HW 4: Multipole Expansion

You are welcome to directly use the vector identities given on the inside flap of Zangwill's book. Draw neat sketches to assist your explanations.

1. Consider a ring of negligible thickness and radius R placed in the xy plane. Find its monopole charge, dipole moment and quadrupolar tensor. It will be useful to identify the volume charge density in cylindrical coordinates.

2. (a) In the multipolar expansion, the quadrupolar moment gives rise to the term

$$Q_{ij}\nabla_j\nabla_i\frac{1}{r}. \tag{1}$$

I would like you to show that

$$\nabla_j\nabla_i\frac{1}{r} = \frac{3r_ir_j - \delta_{ij}r^2}{r^5}. \tag{2}$$

(b) Let's propose the following charge density for a *point* quadrupole moment located at the position $\mathbf{r}_o$

$$\rho_Q = -Q_{ij}\nabla_i\nabla_j\ \delta(\mathbf{r}-\mathbf{r}_o). \tag{3}$$

Using the fundamental definition of the potential, show that the proposed point quadrupole density indeed recovers the potential in Eq. (1).

(c) What is the force on the quadrupole placed inside an electric field $\mathbf{E}(\mathbf{r})$?

(d) What's the electrostatic potential energy of the quadrupole inside the electric field?

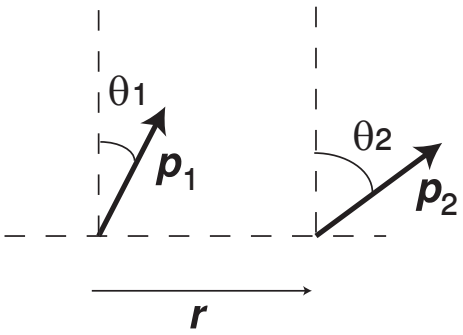
3. Zangwill Eq. (4.24) describes the torque on a dipole. Derive this relationship in a clean and clear manner. First derive the i'th component of the torque.

4. (a) Using the point dipole density $\rho_D(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_o)$, find the electric potential produced by a dipole. Also determine the electric field $\mathbf{Er}$ for $r \neq 0$.

(b) As a result, show that the electric potential energy between two non-overlapping point dipoles is

$$V_E = \frac{1}{4\pi\varepsilon_o r^3}\left(\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\mathbf{p}_1\cdot\hat{r})(\mathbf{p}_2\cdot\hat{r})\right). \tag{4}$$

(c) The diagram shows two point dipoles. What angles will minimize the energy of the system? For example, if $\theta_1 = \pi/4$ what is θ_2?



5. A sphere of radius R carries a spherically symmetric charge

$$\rho(\mathbf{r}) = \frac{R}{r^2}(R - 2r)\sin\theta. \tag{5}$$

What is the lowest multipole moment? Find the electric potential at a distance far away from the sphere along the z axis.