qquad (3)$$

and

(amplitude of second harmonic in W)

$$=W_0\left[\frac{\hat{\theta}^2}{0!2!}-\frac{\hat{\theta}^4}{1!3!}+....\right]|\cos 2\phi|. \qquad (4)$$

In the experimental arrangement, to be described in Sec. III, the modulation $\hat{\theta}\cos\omega t$ for the Faraday rotation is produced by passing a current $\hat{I}\cos\omega t$ through a solenoid which has n turns per unit length and contains an axially mounted Faraday cell of length l. Thus, the amplitudes $\hat{\theta}$ and $\hat{I}$ are related by the equation

$$\hat{\theta}=KnVl\hat{I}, \qquad (5)$$

in which K is a dimensionless factor which can be calculated from the known geometry of the solenoid and Faraday cell. Because a detector is used which produces a voltage output proportional to the incident power, it follows from Eqs. (3) and (5) that

$$u_1=KnVlIU_o[1-(KnVlI)^2+.....]|\sin 2\phi| \qquad (6)$$

and from Eqs. (4) and (5) that

$$u_2=\frac{(KnVlI)^2U_o}{\sqrt{2}}\left[1-\frac{2}{3}(KnVlI)^2+....\right]|\cos 2\phi|, \qquad (7)$$

where u_1 is the rms value of the first-harmonic component of output voltage, u_2 is the rms value of the second-harmonic component of output voltage, U_o is the steady voltage output corresponding to unmodulated incident power W_o on the detector, and I $(\equiv\hat{I}/\sqrt{2})$ is the rms value of the current through the solenoid. Experimentally, U_o is obtained as the output voltage for zero current through the solenoid when the angle ϕ is zero. Equation (6) forms the basis of a method of determination of V from the measured values of u_1 and U_o. Similarly, Eq. (7) forms the basis of another method using measured values of u_2 and U_o. For $\hat{\theta}\ll 1$ rad, the expression in the square brackets in each of Eqs. (6) and (7) may be replaced by unity. If necessary, the approximate value of V obtained in this way from either equation can be used as the initial value in an iterative procedure to obtain a refined value. However, in the experiments to be described (in which $\hat{\theta}$ has a maximum value of 5.9×10^{-3} rad), the iteration is unnecessary as it would lead to a final value of V differing from the initial estimate by less than 0.002%.

Equation (6), with the approximation $\hat{\theta}\ll 1$ rad, was used to determine V in the following two ways: (i) the gradient, du_1/dI, of a least-squares-fitted line to a plot of u_1 against I for $\phi=\pi/4$ and fixed U_o was equated to $KnVlU_o$ and (ii) the gradient of a least-squares-fitted line to a plot of u_1/I against $|\sin 2\phi|$ for fixed U_o was also equated to $KnVlU_o$. Equation (7), again for the condition $\hat{\theta}\ll 1$ rad, was used to determine V by equating the gradient, du_2/dI^2, of the least-squares-fitted line to a plot of u_2 against I^2 for $\phi=\pi/2$ and fixed U_o, to $(KnVl)^2U_o/\sqrt{2}$.

From Eq. (6) it is seen that u_1 vanishes for $\phi=0$ (polarizer and analyzer having parallel transmission axes) and also for $\phi=\pi/2$ (polarizer and analyzer crossed). In some automatic polarimeters the vanishing of u_1 for the crossed setting is used to accurately determine this setting.[18,19]

III. APPARATUS COMMON TO BOTH METHODS

Figure 1 is a schematic diagram of the apparatus. The source of light is a 12-V, 36-W tungsten filament lamp (L) operating from a 250 A h lead-acid battery which continuously receives a small net charge from a stabilized power supply. Approximately monochromatic light of effective wavelength 643.3 nm is selected by an interference filter and then, after collimation by lens L_1, passes successively through polarizer P_1, the water-filled Faraday cell (FC), polarizer P_2, lens L_2, an iris and is finally detected by an EMI 6097B photomultiplier tube (PMT) having an S11 cathode. The polarizers P_1 and P_2 (supplied by Ealing) each consist of polaroid film sandwiched between glass plates and their azimuthal settings can be read, using the vernier scales provided, to $\pm0.1°$. Extinction to within 0.04% of peak transmittance is obtained for the wavelength used. The Faraday cell (FC) is constructed from black plastic tubing (length 282.5 ± 0.1 mm, internal diameter 22 mm) and has a glass window of thickness 0.15 mm cemented by epoxy resin at each end. A pair of opaque

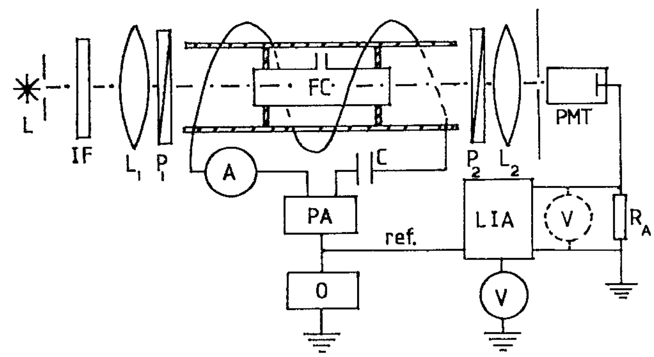

FIG. 1. Schematic diagram of the apparatus used for both methods of determination of the Verdet constant. A, ammeter (digital meter for all 525-Hz measurements); C, capacitor; FC, Faraday cell filled with water; IF, interference filter; L, tungsten filament lamp; L_1 and L_2, lenses; LIA, lock-in amplifier; O, oscillator; P_1 and P_2, polarizers; PA, power amplifier; PMT, photomultiplier tube; R_A anode load resistor for PMT; V, voltmeter.

plastic washers support FC axially and symmetrically within the solenoid. Complete filling of the cell before positioning in the solenoid is via a short vertical side tube cemented in place by epoxy resin.

The solenoid, which consists of two uniform layers of enammelled copper wire of diameter 0.782 mm on a former of internal diameter 53.9 mm has a wound length of 500.5 mm. By counting turns in the outer layer as they were tracked and viewed through a traveling microscope the winding density (n) for the combination of the two layers was found to be 2556 ± 2 turns m^{-1}. A power amplifier (Radiospares model 303-236) deriving its input signal from a variable frequency oscillator (O) feeds the solenoid through a series capacitor (C). The oscillator frequency is adjusted for resonance and this tuning serves the dual functions of giving access to larger solenoid currents and filtering out much of the distortion in the power amplifier output.

The anode current from the PMT flows through a load resistor R_A and the harmonic components u_1 and u_2 in the voltage across R_A are separately measured using an Ithaco model 3921 lock-in amplifier (LIA) which receives its reference input from the oscillator O. To measure the steady voltage U_o across R_A, a digital voltmeter is used while ϕ is set to zero and $I=0$. This meter is disconnected during measurement of u_1 and u_2 as pick-up in its connecting leads was found to be a major effect. The shunting effect of the voltmeter and the LIA when separately connected across R_A automatically cancels when forming the ratios u_1/U_o and u_2/U_o because both instruments have the same input impedance (10 MΩ) and, moreover, this shunting effect is less than 1.2% even for the largest value used for R_A.

IV. PRELIMINARY MEASUREMENTS

The accuracy of the LIA was checked using standard input voltages at a frequency of 525 Hz (the frequency for the majority of Faraday measurements) obtained from a Time Electronics model 9817 calibrator. Accuracy of the LIA was found to be within 0.1% for the 1-mV range and within 1% for the other ranges used; corrections were applied as appropriate to subsequent measurements of u_1 and u_2. The digital ammeter used for measurement of the solenoid current for all 525-Hz experiments was calibrated against another ammeter which a standardizing laboratory had certified as accurate to within 0.2%.

The use of a photomultiplier tube (PMT) as the optical detector gives added flexibility because its gain can be varied by altering the applied voltage; thus overall gain of the measuring system is adjustable by altering either the gain of the PMT or LIA. The relative gain of the PMT as a function of applied voltage was determined from a dc experiment involving a set of several applied voltages at fixed cathode irradiance and a further set with overlapping voltage range for another value of the irradiance.

To determine the effective wavelength (λ_{eff}) passed by the lamp/interference filter combination, the spectrum was obtained using a Zeiss M4 QIII monochromator. The internal photomultiplier in this instrument has similar cathode properties to that of the 6097B photomultiplier. Using Simpson's rule to perform the necessary numerical integration, 45 ordinates were measured from the chart record of the spectrum and the quantity $\lambda_{\mathrm{eff}} = \langle \lambda^{-2.1} \rangle^{-1/2.1}$ evaluated with weighting of each $\lambda^{-2.1}$ value in the calculation of the mean in proportion to the corresponding ordinate. This prescription for λ_{eff} was based on a survey of existing data of V for water which showed $V \propto \lambda^{-2.1}$ over the wavelength range of interest. The properties of the lamp/filter combination as deduced from the transmitted spectrum are as follows: $\lambda_{\mathrm{eff}} = 643.3 \pm 0.5$ nm, centroid wavelength $= 643.5 \pm 0.3$ nm, and 50% of peak ordinate at the wavelengths 637.0 and 650.4 nm.

Second-harmonic distortion in the magnetic field produced by the solenoid was determined by winding a few turns (2–6) of insulated wire around the solenoid and then using the LIA to measure the ratio of second- to first-harmonic components of voltage induced in these sensor turns. Halving this ratio gives the ratio of second- to first-harmonic components of the magnetic field. In this way, for the solenoid circuit containing a nominal 10-μF capacitor and resonating at a frequency of 525 Hz (giving a quality factor $Q=3.15$), the ratio of second- to first-harmonic component in the magnetic field is constant at 0.14% over the tested range (0.5–1.5 A rms) of solenoid current. From a similar test employing instead a 0.11-μF capacitor in series with the solenoid, which raised the resonance frequency to 4.59 kHz and the Q to 22.6, the amplitude of the second-harmonic component in the magnetic field was found to be 0.030% independently of the solenoid current.

A graph of voltage across R_A as ordinate against angular reading of the analyzer P_2 as abscissa was obtained from a dc experiment spanning a range of $\pm 20°$ centered on the crossed condition. The mean of the midpoints of many horizontal chords drawn on the large-scale graph was obtained to give the reading of P_2 corresponding to $\phi = \pi/2$. The appropriate correction, which was reproduced to within 0.1° in a repeat determination, was applied to all subsequent measurements of ϕ.

From measurements of the coil dimensions and the length of the Faraday cell, the numerical constant K in Eq. (5) was calculated to be 0.9904 ± 0.0005.

The retardance introduced by passage of light through each window of the Faraday cell and resulting from accidental stress birefringence in the glass is less than 0.7 deg. This upper limit to the retardance was obtained by placing an identical window between a pair of crossed polarizers and observing the variation in transmitted intensity through the system as the cell window was rotated about an axis perpendicular to its plane.

A rough estimate for the power W_o incident on the cathode of the PMT under conditions of $I=\phi=0$ and the lamp operating at its full rated voltage is $6 \pm 2\ \mu$W. This result was obtained by temporarily replacing the PMT by the silicon receiving cell of a Scientifica Cook laser power monitor intended for measuring power in the red emission from a He–Ne laser.

V. EXPERIMENTS: VERDET CONSTANT (V) FROM FIRST-HARMONIC CONTENT OF MODULATED OPTICAL POWER

A. Check for possible frequency dependence of V

With the angle ϕ between the transmission axes of polarizers P_1 and P_2 set to 45°, the solenoid rms current (I) maintained at 0.5 A and constant values for both the applied voltage to the PMT and the light output from the lamp, the frequency (f) of the solenoid current was varied over the range 15–1500 Hz while readings of u_1 were obtained. To cover this frequency range necessitated using several values of the series capacitance in the solenoid circuit. Measurements of f were by a Global Specialties model 5001 frequency meter and the coil current was obtained using a moving coil ammeter which could be read to an accuracy of 0.3%.

B. Dependence of u_1 on ϕ and its use to determine V

Proportionality between u_1/I and $\sin 2\phi$, which is predicted by Eq. (6) for $\hat{\theta} \ll 1$ rad, was tested over the range $0 < \phi \leqslant \pi/2$ by altering the setting of P_2 while maintaining constant light output from the lamp and using rms solenoid currents of 0.5, 1.0, and 1.5 A all for a frequency of 525 Hz. The steady voltage U_o across R_A corresponding to $\phi=0$ and $I=0$ was obtained both at the beginning and end of the experiment to ensure that its drift was negligible. In addition to testing the relationship $u_1/I \propto \sin 2\phi$, the data were also used to determine V for water.

C. Test of proportionality between u_1 and I for $\phi=\pi/4$

For constant light output from the lamp and with ϕ maintained at $\pi/4$, the voltage u_1 was monitored as I was varied over the range 1.50 A $> I >$ 1 mA for the frequency of 525 Hz. To conveniently cover this wide range of I use was made of the gain adjustment of the PMT through its applied voltage and the measured values of u_1 were converted to the basis of a fixed gain of the PMT. As in Sec. V B, the value of U_o was obtained at the beginning and end of the experiment.

D. Determination of V from dependence of u_1 on I for $\phi=\pi/4$

This experiment was performed eight times, each for about ten different rms solenoid currents in the range $0.1 < I < 1.50$ A and all for a frequency of 525 Hz. Apart from the smaller range of I covered, which therefore required less adjustment of the voltage applied to PMT, each run of the experiment is similar to that described in Sec. C above. A range of different values of W_o (and therefore also U_o) were spanned. Instead of directly measuring U_o from the steady voltage across R_A for $I=0$ and $\phi=0$, it was obtained by doubling the measured voltage across R_A for $I=0$ but with ϕ kept at $\pi/4$. This avoids the need to adjust the analyzer between the ac measurement of u_1 and the dc measurement of U_o although the direct determination of U_o would enhance the precision of its determination. For each run, V was determined from Eq. (6) using the mean value for U_o at the start and finish in conjunction with the gradient, du_1/dI, of the least-squares-fitted line to a plot of u_1 against I.

VI. EXPERIMENTS: VERDET CONSTANT FROM SECOND-HARMONIC COMPONENT IN THE MODULATED OPTICAL POWER

A. Dependence of u_2 on ϕ for constant I

For constant light output from the lamp and for a fixed solenoid current of 1.20 A rms at a frequency of 4.59 kHz, the angle ϕ was altered by adjusting polarizer P_2. For each value of ϕ, the second-harmonic component (u_2) of the voltage across R_A was measured. A capacitor was used in series with the solenoid and the reason for working at a higher frequency than for the other measurements was to sharpen the resonance of the solenoid circuit and thus reduce the distortion in the current it passed.

B. Dependence of u_2 on I for $\phi=\pi/2$ and determination of V

With the polarizers crossed, constant emission from the lamp and the solenoid circuit tuned at a frequency of 525 Hz, the rms value (u_2) of the second-harmonic component of the voltage across R_A was measured for a series of typically ten different values of I. Before and after taking these measurements, P_2 was shifted to the $\phi=0$ setting so that U_o and its drift could be obtained. Because the signal u_2 is small and is measured at $\phi=\pi/2$, thus almost eliminating dc background, it was convenient to use a higher applied voltage to the PMT when measuring u_2 than when measuring U_o although, as always, the measurements were scaled to a fixed gain of the PMT. This experiment was performed 16 times spanning a range of values of W_o and including two widely different values (18 and 118 kΩ) for R_A. The gradients, du_2/dI^2, of least-squares linear regression plots of u_2 against I^2, together with the corresponding

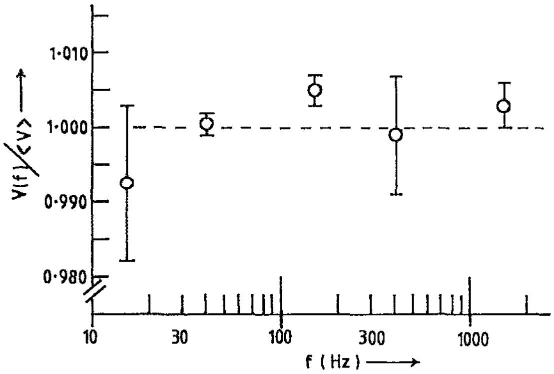

FIG. 2. Plot of $V(f)/\langle V\rangle$ against f where $V(f)$ is the Verdet constant at frequency f and $\langle V\rangle$ is the mean value of $V(f)$ for the frequency range investigated. The horizontal line represents $V(f)/\langle V\rangle = 1$. $V(f)$ is taken as proportional to u_1 for the fixed conditions of $\phi=\pi/4$ and $I=0.50$ A under which the data were obtained as described in Sec. V A. Note the highly expanded scale for the ordinates and that $V(f)/\langle V\rangle$ is within $\pm 1\%$ of unity.

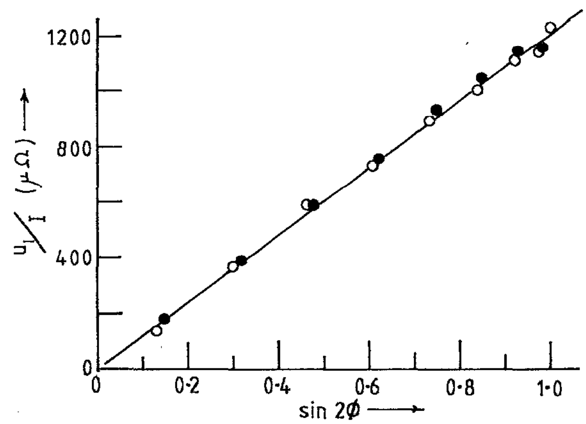

FIG. 3. Plot of u_1/I against sin 2ϕ for a fixed frequency of 525 Hz and fixed output from the lamp. The ordinate for each point is obtained from the mean value of u_1/I from measurements with $I=0.50$, 1.00, and 1.50 A as described in Sec. V B. Points for $0<\phi<\pi/4$ (○); points for $\pi/4<\phi<\pi/2$ (●). The line is fitted by least squares and its gradient, in conjunction with the value (U_0) of the constant voltage across R_A for $I=\phi=0$, yields $V=3.95\pm0.09$ μrad A^{-1}.

values of U_o, were used to determine V from Eq. (7) by neglecting the negligible terms after unity in the square brackets.

VII. RESULTS AND DISCUSSION

Figure 2, which is based on the experiment described in Sec. V A and analysis by Eq. (6), shows that over the tested frequency range, $15<f<1500$ Hz, there is no evidence for frequency dependence of V. This is consistent with expectation because the Faraday effect is due to a perturbing Lorentz force on electrons and these can respond to frequencies far above 1.5 kHz. An assumption inherent in Eq. (5) and therefore also in Eq. (6) is that, even at the highest modulation frequency used, a negligible fraction of the current delivered to the solenoid is capacitively shunted between the layers of windings. The frequency independence of V as calculated from Eq. (6) supports the validity of this assumption because, otherwise, $\hat{\theta}/\hat{I}$ and the apparent value of V would decrease as f is increased. Such a decrease, amounting to 6% for the frequency range $0<f<1$ kHz, was reported by Duffy and Netterfield[6] for their work on Faraday rotation in glass. Further, more direct evidence of negligible capacitive coupling between the two layers of windings of the solenoid used in the present work comes from an experiment in which a few secondary turns of insulated wire were wound over the solenoid and, throughout the tested frequency range $15<f<1500$ Hz, the ratio of the induced rms secondary voltage to the frequency (for a constant amplitude of solenoid current) showed no perceptible frequency dependence.

According to Eq. (6), the ratio u_1/I is proportional to sin 2ϕ for the experimental conditions $\hat{\theta}\ll 1$ rad, and this prediction is confirmed by the linearity of the plot in Fig. 3. From the gradient of the line fitted by least-squares regression, the Verdet constant of water for the measurement conditions (wavelength of 643.3 nm and temperature of 19 ± 3 °C) is found to be 3.95 ± 0.09 μrad A^{-1}. Proportionality between u_1 and I, which is predicted by Eq. (6) for $\hat{\theta}\ll 1$ rad, is supported by the closeness to unity of the gradients of the lines in the double logarithmic plots of u_1 against I shown in Fig. 4. From four such lines of which the figure shows a sample of two, the mean gradient is 0.99 ± 0.01. As shown by Fig. 4, the linearity between u_1 and I is observed to hold down to values of I as small as 1.5 mA and this corresponds to an rms Faraday rotation ($\hat{\theta}/\sqrt{2}$) of 2.4×10^{-4} deg. At the largest solenoid current

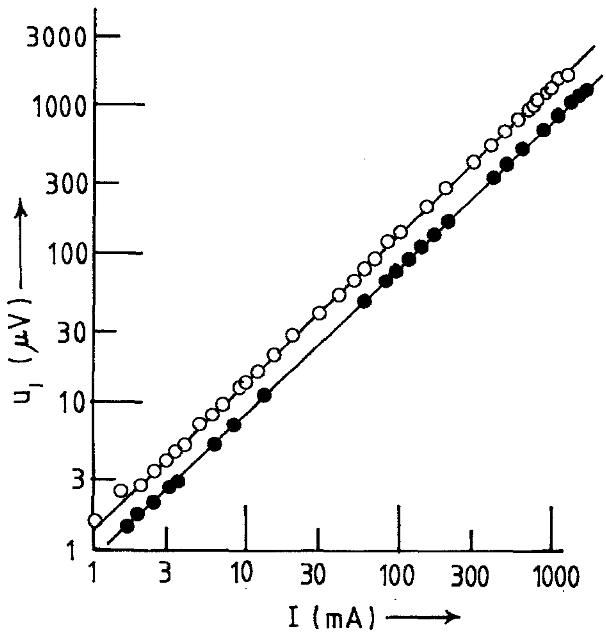

FIG. 4. Two plots, each using logarithmic axes and with lines fitted by least squares regression, of u_1 against I for conditions of $\phi=\pi/4$, $f=525$ Hz, $R_A=118$ kΩ and reduced to 400 V applied across PMT. For each line the light output from the lamp is constant. Linearity is maintained down to $I\simeq1.5$ mA for which the rms Faraday rotation is 2.4×10^{-4} deg. For four similar lines (3 for $R_A=118$ kΩ, 1 for $R_A=18$ kΩ) the mean gradient is 0.99 ± 0.01. The experiment is described in Sec. V C.

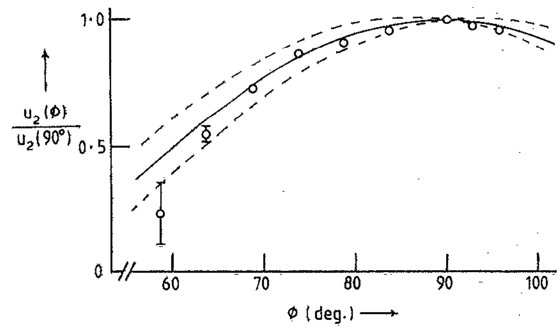

FIG. 5. Plot of u_2 (relative to its value at $\phi=\pi/2$) against ϕ for a fixed output from the lamp and a fixed solenoid current of 1.20 A rms at a frequency of 4.59 kHz. The solid curve represents the theoretical prediction for pure sinusoidal current, i.e., $u_2 \propto |\cos 2\phi|$, and the points show the experimental results from Sec. VI A. Where no error range in the ordinate is indicated, the uncertainty is less than 2%. The range between the two broken curves is accounted for by the observed 0.030% second-harmonic distortion in the magnetic field produced by the solenoid.

used to obtain Fig. 4, $\hat{\theta}/\sqrt{2} \simeq 0.24° = 4.2\times 10^{-3}$ rad. so, under this condition the expression within the square brackets of Eq. (6) differs from unity by less than 0.002%.

For the majority of determinations of V for water by the first-harmonic method, the condition $\phi=\pi/4$ was used and the method has already been described in Sec. V D. The choice of this value for ϕ, as may be seen from the explicit ϕ dependence in Eq. (6), has the dual benefits of maximizing u_1 and simultaneously minimizing the error in V due to a given error in ϕ. There is, however, a further but implicit effect of errors in ϕ on the value obtained for V because the accuracy of U_o is affected by analyzer setting errors. It is estimated that the nominal $\phi=\pi/4$ setting was achieved to within 0.2 deg and this uncertainty in ϕ produces an uncertainty of 0.7% in V if, as was done and described in Sec. V D, U_o is found by doubling the steady voltage across R_A with ϕ unchanged. A direct determination of U_o by setting ϕ to 0.0 ± 0.2 deg, combined with the determination of u_1 for $\phi=45.0\pm0.2$ deg would have reduced the error in V due to errors in ϕ to below 0.1%. If one attempted to use the Duffy and Netterfield[6] or Brevet-Philibert and Monin[7] methods, also with a resolution of 0.2 deg in ϕ (instead of the much superior resolution they obtained), then the error in V due to that in ϕ would be larger than in the present first-harmonic method, e.g. for $\phi=10.0\pm0.2$ deg in their method, the uncertainty in V due to that in ϕ is 2.2%.

From Eq. (7), for conditions of fixed I and U_o, the rms value u_2 of the second-harmonic component of the voltage across R_A is predicted to be proportional to $|\cos 2\phi|$. This prediction is based on the assumption of negligible second-harmonic distortion in the solenoid current (and hence also in the magnetic field produced by the solenoid). Figure 5 shows a plot of u_2, normalized to its value for $\phi=\pi/2$, as a function of ϕ for the conditions $f=4.59$ kHz and $I=1.20$ A with the points representing experimental data (obtained as described in Sec. VI A) and the solid curve showing the above prediction. The region between the two broken curves is the range in which the points are calculated to lie when allowance is made for the observed 0.030% second-harmonic distortion, of unknown phase, in the solenoid current. The calculation is based on the inclusion in Eq. (1) of an additional term in $\cos 2\omega t$ for the Faraday rotation and it is readily shown that the distortion does not alter u_2 for the special case of crossed polarizers. An attempt to extend the range of measurements to lower values of ϕ than for the points in Fig. 5 failed because then the increased first-harmonic component of the photocurrent through R_A and also the enhanced shot noise in this photocurrent causes the LIA to overload. With the exception of the lowest point in Fig. 5, and this point has large uncertainty in its ordinate due to the LIA beginning to overload, the observed results are satisfactorily explained in terms of the theory leading to Eq. (7) after suitable modification for the presence of the observed second-harmonic distortion in the solenoid current.

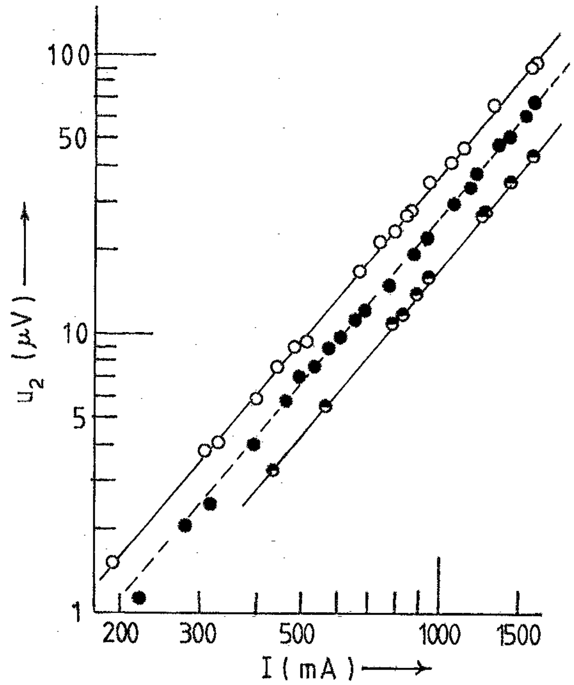

FIG. 6. Three plots, each using logarithmic axes and with lines fitted by least-squares regression, of u_2 against I for conditions of $\phi=\pi/2$, $f=525$ Hz, $R_A=118$ kΩ, and reduced to 600 V across PMT. For each line the light output from the lamp is constant. Linearity is maintained down to $I\simeq 200$ mA for which the rms Faraday rotation is 0.03 deg. For 16 similar lines (11 for $R_A=118$ kΩ, 5 for $R_A=18$ kΩ) the mean gradient is 2.00 $\pm$0.01. The experiment is described in Sec. VI B.

Examples of double logarithmic plots of u_2 against I for $\phi=\pi/2$ are shown in Fig. 6. Each plot is linear and the mean value of the gradients of the least-squares-fitted lines to 16 similar plots is 2.00 ± 0.01 which is in excellent agreement with the theoretical prediction of 2.00 for $\hat{\theta}\ll 1$ rad. With the apparatus used it was not possible to measure u_2 for $\phi=\pi/2$ at Faraday rotation angles smaller than about 0.03 deg rms. The condition $\phi=\pi/2$ is the most favorable to use for measuring u_2 because (i) the desired second harmonic in the optical signal is maximized, (ii) the close approach to ideal extinction minimizes shot noise in the anode current of the PMT, (iii) the effect of undesired

TABLE I. Determination of Verdet constant for deionized water at $\lambda=643.3$ nm and at a temperature of 19 ± 3 °C.

Method	Number of determinations	Result for V (μrad A^{-1})
First-harmonic component ($f=525$ Hz)		
(i) Gradient of u_1 vs I for $\phi=\pi/4$; $R_A=118$ kΩ	8	4.01 ± 0.03
(ii) Gradient of u_1/I vs sin 2ϕ; $R_A=118$ kΩ	1	3.95 ± 0.09
Second-harmonic component ($f=525$ Hz)		
(i) Gradient of u_2 vs I^2 for $\phi=\pi/2$; $R_A=118$ kΩ	11	3.96 ± 0.04
(ii) Gradient of u_2 vs I^2 for $\phi=\pi/2$; $R_A=18$ kΩ	5	3.98 ± 0.04
	Overall mean from the 25 determinations:	3.98 ± 0.03

second-harmonic content in the solenoid current vanishes as previously mentioned, (iv) the u_1 component vanishes thus avoiding the difficulty of measuring u_2 in the presence of a first-harmonic background and (v) the value of V as determined from Eq. (7) has minimum sensitivity to error in ϕ. Even an error in ϕ as large as ±1 deg in attempting to set the analyzer for extinction (to measure u_2) with the same error in ϕ when attempting to set for maximum transmittance (to measure U_0) would cause V to be underestimated by less than 0.05%. In the experiments described here the errors in these settings of ϕ were within 0.2 deg so negligible error in V is attributable to error in ϕ.

The windows of the Faraday cell and, to a lesser extent, the air within the solenoid on each side of the cell, contribute a small Faraday rotation. A correction, which amounts to a reduction of 0.18% from the Verdet constant for water which would be obtained if this effect is neglected, was applied. This correction was calculated using tabulated values[20] of Verdet constants for plate glass and gases and to scale these values to the appropriate wavelength it was assumed that the Verdet constants are proportional to λ^{-2}.

Consideration was given to the possible effect of linear birefringence in the cell windows on the value of V obtained for water. To simplify the analysis, the problem was initially treated as if all birefringence is limited to a single window which has a linear retardance δ. By use of Jones calculus it is readily shown that such birefringence produces no effect on the values of V obtained from either Eqs. (6) or (7) if the electric field of the light incident on the window is polarized along either of the two birefringence eigenaxes. Birefringence in the window has a maximum effect on the values obtained for V if the electric field of the light incident on the window is inclined at an angle of 45 deg to the eigenaxes and, under this condition, it is easily shown that, for small Faraday rotation angles, Eqs. (6) and (7) underestimate V by the respective fractional amounts $(1-\cos\delta)$ and $(1-\sqrt{\cos\delta})$. Because, as stated in Sec. IV, the linear retardance of each window is less than 0.7 deg it follows that the effective total value of δ is less than 1.4 deg and hence error in V resulting from neglect of the birefringence is less than 0.03% for the first-harmonic method and less than 0.015% for the second-harmonic method.

Unfortunately, during the measurements of the Verdet constant of water, the temperature of the water in the Faraday cell was not monitored although the ambient temperature a few centimeters from the solenoid was recorded. The temperature of the water in the cell exceeds the ambient temperature due to Joule heat generation in the windings of the solenoid and it is estimated that the excess temperature acquired during the Faraday experiments was 3 ± 1 °C. This estimate is based on measurement of the

TABLE II. Verdet constant of water for a wavelength of 643.3 nm and temperature of 19 °C as deduced from works of others.

Source	Wavelength range experimentally investigated (nm) min	max	Method used to interpolate or extrapolate to $\lambda=643.3$ nm	V^a (μrad A^{-1})
Ingersoll (Ref. 21)	560	1360	Interpolation by fitting an equation of the form $V\propto\lambda^{-\beta}$	4.00
Stephens and Evans (Ref. 3)	238	600	Extrapolation using an equation given in the original work[b]	3.95
Pierce and Roberts (Ref. 22)	248	578	Extrapolation using an equation given in the original work[b]	3.97
				Mean: 3.97 ±0.02

[a]Each of the cited sources used water at a different temperature but all in the range 15–23 °C. The values of V given in this table have been converted to the basis of a temperature of 19 °C using data from Rodger and Watson (Ref. 2) on the temperature dependence of V for water.

[b]The given equation involves the refractive index of water and data for the wavelength and temperature dependence of this property needed for the extrapolation were obtained from Kaye and Laby (Ref. 23).

thermo-emf produced by a calibrated copper-constantan thermocouple having one junction in the filled Faraday cell and the other junction in air near the apparatus during two runs simulating the earlier conditions during the main experiment. In this way the temperature of the water in the cell during the determination of V is found to be 19 ± 3 °C. According to data obtained by Rodger and Watson[2] the temperature coefficient $(V^{-1}dV/dT)$ of V for water at 19 °C is -1.47×10^{-4} °C^{-1}, thus an uncertainty of 3 °C in the temperature corresponds to an uncertainty of approximately 0.04% in the value of V.

Table I summarizes the results obtained for the Verdet constant (V) for water by the methods proposed in this article. The mean result for V corresponding to the wavelength of 643.3 nm and the temperature of 19 ± 3 °C, obtained by giving equal weight to each of the 25 determinations, is 3.98 ± 0.03 μrad A^{-1}. This is in good agreement with the corresponding result $V=3.97\pm0.02$ μrad A^{-1}, deduced, as shown in Table II, from a survey of other reported determinations.[3,21,22] The method of conversion of V from these other determinations to the wavelength of 643.3 nm is shown in the table and reduction to the temperature of 19 °C was based on the data of Rodger and Watson.[2] In two[3,22] of the three papers surveyed, the wavelength dependence of V was expressed in terms of an equation involving the refractive index of water and data on the wavelength and temperature dependence of this latter property, needed for compiling Table II, were obtained from a book[23] of physical constants. As shown by Table I, the determinations of V from the first-harmonic method for $\phi=\pi/4$ and from the second-harmonic method for $\phi=\pi/2$ both have overall uncertainties in their means of less than 1%. The reproducibility of the former method is 0.8% (although, because eight measurements of V were made by this method, the contribution to the standard error of the mean arising from random errors is reduced to 0.3%). Similarly, the reproducibility of the latter method is 1.4% but, because 16 determinations of V were made by this method, the standard error of the mean arising from random errors is reduced to 0.4%.

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