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Electric fields in the presence of conducting objects

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When a conducting object is placed in a region where there is an electric field, charges are induced on its surface. We seek the unique surface charge density that produces an electric field that cancels the original field inside the conductor. When the external sources are point charges or uniform fields, it is easy to determine the field that the induced charges must produce inside the conducting object. Up to a constant, this field gives the potential on the conducting surface, which suffices to determine the potential function outside the conductor. The perturbing field produced by the induced charges is obtained from this potential, and a simple boundary condition gives us the induced surface charge density.
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I. INTRODUCTION

When a conducting object is placed in a region where there is an external electric field, induced charges in the conductor pile up on its surface until the electric field produced by the surface charges cancels the original field inside the conducting body and distort the external electric field.[1,2] The problem of electrostatics for conductors is to determine the electric field outside the conductor and the distribution of charges on its surface.[3] Thomson's theorem states that the introduction of an uncharged conductor into the field of given charges reduces the total energy of the field.[3,4] This theorem can be applied to determine the induced charge on the conducting surface, but its use is difficult even in simple cases.[5]

Many problems in electrostatics involve boundaries on which the potential is specified. Two special techniques used to solve these problems are the method of images and the expansion in orthogonal functions. The idea of the method of images is to find a set of fictitious charges that, together with the given charges, produce a field such that the surface of the conductor is an equipotential surface.[2–4,6] However, it is not always easy to determine the location and the magnitude of the imaginary charges, especially in problems with curved surfaces. The representation of the potentials in terms of orthogonal functions is a powerful technique that can be used to solve a large class of problems, but its study requires a higher level of mathematical proficiency, and it is postponed to intermediate and advanced courses in electrodynamics.[2,4,6]

In the initial study of electrostatics of conductors, much attention is paid to the role played by the induced charges in canceling the electric field inside the conductor.[7] However, this fact is seldom used in obtaining the electric fields surrounding the conductors or the induced charges. Instead, the methods used to solve these problems rely on the fact that conductors are equipotential surfaces. We propose to determine the unique surface charge density that produces a known electric field inside the conductor to cancel the field of the external sources. When these sources are point charges or uniform fields, it is easy to determine the fields that the induced charges must produce inside the conducting object. Up to a constant, this field gives the potential on the conducting surface, which suffices to determine the potential function outside the conductor. A single boundary condition gives the induced surface charge density. This idea is intuitive because it is connected with the electrostatics of conductors. It is also a straightforward and general method that if combined with a basic knowledge of expansions in orthogonal functions provides another approach to solving a large class of complex problems. Similar approaches have been proposed for electrostatics in the presence of dielectrics.[8,9]

Many problems of electric fields in the presence of uncharged conductors can be represented by the general case in Fig. 1. The conducting object is in a region where both an external electric field $\vec{E}_0$ and the induced charge density σ produce the field $\vec{E}$. Consider a closed surface of the same shape as the conductor with an unknown surface charge density σ. This charge produces a known electric field $-\vec{E}_0$ inside the surface, as shown in Fig. 2. Up to a constant, this field gives the potential function at all points on the boundary.[10] Outside this surface the potential must satisfy Laplace's equation. According to the uniqueness theorem, the knowledge of the potential on the boundary suffices to determine the potential function in the external region, from which the outside field $\vec{E}_1$ is obtained. Finally, if we add a field $\vec{E}_0$ to that due to the charge distribution in Fig. 2, we obtain the electric field distribution shown in Fig. 1, where $\vec{E}=\vec{E}_1+\vec{E}_0$. Figure 3 summarizes this result. The surface charge density σ is obtained from the boundary condition of the normal components of the electric field,

$$\hat{n} \cdot (\vec{E}_1 + \vec{E}_0) = \sigma/\epsilon_0, \tag{1}$$

where $\hat{n}$ is a unit vector outwardly directed and normal to the surface of the conductor. Note that $\vec{E}_0$ is the electric field produced by sources not shown in the figure with the conductor absent and $\vec{E}_1$ is the perturbing field produced by the surface charge density σ. Thus, the key to solving the problem is to determine $\vec{E}_0$.

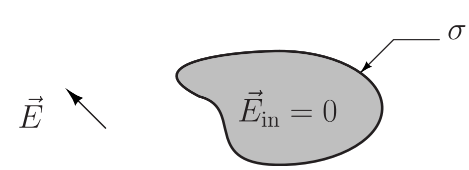

Fig. 1. A conducting object is in a region where external sources not shown in the figure and the induced surface charge density σ produce an electric field $\vec{E}$.

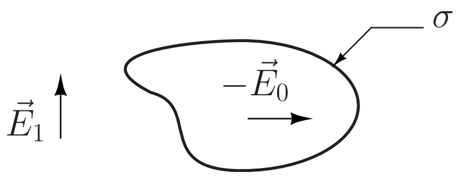

Fig. 2. The induced surface charge density σ produces an electric field $-\vec{E}_0$ inside the conducting surface to cancel the electric field $\vec{E}_0$ of the external sources, with the conductor absent. It also produces a perturbing electric field $\vec{E}_1$ outside the surface.

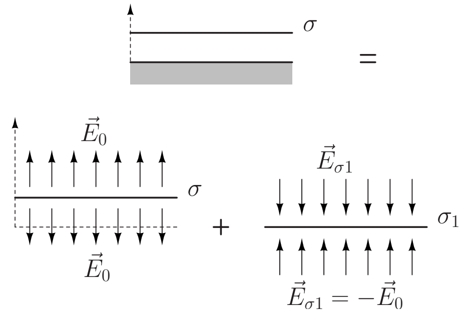

Fig. 4. A uniform charged sheet above an infinite conducting plane and the equivalent system of charges. The electric field inside the conducting plane is canceled if $\sigma_1 = -\sigma$.

II. EXAMPLES

Figure 4 illustrates a flat sheet infinite in extent above an infinite conducting plane and the equivalent system. The sheet has a uniform surface charge density σ. Negative charges are induced on the surface of the conducting plane to cancel the electric field inside the conductor. These charges are represented in the equivalent system by a sheet with uniform charge density σ_1 placed at the conductor's surface. Because $E_0 = \sigma/(2\epsilon_0)$, the electric field inside the conductor is canceled if $\sigma_1 = -\sigma$. The electric field between the sheet and the conducting plane is σ/ϵ_0 and zero both above the sheet and inside the conducting plane.

Now consider a point charge $+Q$ located near an infinite conducting plane. Let us place a surface density σ on the conducting plane that cancels the electric field inside the conductor (see Fig. 5). The electric field $\vec{E}_\sigma$ produced by the surface charge density must cancel the electric field $\vec{E}_0$ produced by the point charge $+Q$ at all points below the surface. That is, below the surface of the conductor,

$$\vec{E}_0 + \vec{E}_\sigma = 0. \tag{2}$$

The surface charge density depends only on the distance r (see Fig. 5). Also, given the symmetry of the charge distribution, the electric field $\vec{E}_\sigma$ has mirror symmetry as well. The vertical component of $\vec{E}_\sigma$ is discontinuous because of the surface charge density σ. Hence,

$$\sigma = -2\epsilon_0 |E_{\sigma z}|. \tag{3}$$

From Eqs. (2) and (3) we obtain

$$\sigma = \frac{-Qh}{2\pi(r^2 + h^2)^{3/2}}, \tag{4}$$

which is a well known result.[11]

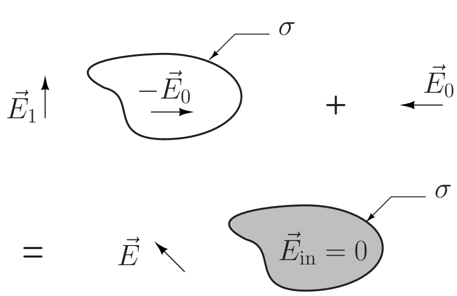

Fig. 3. Illustration of the proposed approach: The electric fields due to the surface charge density σ plus the external field $\vec{E}_0$ are equivalent to the electric field distribution in Fig. 1, where $\vec{E} = \vec{E}_1 + \vec{E}_0$.

We next study the charge distribution on a conducting sphere of radius a placed in a uniform electric field $E_0\hat{z}$. Figure 6 shows both the original system and the equivalent system made up of the uniform external field plus a spherical charge distribution $\sigma(\theta)$ that produces a uniform field $-E_0\hat{z}$ inside the spherical surface.

The charge density $\sigma(\theta)$ can be found if we know the electric fields at both sides of the boundary. Because the field inside the spherical surface is $-E_0\hat{z}$, the potential function in this region is $\phi_{\text{in}}(r,\theta) = E_0 z = E_0 r \cos\theta$. That is, the potential is known at the boundary $r=a$. Because the solutions of Laplace's equation are unique, this information suffices to determine the potential outside the spherical surface, which must have the form[12]

$$\phi_{\text{out}}(r,\theta) = \sum_{k=0}^{\infty} \frac{A_k}{r^{k+1}} P_k(\cos\theta), \tag{5}$$

where P_k are the Legendre polynomials. The boundary condition at $r=a$,

$$E_0 a \cos\theta = \sum_{k=0}^{\infty} \frac{A_k}{a^{k+1}} P_k(\cos\theta), \tag{6}$$

must hold for all values of the angle θ. Then, only one term is present in the sum, from which we obtain $A_1 = E_0 a^3$. That is, the surface charge distribution outside the sphere corresponds to a dipole field,

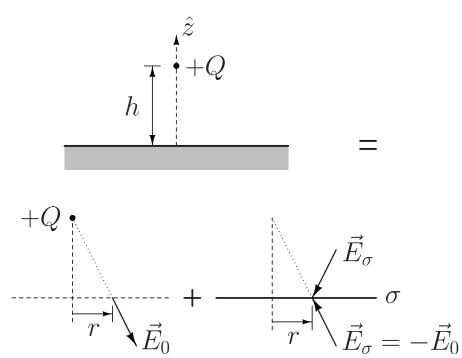

Fig. 5. A point charge $+Q$ located near an infinite plane conductor and the equivalent system of charges. The electric field produced by the surface charge density σ cancels the electric field due to the point charge $+Q$ inside the conductor.

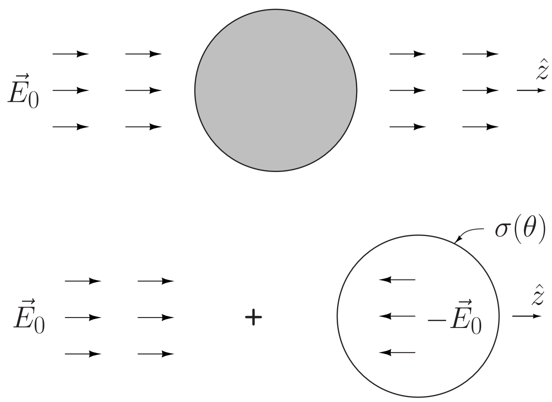

Fig. 6. Conducting sphere placed in a uniform electric field $\vec{E}_0$ and the equivalent system made up of the uniform external field plus a surface charge density $\sigma(\theta)$. This surface charge produces a field $-\vec{E}_0$ inside the conducting surface that cancels the external electric field.

$$\phi_{\text{out}}(r,\theta) = \frac{E_0 a^3}{r^2}\cos\,\theta = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}, \tag{7}$$

where $\vec{p}=4\pi\epsilon_0 E_0 a^3 \hat{z}$ is parallel to the external electric field. The electric fields inside and outside the charge distribution are obtained from the potentials. The surface charge density $\sigma(\theta)=3\epsilon_0 E_0 \cos\,\theta$ is obtained from the boundary condition in Eq. (1).[13] Finally, the total electric field outside the conductor is the sum of the uniform and the dipole fields, $\vec{E}=\vec{E}_0+\vec{E}_{\text{dipole}}$.

We next consider a charge $+Q$ placed inside a conducting spherical cavity. Figure 7 shows both the original and the equivalent problems. The induced surface charge density σ_a must produce an electric field inside the conductor of opposite sign as that produced by the charge $+Q$. If $\vec{r}_0$ represents the vector position of the point charge $+Q$, the potential function due to σ_a for $r \geq a$ must be[14]

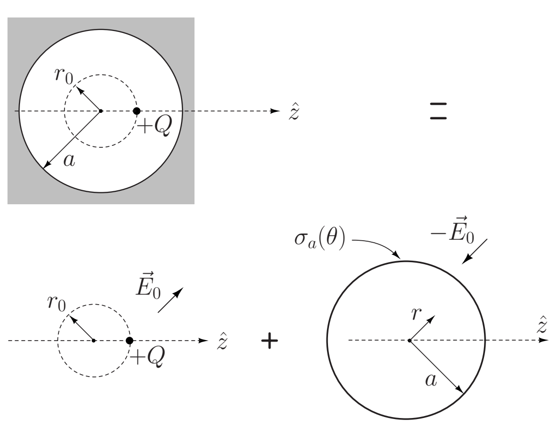

Fig. 7. A point charge $+Q$ placed at $\vec{r}_0$ inside a conducting spherical cavity of radius a and the equivalent system of charges. The surface charge density $\sigma_a(\theta)$ produces the same electric field outside the cavity as a point charge $-Q$ placed at $\vec{r}_0$.

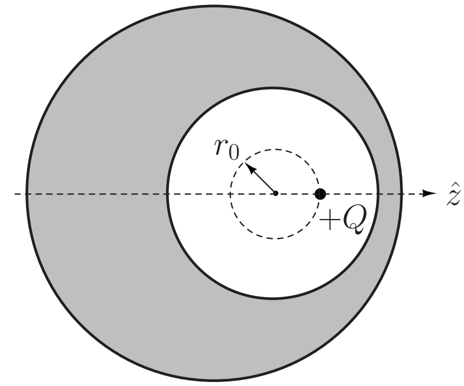

Fig. 8. Neutral spherical conductor of radius b and charge $+Q$ inside a cavity of radius a.

$$\phi_{\text{out}}(r,\theta) = -\frac{Q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|} = -\frac{Q}{4\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{r_0^k}{r^{k+1}}P_k(\cos\,\theta). \tag{8}$$

Because the potential is known at the boundary $r=a$, this information suffices to determine the potential inside the cavity, which must be of the form[12]

$$\phi_{\text{in}}(r,\theta) = \sum_{k=0}^{\infty} B_k r^k P_k(\cos\,\theta). \tag{9}$$

The boundary condition for the potentials at $r=a$ is

$$\sum_{k=0}^{\infty} B_k a^k P_k(\cos\,\theta) = -\frac{Q}{4\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{r_0^k}{a^{k+1}}P_k(\cos\,\theta). \tag{10}$$

This condition must be satisfied for all values of the angle θ, which makes it possible to obtain the coefficients

$$B_k = -\frac{Q}{4\pi\epsilon_0}\frac{r_0^k}{a^{2k+1}}. \tag{11}$$

The use of Eq. (1) at $r=a$ gives the surface charge density

$$\sigma_a(\theta) = -\frac{Q}{4\pi a^2}\sum_{k=0}^{\infty}(2k+1)\left(\frac{r_0}{a}\right)^k P_k(\cos\,\theta)$$
$$= -\frac{Q}{4\pi a^2} + \Delta\sigma(\theta). \tag{12}$$

The total charge induced on the spherical surface is $-Q$ because the term $\Delta\sigma(\theta)$ integrates to zero over the surface of the sphere.[15] The term $\Delta\sigma(\theta)$ also vanishes when $r_0 \to 0$, as expected.

The results of the previous example can be used to solve the problem illustrated in Fig. 8, where a sphere of radius b has been carved out of the conductor of Fig. 7. Assume that the conductor has no net charge, and let σ_a and σ_b be the induced surface charge densities on the cavity and the outer surface, respectively. Equations (8) and (9) still describe the potentials due to σ_a in the region $r > a$ and the cavity, respectively, with the same boundary condition. Then, Eq. (12) gives the induced charge density σ_a on the cavity surface. The surface charge density σ_b will be induced on the outer surface because the conductor is neutral. Because $+Q$ and σ_a produce zero field for $r > a$ and there is no field in the conductor, the induced charge density σ_b must produce zero field within its surface. Given that the outer surface is spherical, σ_b must be a constant. Then, $\sigma_b = +Q/(4\pi b^2)$, and the field outside the spherical conductor is the same as if the charge $+Q$ were concentrated at its center. An important lesson of this example is that the potential in the cavity is determined entirely by both $+Q$ and the induced charge σ_a[17] and that σ_b arises to fix the net charge in the conductor, with no effect on the fields within its surface.[18]

III. CONCLUSION

The use of an expansion in orthogonal functions in the proposed approach requires less work than the usual methods because only one simple boundary condition is needed to obtain the perturbing fields outside the conductor. If the electric field (or, equivalently, the potential function) produced by the sources with the conductor absent is known, this method is straightforward and can be applied to more advanced problems, as problem (3) in the Appendix shows. Problems of the conductor with a cavity, both in the examples and the suggested problems, show that the method also provides physical insight because it helps to understand how charges distribute on the conducting surface to cancel the field of the external sources within the conductor.

ACKNOWLEDGMENTS

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APPENDIX: SUGGESTED PROBLEMS

The following problems can be assigned to intermediate level students to help reinforce the ideas presented in the paper.

(1) Consider a nonspherical cavity in the spherical conductor in Fig. 8. Compare the charge distribution in both the cavity and the outer surface with the spherical cavity studied in the examples.

(2) Consider a nonspherical conductor with a spherical cavity. Somewhere within the cavity is a charge $+Q$. What can be said about the surface charge densities on both the cavity surface and the outer surface?

(3) Figure 9 illustrates a ring of total charge $+Q$ inside a conducting spherical cavity. The ring of charge is located in the x-y plane, and it is concentric with the cavity.[19] This problem is equivalent to a ring of charge of radius r_0 and total charge $+Q$ and a charge density $\sigma(\theta)$ on a spherical surface of radius a. This surface charge produces an electric field for $r > a$ that cancels the field due to the charged ring. To find $\sigma(\theta)$, we need to know the electric field (or the potential function) of the ring of charge in free space. The potential function of the charged ring for $r \geq a$ is[20]

$$\phi_{\text{ring}} = \frac{Q}{4\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{r_0^k}{r^{k+1}} P_k(0) P_k(\cos\theta). \qquad \text{(A1)}$$

Use this result to show that the induced charge density is given by

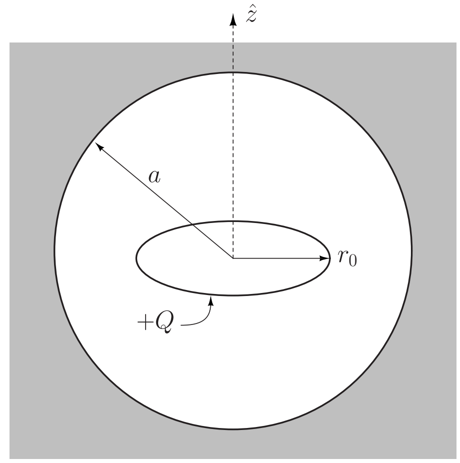

Fig. 9. Ring of total charge $+Q$ inside a conducting spherical cavity.

$$\sigma(\theta) = -\frac{Q}{4\pi a^2} \sum_{k=0}^{\infty} (2k+1)\left(\frac{r_0}{a}\right)^k P_k(0) P_k(\cos\theta). \qquad \text{(A2)}$$

Check that the net charge induced on the sphere is $-Q$ and that the induced charge density gives the expected answer in the limits $r_0 \to 0$ and $a \to \infty$.

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[10] We use this constant to set the zero of potential at infinity.
[11] See, for example, Ref. 1, Eq. (8), p. 100, and Ref. 2, Eq. (3.10).
[12] See, for example, Ref. 2, pp. 137–140, and Ref. 6, pp. 90–93.
[13] See, for example, Ref. 2, Example 3.8, and Ref. 3, Problem 1, p. 14.
[14] See, for example, Ref. 4, Eq. (5-12), and Ref. 6, Eq. (3.38).
[15] This result can be shown by using the orthogonality property of the Legendre polynomials. See, for example, Ref. 2, Eq. (3.68), and Ref. 6, Eq. (3.21).
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[17] See Ref. 16 for a general proof.
[18] Reference 2, Sec. 2.5.2, gives a good qualitative discussion of this problem.
[19] See Ref. 6, pp. 113–115, where the problem is solved using the spherical Green function expansion.
[20] The potential of a ring of charge is given in Ref. 6, p. 93, and Ref. 4, p. 87.