Chapter 3

▶ The water droplets that fly up from this beautiful fountain in Monaco follow parabolic paths. Such paths are a consequence of the constant vertical acceleration associated with local gravity.

Motion in Two and Three Dimensions

In Chapter 2 we considered the description of motion in one dimension—linear motion. We will now extend this description to motion in space. We observe such motion in the curved path of a thrown ball, in the swing of the pendulum of a grandfather clock, or in the orbits of the planets around the Sun. While for one-dimensional motion the directional aspect is encapsulated in signs, for motion in two or three dimensions we must use vectors to describe the directional aspect properly. With the help of the mathematical apparatus provided by vectors, we'll find that it is straightforward to describe the motion of objects in a plane or in space in a manner that builds on our earlier work with one-dimensional motion.

3–1 Position and Displacement

The motion of a planet orbiting the Sun traces out a path in space. Similarly, a rock thrown off a cliff follows a certain path, or **trajectory**, as does any pointlike object as it moves through space. For motion in a plane, think of a skater on a lake whose skates leave marks that specify the trajectory of the motion. Figure 3–1 depicts a particle, for example the skater, moving in a two-dimensional plane. We label the plane as the xy-plane and introduce a Cartesian coordinate system that contains an origin and x- and

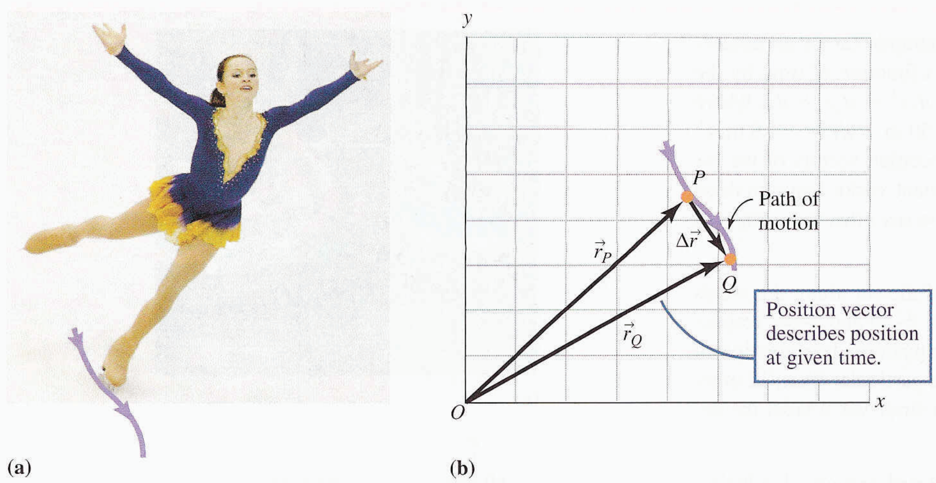


FIGURE 3–1 (a) The ice skater is gliding over the path indicated by the blue trajectory line. (b) Position vectors $\vec{r}_P$ and $\vec{r}_Q$ point from the origin to the positions P and Q at the two times t_1 and t_2, respectively, along the skater's path of motion. The displacement vector between these times is $\Delta\vec{r} \equiv \vec{r}_Q - \vec{r}_P$.

y-axes. The particle is at the position P at time t_1; this position is described by the position vector $\vec{r}_P$, which points from the origin to the point P. At a later time t_2 the particle is located at position Q and is described by the position vector $\vec{r}_Q$. The change in the particle's position between times t_1 and t_2—the final position minus the initial position—can be described by the displacement vector $\Delta\vec{r}$; this vector is defined by

$$\Delta\vec{r} \equiv \vec{r}_Q - \vec{r}_P. \tag{3–1}$$

The vector $\Delta\vec{r}$ points from the tip of vector $\vec{r}_P$ to the tip of vector $\vec{r}_Q$ and describes the direction of the displacement as well as its magnitude. Whereas the position vectors $\vec{r}_Q$ and $\vec{r}_P$ depend on the choice of origin, the displacement vector $\Delta\vec{r}$ is independent of the choice of origin. To see this clearly, let's imagine that there is a new origin O' such that the vector from O' to O is the fixed vector $\vec{b}$ (Fig. 3–2). The position vector of point P in the new coordinate system is $\vec{r}_P + \vec{b}$ and that of point Q in the new system is $\vec{r}_Q + \vec{b}$. If we calculate the displacement, which is the difference $(\vec{r}_Q + \vec{b}) - (\vec{r}_P + \vec{b})$, the vector $\vec{b}$ cancels. In other words, we have again arrived at the displacement vector defined in Eq. (3–1). This can also be seen in the graphical representation in Fig. 3–2. The displacement is independent of our choice of origin.

As a particle moves, the components of its position vector (with respect to the Cartesian coordinate axes) change with time:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}. \tag{3–2}$$

For three-dimensional motion, we would proceed exactly as in Section 1–6: We set up three axes, define three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, and write a position vector in the form

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}. \tag{3–3}$$

The fact that there is more than one vector component to the motion is the only difference between one-dimensional motion and two- or three-dimensional motion.

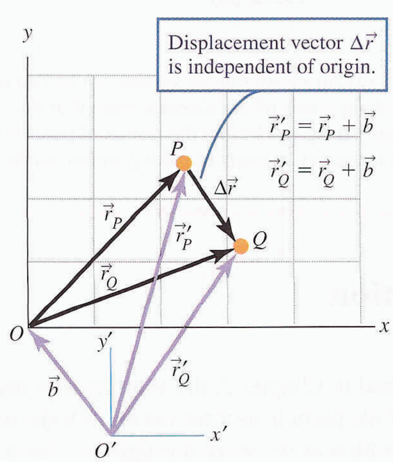


FIGURE 3–2 A displacement vector $\Delta\vec{r}$ is independent of the origin. Here O and O' are two origins, and although the initial and final position vectors to points P and Q do depend on the origins, the difference between these position vectors does not.

EXAMPLE 3–1 The position of a bumper car in an amusement park ride (Fig. 3–3a) is described as a function of time by the coordinates $x = c_1t^2 + c_2t + c_3$ and $y = d_1t^2 + d_2t + d_3$, where $c_1 = 0.20\ \text{m/s}^2$, $c_2 = 5.0\ \text{m/s}$, $c_3 = 0.50\ \text{m}$, $d_1 = -1.0\ \text{m/s}^2$, $d_2 = 10.0\ \text{m/s}$, and $d_3 = 2.0\ \text{m}$. Find the position vectors of the car at $t = 3.0$ s and $t = 6.0$ s and the displacement vector between these times. Plot the trajectory, that is, a curve of y versus x that traces the path of the car on the floor.

Setting It Up We plot the locations of the car along the x-axis and the y-axis as a function of time in Figs. 3–3b and 3–3c, respectively. In the first instance we are given the position as a function of time and are asked to find the position at a particular time; in other words, we are actually given what we must find! We'll label the desired displacement vector $\Delta\vec{r}$.

Strategy The first part is a straightforward numerical substitution. For the displacement, we calculate the difference between the position vectors (components x and y) at 6.0 and 3.0 s. This difference is the displacement vector. As for plotting the trajectories, the simplest way to proceed is to start with an xy-plane. We can then mark the x- and y-values at a given time as a point on this graph. A half-second later, say, there is another point that can be marked, and so forth. By connecting those consecutive points starting from the one at the earliest time, we mark out the trajectory.

Working It Out We insert the two values of time (3.0 and 6.0 s) into the equations for x and y:

for $t = 3.0$ s:

$$x(t) = (0.20\ \text{m/s}^2)(3.0\ \text{s})^2 + (5.0\ \text{m/s})(3.0\ \text{s}) + 0.50\ \text{m} = 17\ \text{m},$$

$$y(t) = (-1.0\ \text{m/s}^2)(3.0\ \text{s})^2 + (10.0\ \text{m/s})(3.0\ \text{s}) + 2.0\ \text{m} = 23\ \text{m};$$

for $t = 6.0$ s:

$$x(t) = (0.20\ \text{m/s}^2)(6.0\ \text{s})^2 + (5.0\ \text{m/s})(6.0\ \text{s}) + 0.50\ \text{m} = 38\ \text{m},$$

$$y(t) = (-1.0\ \text{m/s}^2)(6.0\ \text{s})^2 + (10.0\ \text{m/s})(6.0\ \text{s}) + 2.0\ \text{m} = 26\ \text{m}.$$

With these components, Eq. (3–2) gives us the position vectors of the car at the two times:

$$\text{for } t = 3.0\ \text{s: } \vec{r}(t) = (17\hat{i} + 23\hat{j})\ \text{m};$$

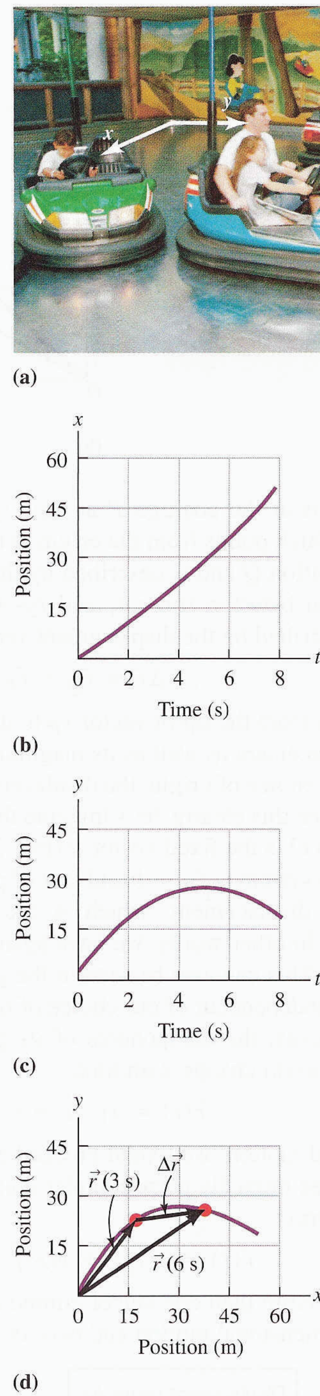
$$\text{for } t = 6.0\ \text{s: } \vec{r}(t) = (38\hat{i} + 26\hat{j})\ \text{m}.$$

Thus the displacement vector of the car between 3.0 and 6.0 s is [Eq. (3–1)]

$$\Delta\vec{r} = \vec{r}(t = 6.0\ \text{s}) - \vec{r}(t = 3.0\ \text{s})$$
$$= (38\hat{i} + 26\hat{j})\ \text{m} - (17\hat{i} + 23\hat{j})\ \text{m} = (21\hat{i} + 3\hat{j})\ \text{m}.$$

Finally, we plot y versus x, moment by moment, in Fig. 3–3d. This curve is the trajectory of the car.

What Do You Think? Why does the trajectory curve look so similar to the curve of y versus time? There are other vector descriptions of motion in a plane. Can you think of another such set? *Answers to **What Do You Think?** questions are given in the back of the book.*



▲ **FIGURE 3–3** (a) An xy-coordinate system can be laid out on the floor. (b) Location along x-axis of car's motion from 0 to 8 s. (c) Car's location along y-axis. (d) By combining the results of part (b) and part (c), we can plot the trajectory, a graph of the y-position versus the x-position.

3–2 Velocity and Acceleration

Velocity

As for the one-dimensional motion described in Chapter 2, the velocity of a particle describes the rate of change of the position of the particle as it moves on its trajectory. Generally we will consider two-dimensional motion as we work through the chapter, as it is simpler than considering three-dimensional motion, but the approach applies perfectly well to three dimensions. Using Eq. (3–1) for the particle's displacement, the average velocity $\vec{v}_{\text{av}}$ over the finite time interval from t to $t + \Delta t$ is accordingly defined by

$$\vec{v}_{\text{av}} \equiv \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t}. \tag{3–4}$$

Equation (3–4) shows that the direction of $\vec{v}_{\text{av}}$ is the same as the direction of the displacement vector $\Delta\vec{r}$.

As the time Δt tends towards zero, the displacement over that interval becomes smaller and smaller, and as we'll describe in more detail later, the displacement vector $\Delta\vec{r}$ becomes tangent to the particle's trajectory at the location of the moving particle. Then, as in Eq. (2–11), the *instantaneous velocity* $\vec{v}(t)$ is obtained by letting Δt become infinitesimally small:

$$\vec{v}(t) \equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt}. \tag{3–5}$$

We have recognized that in the limit $\Delta t \to 0$ we arrive at the time derivative of the position vector. The instantaneous velocity can change from moment to moment. The direction of $\vec{v}$ at time t is tangent to the trajectory curve at that time (Fig. 3–4). Of course, we already know that its magnitude is by definition the particle's speed.

We can write the velocity vector in terms of components by using Eqs. (3–2) and (3–5):

$$\vec{v} = \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}[x(t)\hat{i} + y(t)\hat{j}] \tag{3–6}$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}. \tag{3–7}$$

(The unit vectors $\hat{i}$ and $\hat{j}$ are constant in magnitude and direction, so their derivatives are zero.) We write Eq. (3–7) in the form

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \tag{3–8}$$

$$= \vec{v}_x + \vec{v}_y, \tag{3–9}$$

where

$$v_x = \frac{dx}{dt}, \tag{3–10a}$$

$$v_y = \frac{dy}{dt}, \tag{3–10b}$$

and the component vectors are

$$\vec{v}_x = \frac{dx}{dt}\hat{i}, \tag{3–11a}$$

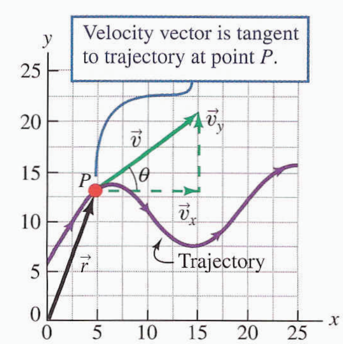
$$\vec{v}_y = \frac{dy}{dt}\hat{j}. \tag{3–11b}$$

The component vectors $\vec{v}_x$ and $\vec{v}_y$ of the velocity vector $\vec{v}$ are drawn in Fig. 3–4. The magnitude of the velocity $\vec{v}$ can be written in terms of the components of $\vec{v}$:

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}. \tag{3–12}$$

The angle θ that the velocity vector $\vec{v}$ makes with the x-axis is determined in terms of the components of the velocity by

$$\tan\theta = \frac{v_y}{v_x}. \tag{3–13}$$



▲ **FIGURE 3–4** The velocity vector $\vec{v}$ at point P is tangent to the particle's trajectory at that point. The component vectors $\vec{v}_x$ and $\vec{v}_y$ of the velocity vector $\vec{v}$ at that point are also included.

EXAMPLE 3–2 Use the data presented in Example 3–1 to find the bumper car's average velocity over the period from 3.0 to 6.0 s and the car's instantaneous velocity at $t = 3.0$ s.

Setting It Up The data of Example 3–1 give the position of the car as a function of time.

Strategy The average velocity is given by Eq. (3–4), and this requires us to know the displacement for a given time interval. That information is available from Example 3–1. For the instantaneous velocity we use Eq. (3–7) and evaluate the derivatives of $x(t)$ and $y(t)$.

Working It Out Given the result of Example 3–1, that the displacement vector of the bumper car between $t = 3.0$ s and $t = 6.0$ s is $\Delta\vec{r} = (21\hat{i} + 3.0\hat{j})$ m, we have

$$\vec{v}_{\mathrm{av}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{(21\hat{i} + 3.0\hat{j})\ \mathrm{m}}{6.0\ \mathrm{s} - 3.0\ \mathrm{s}} = (7.0\hat{i} + 1.0\hat{j})\ \mathrm{m/s}.$$

As for the instantaneous velocity, we require

$$\frac{dx}{dt} = \frac{d}{dt}(c_1t^2 + c_2t + c_3) = 2c_1t + c_2,$$

$$\frac{dy}{dt} = \frac{d}{dt}(d_1t^2 + d_2t + d_3) = 2d_1t + d_2.$$

Substituting the numerical values of c_1, c_2, d_1, and d_2 at $t = 3.0$ s from Example 3–1, we have

$$\frac{dx}{dt} = 2(0.20\ \mathrm{m/s^{\cancel{2}}})(3.0\ \cancel{\mathrm{s}}) + (5.0\ \mathrm{m/s}) = 6.2\ \mathrm{m/s},$$

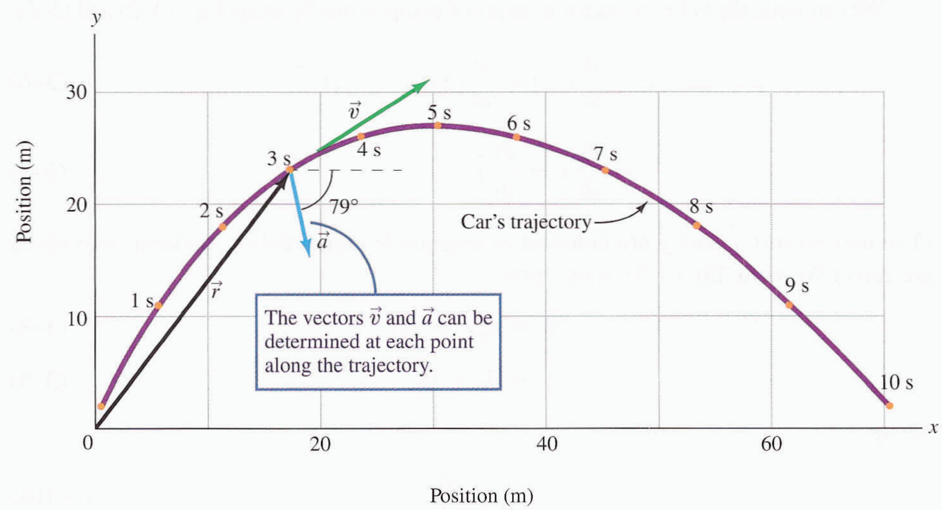
$$\frac{dy}{dt} = 2(-1.0\ \mathrm{m/s^{\cancel{2}}})(3.0\ \cancel{\mathrm{s}}) + (10.0\ \mathrm{m/s}) = 4.0\ \mathrm{m/s}.$$

Thus the velocity at $t = 3.0$ s is

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = (6.2\ \mathrm{m/s})\hat{i} + (4.0\ \mathrm{m/s})\hat{j}.$$

This velocity vector is shown in Fig. 3–5.

What Do You Think? Could you have used the graph of the trajectory in Fig. 3–3d to read off the velocity of the car?



▶ **FIGURE 3–5** The trajectory of the car's path in Example 3–1 is plotted for times up to 10 s; also shown are the position vector $\vec{r}$, the velocity $\vec{v}$, and the acceleration $\vec{a}$ at $t = 3.0$ s.

Acceleration

Acceleration describes how rapidly velocity changes with time. This "change" could be in the magnitude (the speed) or the speed could remain the same while the *direction* of the velocity vector changes or both magnitude and direction may change. As for motion in one dimension, acceleration is found from velocity in the same way that velocity is found from displacement. For a finite time interval Δt, the average acceleration is defined as

$$\vec{a}_{\mathrm{av}} \equiv \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}. \tag{3–14}$$

The instantaneous acceleration at time t is the limit of the average acceleration as Δt approaches zero, which is a derivative:

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt}. \tag{3–15}$$

The instantaneous acceleration is in principle a function of time, meaning that its three components are generally functions of time. As for velocity, we can express acceleration in terms of its components; for two dimensions (again for economy) we have

$$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} \qquad (3\text{–}16)$$

$$= a_x\hat{i} + a_y\hat{j}. \qquad (3\text{–}17)$$

Here, the components of the acceleration vector are

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \qquad (3\text{–}18a)$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}. \qquad (3\text{–}18b)$$

EXAMPLE 3–3 Calculate the instantaneous acceleration, magnitude and direction, of the bumper car in Example 3–1 at $t = 1.0$ s and $t = 3.0$ s.

Setting It Up We will want to use the known velocity vector of the car calculated in Example 3–2 using data for the position vectors from Example 3–1.

Strategy The acceleration vector is the time derivative of the known velocity vector, Eqs. (3–18a) and (3–18b). The acceleration is a function of time, into which we will then substitute particular values of time.

Working It Out From Eqs. (3–18),

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(2c_1t + c_2) = 2c_1$$

and

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(2d_1t + d_2) = 2d_1.$$

Thus [Eq. (3–17)]

$$\vec{a} = 2c_1\hat{i} + 2d_1\hat{j}.$$

In this case, the car's acceleration is a constant—it is independent of time—and so is exactly the same for $t = 1.0$ s and for $t = 3.0$ s. Given the values of c_1 and d_1 from Example 3–1 (0.20 and $-1.0\ \mathrm{m/s^2}$, respectively), the numerical value of the acceleration is

$$\vec{a} = (0.40\hat{i} - 2.0\hat{j})\ \mathrm{m/s^2}.$$

The magnitude of the acceleration is

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{(0.40)^2 + (-2.0)^2}\ \mathrm{m/s^2}$$
$$= \sqrt{4.2}\ \mathrm{m/s^2} = 2.0\ \mathrm{m/s^2}.$$

The acceleration vector makes an angle θ with the x-axis, which is shown in Fig. 3–5; the angle θ is derived from

$$\tan\theta = \frac{a_y}{a_x} = \frac{-2.0\ \mathrm{m/s^2}}{0.40\ \mathrm{m/s^2}} = -5.0;$$

so $\theta = -79°$; that is, the direction of the acceleration is at $-79°$ to the horizontal, almost directly toward the $-y$-direction.

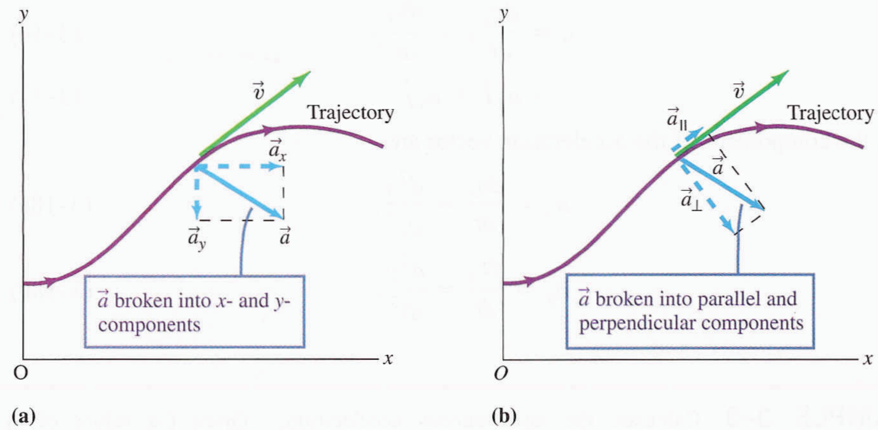
What Do You Think? We started this example by taking the time derivative of the known velocity vector. Could we instead have started with the position vector as a function of time (given in Example 3–1)?

Representing Trajectories

Look again at the trajectory of the bumper car discussed in Examples 3–1, 3–2, and 3–3 (Fig. 3–5); this trajectory is a curve representing the car's position on the floor of the amusement park ride (its x-position versus its y-position). In Fig. 3–5, we show a position vector $\vec{r}$, a velocity vector $\vec{v}$, and an acceleration vector $\vec{a}$ at $t = 3$ s. Although the figure shows $\vec{a}$ at the point corresponding to $t = 3.0$ s, we in fact saw that the acceleration of the car is independent of time and hence would be drawn as the same vector everywhere along the curve.

We can create a graphical representation like that in Fig. 3–5 for any motion. As an object moves, its trajectory will be traced out by the tip of the position vector $\vec{r}$ as $\vec{r}$ changes with time. The velocity vector $\vec{v}$ at any time t is a vector of magnitude $|\vec{v}|$ that is *tangential* to the trajectory at time t. This is quite intuitive. However, it is not quite so obvious how to think about the acceleration.

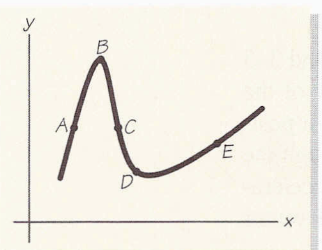
Since the acceleration is to the velocity as the velocity is to the displacement, one thing we could do is to repeat the procedure of the preceding paragraph with velocity and acceleration. A plot of the tip of the vector $\vec{v}$ can be drawn; it is the curve of the points whose *horizontal coordinate* at time t is v_x and whose *vertical coordinate* at that time is v_y. We might call this the "velocity trajectory." The acceleration at time t is given by a vector whose magnitude is $|\vec{a}|$ and whose direction is tangential to the velocity trajectory at time t. *The acceleration vector $\vec{a}$ is tangent to the velocity trajectory but not to the trajectory itself.*



▲ **FIGURE 3–6** Velocity $\vec{v}$ and acceleration $\vec{a}$ of a particle following some trajectory. (a) The acceleration of the particle is separated into x- and y-components. (b) The acceleration of the particle is separated into components parallel and perpendicular to the path.

We can illustrate the consequences of these facts using Fig. 3–6, which shows the path of an object with $\vec{v}$ and $\vec{a}$ indicated at one time along the path. In Fig. 3–6a, the acceleration is separated into its a_x and a_y components. Alternatively, we can separate the acceleration $\vec{a}$ into components that are parallel (tangential) and perpendicular (normal) to the velocity vector (Fig. 3–6b). We label these components $a_{\parallel}$ and $a_{\perp}$, respectively. The component $a_{\parallel}$ of $\vec{a}$ that is parallel to $\vec{v}$ affects the magnitude but not the direction of $\vec{v}$. Similarly, the $a_{\perp}$ component changes the direction but not the magnitude of $\vec{v}$. It is useful to refer separately to the parallel and perpendicular components of an object's acceleration because they affect the velocity differently.

CONCEPTUAL EXAMPLE 3–4 The motion of bumper cars is extremely erratic: You are colliding with other cars or you are trying to use evasive techniques. Figure 3–7 shows the path of a bumper car. Consider the points *A, B, C, D*, and *E*. (a) At which point did a collision most likely take place? (b) At which point did evasive action most likely take place? (c) Can you determine where the magnitude of the velocity is the greatest? (d) Can you determine where the magnitude of the acceleration is the greatest?



▲ **FIGURE 3–7** The motion of a bumper car can be quite erratic as it slams into other cars and takes evasive action to avoid collisions. Here we have the trajectory, or path, of such a car.

Answer (a) A collision most likely took place at point *B* because there is an abrupt change in direction. The driver could not change the direction so quickly without some outside effect.

(b) Evasive action probably took place at point *D* because there is a rapid but smooth change in direction as the driver turned quickly. The motion change is typically more abrupt at collisions.

(c, d) We can't tell where the magnitudes of the velocity and acceleration are the greatest from the trajectory alone because we do not know the times associated with points along the trajectory. We might guess that the acceleration was a maximum during the collision at point *B* because the velocity would change dramatically during the collision. On the other hand, if the bumper car were traveling very slowly at the time of a collision at point *B* then the collision might not be a very violent one and the acceleration would not necessarily be very large. You simply do not have enough information on a trajectory to tell. A plot such as Fig. 3–7 does not contain *all* the information about the motion.

3-3 Motion with Constant Acceleration

When an object moves with *constant acceleration*—meaning constant in both magnitude and direction—it can move only in a straight line (one dimension) or a plane (two dimensions). The plane of motion is formed by the initial velocity vector and the acceleration vector $\vec{a}$. The motion remains in this plane because, as Fig. 3–8 illustrates, the initial velocity vector has no component $v_{0\perp}$ perpendicular to the specified plane, and since the acceleration is in the plane, $v_{\perp}$ can never change and become nonzero. Motion near

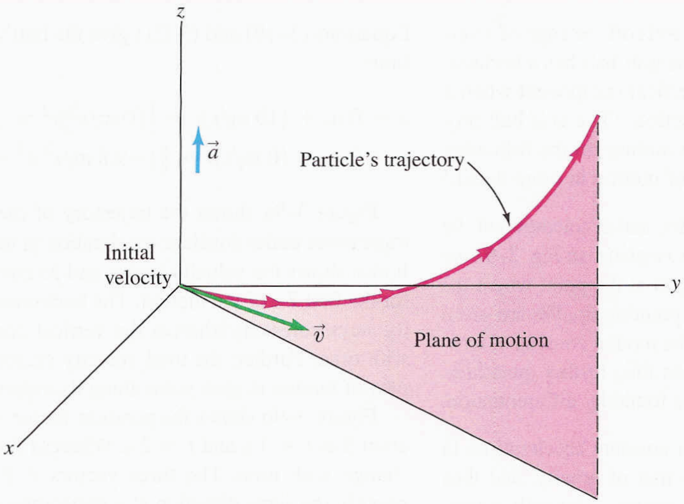


FIGURE 3–8 Projectile motion lies in a plane, the plane formed by the initial velocity vector and the acceleration vector.

Earth's surface, that is, motion under the sole influence of gravity, with air resistance neglected, provides an everyday example. If we throw a rock, it moves in a plane defined by the initial direction of the motion and the constant (vertical) acceleration of gravity.

At this point we'll simplify our notation by defining the plane of the motion as the xy-plane. We'll suppose for the moment that the initial velocity can have both x- and y-components, as can the (constant) acceleration. We can then use the results for one dimension from Chapter 2 to write *independently* the x- and y-components of position $\vec{r}$ and velocity $\vec{v}$ in terms of the constant-acceleration components. In other words, we can think of the x- and y-motions as separate from each other, governed only by their own separate constant accelerations. We use Eqs. (2–17) and (2–21) to find

$$x\text{-component of } \vec{r}: \quad x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2, \tag{3–19}$$

$$x\text{-component of } \vec{v}: \quad v_x = v_{0x} + a_x t; \tag{3–20}$$

$$y\text{-component of } \vec{r}: \quad y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2, \tag{3–21}$$

$$y\text{-component of } \vec{v}: \quad v_y = v_{0x} + a_x t. \tag{3–22}$$

Here, x_0 and y_0 are the components of $\vec{r} = \vec{r}_0$ at an initial time $t = 0$ and v_{0x} and v_{0y} are the components of $\vec{v} = \vec{v}_0$ at time $t = 0$. Together these quantities are the given **initial conditions**. In vectorial form, the initial conditions are

$$\vec{r}_0 = x_0\hat{i} + y_0\hat{j} \tag{3–23}$$

and

$$\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j} \tag{3–24}$$

at $t = 0$.

Equations (3–19) through (3–22), which give position and velocity for motion with constant acceleration $\vec{a}$, can be written more compactly in vector form:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2 \tag{3–25}$$

MOTION WITH CONSTANT ACCELERATION

$$\vec{v} = \vec{v}_0 + \vec{a}t \tag{3–26}$$

This form of the kinematic equations has the additional benefit that it does not refer to any particular set of axes. Remember that these important and useful results are valid *only* when $\vec{a}$ is constant. We can easily see the important features of these compact equations. In particular, you can see that for any direction for which the acceleration component is zero the position (or, equivalently, the displacement) component changes *linearly* with time, corresponding to a *constant*-velocity component. For any direction for which the acceleration component is not zero, the position component changes *quadratically* in time, corresponding to a *linearly* changing velocity component.

EXAMPLE 3–5 A wayward golf ball rolls off the edge of a vertical cliff overlooking the Pacific Ocean. The golf ball has a horizontal velocity component of 10 m/s and no vertical component when it leaves the cliff. Describe the subsequent motion. (The golf ball provides us with our first glimpse of projectile motion. In the following section we will look at this important type of motion in more detail.)

Setting It Up The displacement, velocity, and acceleration all lie in the same plane, which we assign to be the xy-plane. In Fig. 3–9a we include a coordinate system, placing the origin at the point where the ball leaves the cliff and with the y-direction pointing up. We are given initial values of velocity. A "description" of the motion consists of writing the position as a function of time. Given this, further quantities, such as velocity as a function of time, can be found by differentiation.

Strategy This is a case of motion under constant acceleration. In this case the ball's constant acceleration is that of gravity, and thus $\vec{a} = \vec{g}$. The vector $\vec{g}$ points toward Earth's center—vertically downward—and has magnitude 9.8 m/s^2. Because we have constant acceleration, we can use Eqs. (3–19) through (3–22) to describe the motion, for which we know the initial values (at $t = 0$, the moment when the ball rolls off the cliff). As emphasized above, we can say that because there is no component of acceleration in the x-direction, the horizontal velocity component is constant and will remain at its initial value.

Working It Out We start with initial values: The golf ball's initial position and velocity in our chosen coordinate system are $x_0 = 0\ \text{m}$, $v_{0x} = 10\ \text{m/s}$, $y_0 = 0\ \text{m}$, and $v_{0y} = 0\ \text{m/s}$. Next we specify that in our coordinate system the acceleration has components $a_x = 0\ \text{m/s}^2$ and $a_y = -9.8\ \text{m/s}^2$. We determine the velocity components as a function of time from Eqs. (3–20) and (3–22):

$$v_x = 10\ \text{m/s}$$

and

$$v_y = 0\ \text{m/s} + (-9.8\ \text{m/s}^2)t = (-9.8\ \text{m/s}^2)t. \tag{3–27}$$

Equations (3–19) and (3–21) give the ball's position as a function of time:

$$\begin{aligned} x &= 0\ \text{m} + (10\ \text{m/s})t + \tfrac{1}{2}(0\ \text{m/s}^2)t^2 = (10\ \text{m/s})t, \\ y &= 0\ \text{m} + (0\ \text{m/s})t + \tfrac{1}{2}(-9.8\ \text{m/s}^2)t^2 = (-4.9\ \text{m/s}^2)t^2. \end{aligned} \tag{3–28}$$

Figure 3–9a shows the trajectory of the golf ball. (We'll discuss trajectories under constant acceleration in more detail in Section 3–4.) It also shows the velocity vector and its components at 0.5-s intervals for the first 2 s of the motion. The horizontal component of the velocity stays constant, whereas the vertical component changes linearly with time. Further, the total velocity vector is a tangent to the ball's path of motion at each point along its trajectory.

Figure 3–9b shows the position vector $\vec{r}$, velocity $\vec{v}$, and acceleration $\vec{a}$ at $t = 1$ s and $t = 2$ s. Whereas $\vec{a}$ remains constant, $\vec{r}$ and $\vec{v}$ change with time. The three vectors $\vec{r}$, $\vec{v}$, and $\vec{a}$ do not generally point in the same direction at a given time during the golf ball's motion. The directions of $\vec{r}$ and $\vec{v}$ are specified by angles θ and θ', respectively, with respect to the x-axis. These angles are

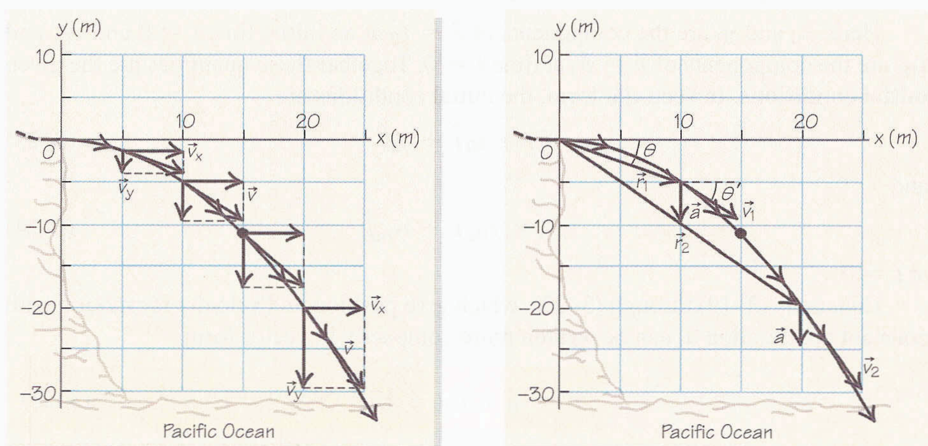
$$\tan\theta = \frac{y}{x} = \frac{(-4.9\ \cancel{\text{m}}/\text{s}^{\cancel{2}})t^2}{(10\ \cancel{\text{m}}/\cancel{\text{s}})t} = (-0.49\ \text{s}^{-1})t$$

and

$$\tan\theta' = \frac{v_y}{v_x} = \frac{(-9.8\ \cancel{\text{m}}/\text{s}^{\cancel{2}})t}{(10\ \cancel{\text{m}}/\cancel{\text{s}})} = (-0.98\ \text{s}^{-1})t,$$

respectively. Both angles vary with time.

What Do You Think? According to Fig. 3–9, the ball appears to drop into the ocean about 25 m from the cliff. In this problem what determines how far from the base of the cliff the ball enters the water?



(a) (b)

FIGURE 3–9 (a) The velocity vector $\vec{v}$ and components $\vec{v}_x$ and $\vec{v}_y$ of the golf ball are shown at 0.5-s intervals up to 2.0 s. (b) Position $\vec{r}$, velocity $\vec{v}$, and acceleration $\vec{a}$ of the golf ball for $t_1 = 1$ s and $t_2 = 2$ s.

3–4 Projectile Motion

A golf ball in motion is an example of a *projectile* that moves under the effect of gravity. In the absence of air resistance, what is the trajectory of a projectile? The motion is that of constant acceleration due to gravity, and this constant acceleration $\vec{g}$ has only a vertical component; we can use all the constant-acceleration results of the previous section to find the trajectory. The ball's motion is best described by separating it into horizontal and vertical components—as we have already emphasized, the horizontal motion is *independent* of the vertical motion—and then applying the kinematic equations for constant acceleration.

Usually it is easiest to place the origin at the starting point, assigning the y-direction vertically and the x-direction along the horizontal (Fig. 3–10), as we did in Example 3–5. The initial position of the ball is $x_0 = y_0 = 0$; the initial velocity at $t = 0$ is $\vec{v}_0$. The flight of the golf ball starts at an initial angle to the horizontal that we call the *elevation angle* θ_0. Then $\vec{v}_0$ has components

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0. \tag{3–29}$$

The components of the acceleration are the constants

$$a_x = 0 \quad \text{and} \quad a_y = -g. \tag{3–30}$$

Using Eqs. (3–19) through (3–22), the components of $\vec{r}$ and $\vec{v}$ (the position and velocity of the ball, respectively) are

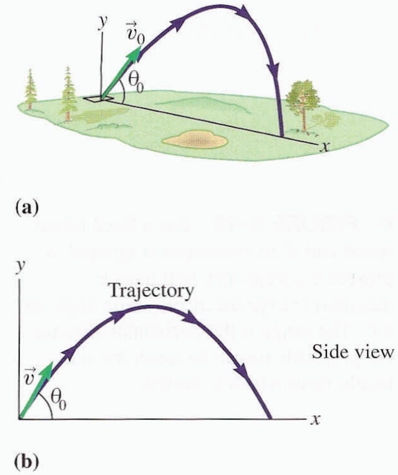
$$x = 0 + (v_0 \cos\theta_0)t + \tfrac{1}{2}(0)t^2 = (v_0 \cos\theta_0)t, \tag{3–31}$$

$$y = 0 + (v_0 \sin\theta_0)t + \tfrac{1}{2}(-g)t^2 = (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2, \tag{3–32}$$

and

$$v_x = v_0 \cos\theta_0 + (0)t = v_0 \cos\theta_0, \tag{3–33}$$

$$v_y = v_0 \sin\theta_0 - gt. \tag{3–34}$$



▲ **FIGURE 3–10** (a) A golf ball leaves a tee with an initial velocity of magnitude v_0 at an elevation angle θ_0. (b) The side view of the motion shows a parabolic trajectory.

The Trajectory

We can find the trajectory of the golf ball by plotting its height y versus its x-position. We know both x and y as functions of time, and we can eliminate the time dependence by using Eq. (3–31) to find the time t as a function of x. We then insert the result for t into Eq. (3–32) to find the trajectory, that is, the height y as a function of x, with the time dependence eliminated:

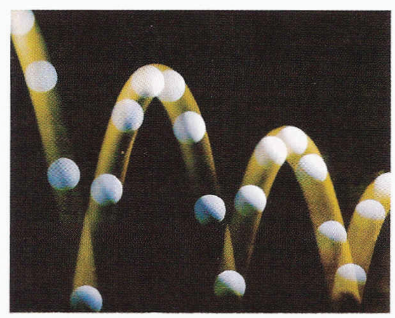
$$t = \frac{x}{v_0 \cos\theta_0}; \tag{3–35}$$

$$\begin{aligned} y &= (v_0 \sin\theta_0)\frac{x}{v_0 \cos\theta_0} - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta_0}\right)^2 \\ &= (\tan\theta_0)x - \left(\frac{g}{2v_0^2 \cos^2\theta_0}\right)x^2. \end{aligned} \tag{3–36}$$

The coefficients of x and x^2 in Eq. (3–36) are both constants, so the trajectory has the form

$$y = C_1 x - C_2 x^2. \tag{3–37}$$

This is the equation of a parabola passing through the origin with its axis parallel to the y-axis. *The trajectory of all objects moving with constant acceleration is parabolic.* Parabolic motion is illustrated in the chapter-opening photograph and Fig. 3–10 as well as in Fig. 3–11, which shows the position of a ball at equal time intervals.

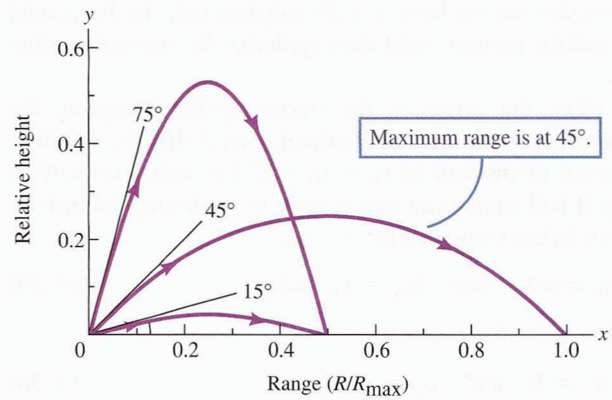


▲ **FIGURE 3–11** Motion of a ball bouncing along the floor and moving under the influence of gravity. In the air, the ball moves with constant acceleration, which in this case is directed downward due to gravity. The velocity vector changes throughout the motion, although its horizontal component does not. The velocity's vertical component changes linearly with time. The resulting trajectory forms a series of parabolas.

The trajectory and the time dependence of the components of displacement have some simple characteristics that can be useful in our study of projectile motion—range, flight time, and maximum height. These are easily extracted from the motion, and we discuss them further below.

Range: We define the **range** R of a projectile launched from the ground ($y = 0$) to be the horizontal distance that the projectile travels over level ground; that is, it lands at the same height from which it started. The quantity R is the value of x when the projectile has returned to the ground, that is, when y again equals zero. If we insert $y = 0$ into Eq. (3–37), we have

$$y = 0 = R(C_1 - C_2 R), \tag{3–38}$$



▶ **FIGURE 3–12** For a fixed initial speed and if air resistance is ignored, a projectile's trajectory will have a maximum range for an elevation angle of 45°. The range is the horizontal distance the projectile travels to reach the same height from which it started.

where we have set $x = R$. To find R, set the factor $C_1 - C_2R = 0$ in Eq. (3–38), or $R = C_1/C_2$. Inserting the values of C_1 and C_2 from Eq. (3–36) yields

$$R = \frac{C_1}{C_2} = \frac{\tan\theta_0(2v_0^2\cos^2\theta_0)}{g} = \frac{2v_0^2}{g}\left(\frac{\sin\theta_0}{\cancel{\cos\theta_0}}\right)\cos^{\cancel{2}}\theta_0 = \frac{v_0^2}{g}\,2\sin\theta_0\cos\theta_0.$$

From trigonometry, $\sin(2\theta_0) = 2\sin\theta_0\cos\theta_0$, and we find

$$R = \frac{v_0^2}{g}\sin 2\theta_0. \tag{3–39}$$

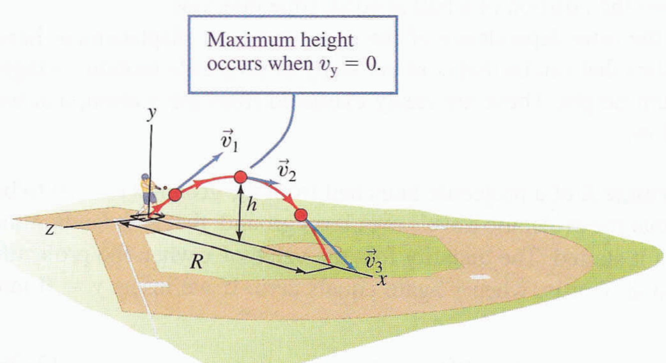
The range R depends on the initial speed v_0 and the elevation angle (the initial angle) of the projectile. As θ_0 increases progressively from 0° to 45° and then to 90°, the range $R\,[\propto \sin(2\theta_0)]$ starts out at zero, increases to a maximum at $\theta_0 = 45°$ [i.e., $\sin(2\theta_0) = 1$], then decreases back down to zero at $\theta_0 = 90°$. So, to throw or kick a ball over level ground as far as you can, send it upward at a 45° angle. For this case, which gives the maximum range, we have

$$R_{max} = \frac{v_0^2}{g}. \tag{3–40}$$

If the projectile is launched at an angle higher or lower than 45°, the range is shorter (Fig. 3–12). Note that according to Eq. (3–39) there are *two* initial angles for which a projectile has the same range for a given initial speed (Fig. 3–12). For example, in softball a pop fly at 75° and a line drive at 15° can both be caught by the shortstop (compare the two trajectories in Fig. 3–12).

Flight Time: Let T be the total flight time of a ball. Figure 3–13 shows that the ball reaches its maximum height exactly halfway through its trajectory, at time $t = T/2$. At this point, its motion is horizontal and the vertical component of velocity is zero. We can find $T/2$ by setting $v_y = 0$ in Eq. (3–34), $0 = v_0\sin\theta_0 - g(T/2)$. We solve for T to find that

$$T = \frac{2v_0}{g}\sin\theta_0. \tag{3–41}$$



▶ **FIGURE 3–13** A projectile (a ball) moving under the force of gravity is at its maximum height when $v_y = 0$. At that moment, the ball is traveling horizontally. We have marked the velocity at this point as $\vec{v}_2$.

This time the simple factor $\sin\theta_0$ enters. Look again at the motion at 75° and 15° in Fig. 3–12 and you will understand that in softball the fly ball's flight time is greater than that of the line drive.

Maximum Height: The maximum height $y_{max} = h$ is reached at time $T/2$. From Eq. (3–32), we find the height at this time,

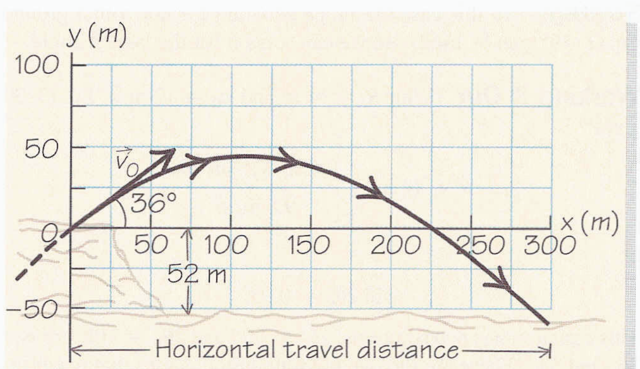
$$h = (v_0 \sin\theta_0)\frac{2v_0}{2g}\sin\theta_0 - \frac{1}{2}g\left(\frac{2v_0}{2g}\sin\theta_0\right)^2 = v_0^2\frac{\sin^2\theta_0}{g} - gv_0^2\frac{\sin^2\theta_0}{2g^2}$$

$$= v_0^2\frac{\sin^2\theta_0}{2g}. \qquad (3\text{–}42)$$

We use Eqs. (3–36), (3–39), (3–41), and (3–42) to determine a projectile's trajectory, range, flight time, and maximum height, respectively. The range and flight time refer to the special case where the ball returns to its original height. These equations need not be memorized; instead, it is important to understand how they were obtained. We apply these methods again in Examples 3–6 through 3–10.

EXAMPLE 3–6 To win a bet that he can drive a golf ball a horizontal distance of 250 m, an amateur golfer goes to a cliff overlooking the ocean. The cliff is 52 m above the ocean. The golfer strikes the golf ball so that the ball's initial speed is 48 m/s and the elevation angle (from the horizontal) is 36°. Does he win his bet? What is the horizontal distance actually covered by the ball?

Setting It Up Figure 3–14 shows the situation. We place the origin of our coordinate system at the tee where the ball's motion starts, letting y extend upward. We know the initial conditions ($t = 0$ when the ball is struck), which with our coordinate system are $x_0 = 0$ m, $y_0 = 0$ m, $v_0 = 48$ m/s, and $\theta_0 = 36°$. We want to determine the distance R' from the tee to the point at which the golf ball reaches the ocean ($y = -52$ m).



▲ **FIGURE 3–14** A golf ball is driven off a cliff into the ocean.

Strategy We use the trajectory equation to find the value of x at which the golf ball reaches the ocean surface. Note that we cannot use Eq. (3–39) to calculate the range because that result applies only to level ground; we don't want the horizontal distance when the ball returns to $y = 0$ m. However, we can still use Eq. (3–36) to find the value of x when $y = -52$ m.

Working It Out Equation (3–36) reads in our case

$$y = -52\text{ m} = (\tan\theta_0)R' - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right)R'^2.$$

Rearranging this equation yields

$$R'^2 - \frac{2v_0^2\cos^2\theta_0\tan\theta_0}{g}R' + \frac{2yv_0^2\cos^2\theta_0}{g} = 0,$$

$$R'^2 + bR' + c = 0,$$

where

$$b = -\frac{2v_0^2\cos^2\theta_0\tan\theta_0}{g} \quad\text{and}\quad c = \frac{2yv_0^2\cos^2\theta_0}{g}.$$

Solving this quadratic equation to find R' gives

$$R' = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Inserting the values of b and c, we obtain

$$R' = \frac{v_0^2\cos^2\theta_0\tan\theta_0}{g} \pm \frac{1}{2}\sqrt{\frac{4v_0^4\cos^4\theta_0\tan^2\theta_0}{g^2} - \frac{8yv_0^2\cos^2\theta_0}{g}}.$$

Now inserting $y = -52$ m and the initial values to determine R' yields

$$R' = \frac{(48\text{ m/s})^2\cos^2 36°\tan 36°}{9.8\text{ m/s}^2}$$

$$\pm\frac{1}{2}\sqrt{\frac{4(48\text{ m/s})^4\cos^4 36°\tan^2 36°}{(9.8\text{ m/s}^2)^2} - \frac{8(-52\text{ m})(48\text{ m/s})^2\cos^2 36°}{9.8\text{ m/s}^2}}$$

$$= 281\text{ m or } -57\text{ m}.$$

Now, did the golfer drive the ball a distance of 281 m or −57 m? The positive value must be correct. The golfer wins his bet.

What Do You Think? We stated the positive solution (281 m) must be the correct solution to the problem, but the negative solution (−57 m) also is a solution. What is the physical meaning of the negative solution?

EXAMPLE 3–7 What was the maximum height above the ocean of the golf ball in Example 3–6, and how long was the golf ball in flight?

Setting It Up We can again refer to Fig. 3–14. We denote the maximum height above the ground by h. In this case we want the maximum height *above the ocean*. We also want to find the total time T of the trip.

Strategy The maximum height of the golf ball occurs when the vertical component of the velocity is zero, and Eq. (3–42) will give h. The value we seek is h + 52 m. As for T, we can find it by using Eq. (3–31), together with the knowledge that the total horizontal distance traveled is 281 m.

Working It Out From Eq. (3–42),

$$h = \frac{v_0^2 \sin^2 \theta_0}{2g} = \frac{(48\ \text{m/s})^2 \sin^2 36°}{2(9.8\ \text{m/s}^2)} = 41\ \text{m}.$$

The answer is therefore 41 m + 52 m = 93 m.

From Eq. (3–31) with a horizontal distance traveled of 281 m,

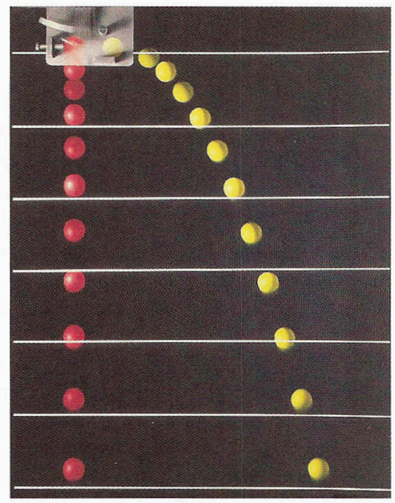
$$281\ \text{m} = (48\ \text{m/s})(\cos 36°)T;$$

$$T = \frac{281\ \cancel{\text{m}}}{(48\ \cancel{\text{m}}/\text{s}) \cos 36°} = 7.2\ \text{s}.$$

What Do You Think? On level ground the horizontal distance covered by the golf ball depends on $\sin 2\theta_0$ [see Eq. (3–39)], where θ_0 is the initial elevation angle, and the range is a maximum for $\theta_0 = 45°$. However, in this example we do not have level ground. Will the maximum horizontal travel distance still occur for $\theta_0 = 45°$?

CONCEPTUAL EXAMPLE 3–8 A major league pitcher and you, the student, compete in throwing a baseball as far as possible in an initially horizontal direction. Assume that you each throw so that each ball leaves the hand at exactly the same height. Whose ball will go further and why?

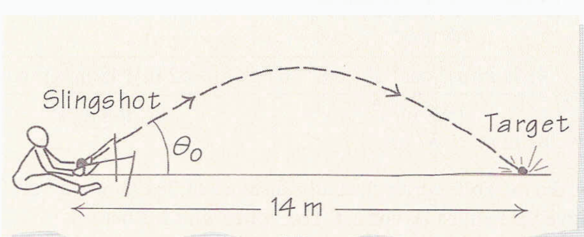
Answer The time it takes the ball to hit the ground is determined by the height from which the ball starts. Since the ball leaves the hand horizontally, the initial vertical velocity component is the same for both of you, namely zero, and hence the time it takes to hit the ground is the same for both of you. But the initial horizontal velocity component of the ball is expected to be larger for the professional pitcher, so that in the same time it covers a larger distance. The independence of the two components of the motion is again key here. Figure 3–15 illustrates the equal fall time for two projectiles that fit this description; in this photograph one of the projectiles has an initial speed of zero.



▶ **FIGURE 3–15** Two balls released simultaneously have two different trajectories, but in a given time each moves the same vertical distance. The difference in their motions is the magnitude of their (constant) x-components of velocity.

EXAMPLE 3–9 A group of engineering students constructs a slingshot device that lobs water balloons. The device is constructed so that the angle of the lob can be adjusted, and it has a launch speed (the balloon's initial speed) of 12 m/s. There is a target 14 m away at the same elevation. How should they adjust the initial angle so that they reach the target?

Setting It Up The slingshot setup is shown in Fig. 3–16. The students must find a value of launch angle θ_0 that will produce a given range R for a given initial speed v_0.



▲ **FIGURE 3–16** The students can orient their slingshot in two ways to get the same range for the same initial speed—just one is shown here.

Strategy In this case the range equation for horizontal ground, Eq. (3–39), can be used, and we can solve it for the launch angle.

Working It Out With R = 14 m and v_0 = 12 m/s, Eq. (3–39) gives

$$R = 14\ \text{m} = \frac{(12\ \text{m/s})^2 \sin 2\theta_0}{9.8\ \text{m/s}^2},$$

or

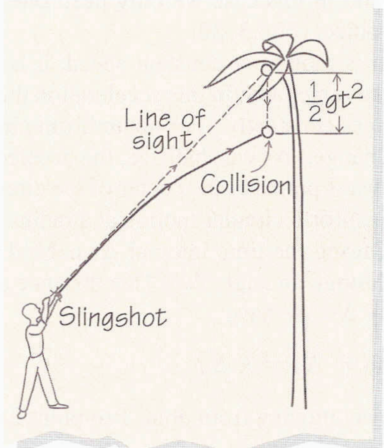
$$\sin 2\theta_0 = 0.95.$$

This equation has *two* solutions, $2\theta_0 = 72°$ and $2\theta_0 = 108°$, or θ_0 is 36° and 54°. These are the two possible initial angles that result in a given range, as in Fig. 3–12. We have drawn one of these trajectories (Fig. 3–16). *A reminder*: There will always be two initial angles that generate the same range, except for maximum range, which is produced only by the limiting angle 45°.

What Do You Think? From the standpoint of surprise, which of the two solutions, 36° or 54°, might be best for the students to use if the target were human?

EXAMPLE 3–10 A boy would rather shoot coconuts down from a tree than climb the tree or wait for the coconuts to drop. The boy aims his slingshot directly at a coconut, but at the same moment that his rock leaves the slingshot, the coconut falls from the tree. Show that the rock will hit the coconut.

Setting It Up We establish the launch point of the rock as the origin of a suitable xy-coordinate system in Fig. 3–17. The coconut is at the point (x_0, y_0). We are asked if the two objects moving under the influence of gravity will be at the same spot at the same time. For one of the trajectories, the initial velocity is given by an angle that would take it to the coconut if there were no gravity.



▲ **FIGURE 3–17** If the coconut falls at the same time the rock leaves the slingshot, both the coconut and rock fall the same distance.

Strategy We must compare a trajectory that includes both a horizontal and a vertical component (the rock) versus one that has only a vertical component (the coconut). For that reason it will be useful to think in terms of these components. It is useful to first consider what would happen if there were no gravity, then to see how the presence of gravity modifies the positions of both the rock and coconut.

Working It Out The rock has an initial velocity (v_{x0}, v_{y0}). If there were no gravity acting, the rock would follow a straight-line path that would place it at the point $(x_0 = v_{x0}t, y_0 = v_{y0}t)$ after a time t. This is the time necessary for the rock to reach the coconut (which is still at the tree since gravity has been ignored so far). Now let's include the effect of gravity. First, consider what happens to the coconut. During the time t that the rock travels toward the coconut, the coconut falls the distance $gt^2/2$ (Fig. 3–17). In other words, the height of the coconut after time t is [Eq. (3–21)]

$$y = y_0 - \tfrac{1}{2}gt^2.$$

Next, consider what happens to the rock when we include gravity. The rock's horizontal velocity component remains constant at v_{x0}. However, the vertical velocity component of the rock is changing under the effect of gravity and, after time t, Eq. (3–21) shows us that the rock's height is not $v_{y0}t$ but rather

$$y = v_{y0}t - \tfrac{1}{2}gt^2.$$

The rock is a height $gt^2/2$ below the height it would have if it followed a straight-line motion, which is precisely the distance the coconut falls (Fig. 3–17). Thus the rock will hit the coconut at the common point $\frac{1}{2}gt^2$ below the coconut's starting point. In effect, the parabolic path of the rock "tracks" the falling coconut.

What Do You Think? The real world is usually somewhat different than the idealized case discussed in textbooks. What are some reasons why the rock may not hit the coconut?

THINK ABOUT THIS . . .

IS IT POSSIBLE TO EXPERIENCE FREE FALL FOR LONG PERIODS?

Every jump puts you in free fall. Some of you may have done bungee jumping, where you can be in free fall for a couple of seconds until the cord starts to pull. What would it be like to be in free fall for longer periods? NASA has equipped a KC-135 airplane that allows training astronauts and others to experience longer periods of free fall. The plane is equipped to coast following a parabolic trajectory identical to that of a projectile. For the 25 s of the dive, the occupants are in free fall along with the airplane (see Fig. 3–18). Much of the film *Apollo 13*, which recounts the dramatic story of a mission to the Moon that barely made it back to Earth, was shot within the NASA plane. The best place to experience free fall is the International Space Station, which is orbiting around Earth in a free fall in which Earth's curvature allows the surface to "fall away" from the projectile's path as the projectile proceeds. The personnel inhabiting the station may be in free fall for months.

◀ **FIGURE 3–18** The interior of the airplane used by NASA for a free-fall environment during that part of the flight where the plane follows the same parabolic path taken by a projectile in free fall. In this photograph, astronauts in training are experiencing some of the same effects they will feel during a stay in the International Space Station.