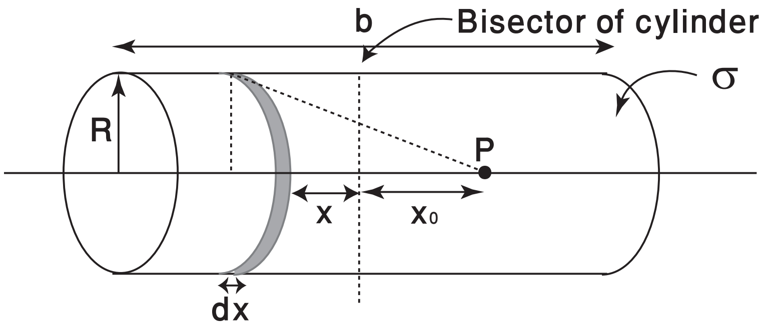
Mid Term Solution

1. **Answer:**



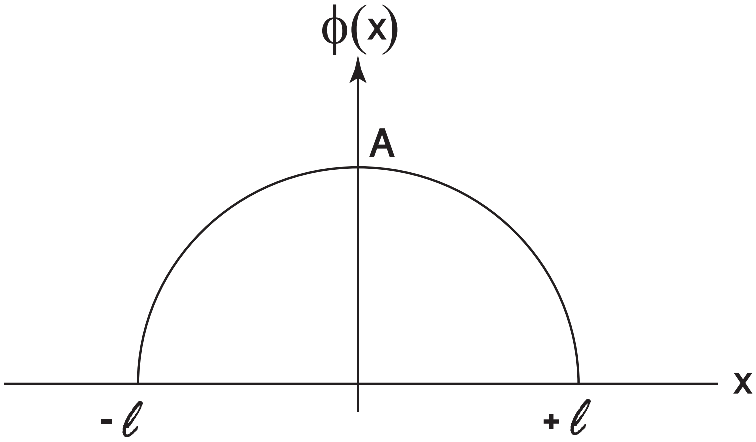
Consider a point P at a distance x_0 from the transverse bisector of the cylinder. Consider a thin strip at a distance x from the bisector. This strip results in a potential,

$$d\phi = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi R}{\sqrt{(x_0+x)^2+R^2}}dx$$

$$\phi(x_0) = \frac{\sigma 2\pi R}{4\pi\varepsilon_0}\int_{x=-b/2}^{b/2}\frac{dx}{\sqrt{(x_0+x)^2+R^2}}.$$

2. **Answer:**

(a)

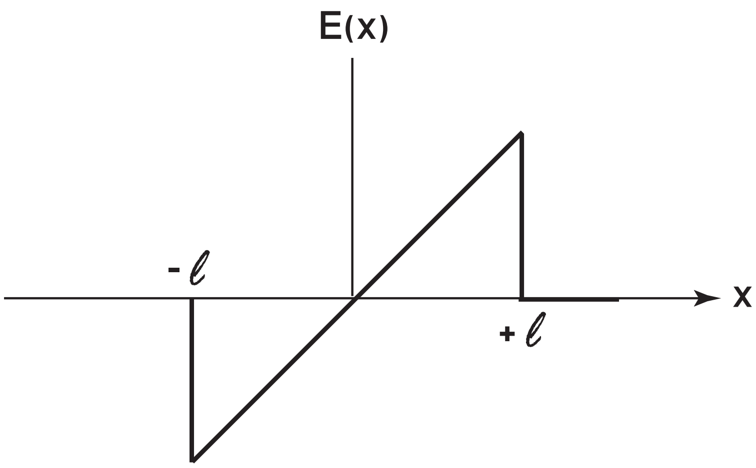


$$\phi(x) = A(\ell^2 - x^2)$$

$$\mathbf{E}(x) = -\nabla\phi(x) = 2Ax\hat{\mathbf{e}}_x, \quad |x| < \ell$$

$$\text{and} \quad \mathbf{E} = 0, \quad |x| > \ell.$$

The electric field is therefore discontinuous at $x = \pm\ell$.



(b) For $x > -\ell$ and $x < \ell$, i.e., $|x| < \ell$, $\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = 2A\varepsilon_0$. For $|x| > \ell$, $\rho = 0$.

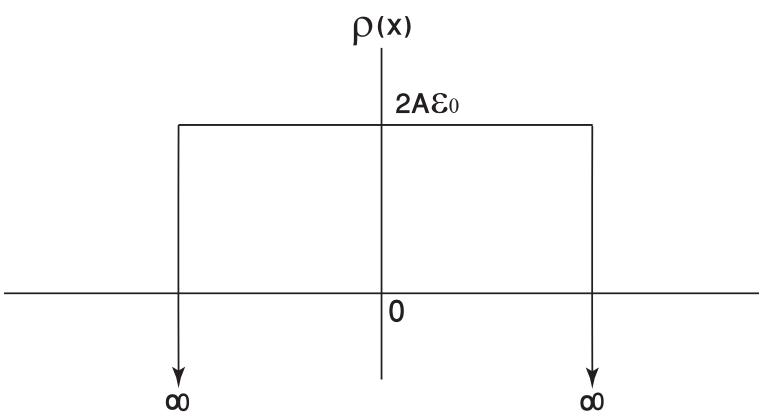
At the boundaries, the electric field is discontinuous. For the discontinuity at the left boundary ($x = -\ell$), we have the boundary condition

$$(-2A\ell)\hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_x = -2A\ell = \frac{\sigma}{\varepsilon_0}$$

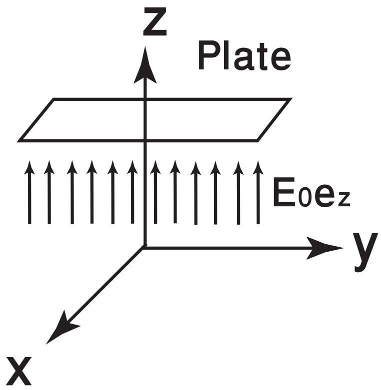
where σ is the surface charge density accumulating on the boundary. Hence at $x = -\ell$, we obtain $\sigma = -2\varepsilon_0 A\ell$. A similar result holds at $x = \ell$. Therefore the surface charge density can overall be written as

$$\rho = -2\varepsilon_0 A\ell\Big(\delta(x+\ell) + \delta(x-\ell)\Big) + 2A\varepsilon_0\Big(\Theta(x+\ell) - \Theta(x-\ell)\Big).$$

The figure shows a plot of the charge density,



3. **Answer:**



Suppose the plate is horizontal (parallel to xy) and the electric field is vertically upward. We have $\mathbf{E} = E_0\hat{\mathbf{e}}_z$. We can compute the elements of the stress tensor.

$$\begin{aligned}
T_{xy} &= T_{yx} = T_{xz} = T_{yz} = T_{zx} = T_{zy} = 0 \\
T_{xx} &= T_{yy} = -\frac{\varepsilon_0}{2}E_0^2 \\
T_{zz} &= \varepsilon_0\left(E_0^2 - \frac{1}{2}E_0^2\right) = \frac{\varepsilon_0}{2}E_0^2 \\
\overleftrightarrow{T} &= \frac{\varepsilon_0}{2}E_0^2\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
f &= \nabla \cdot \overleftrightarrow{T} \\
f_z &= (\overleftrightarrow{T} \cdot d\mathbf{a})_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z \\
&= \frac{\varepsilon_0 E^2}{2}da_z, \quad \text{(since } da_y = da_z = 0).
\end{aligned}$$

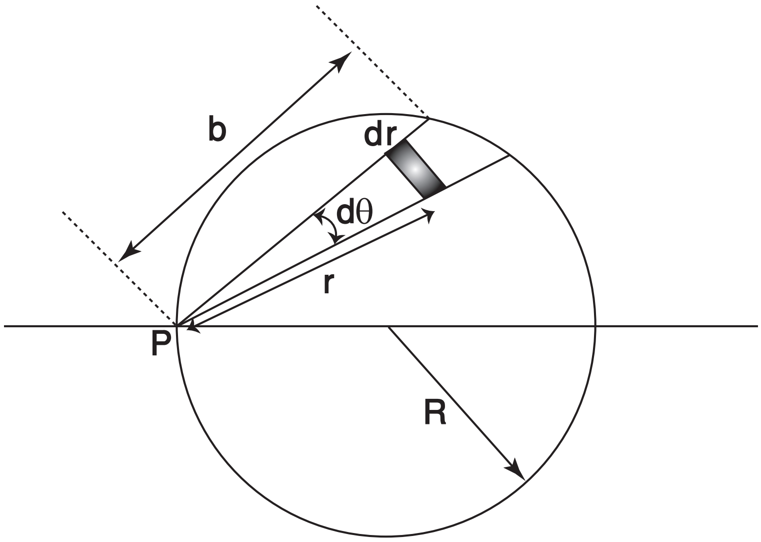
$$\begin{aligned}
\text{Therefore the force per unit area is } \frac{\varepsilon_0 E_0^2}{2} &= \frac{\varepsilon_0}{2} \cdot \frac{\sigma^2}{\varepsilon_0^2} = \frac{\sigma^2}{2\varepsilon_0} \cdot \\
\text{since } E_0 &= \frac{\sigma}{\varepsilon_0}\cdot
\end{aligned}$$

This also matches the result

$$\mathbf{f} = \frac{\sigma}{2}(\mathbf{E}_1 + \mathbf{E}_2) = \frac{\sigma}{2}E_0\hat{\mathbf{e}}_z = \frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{e}}_z\cdot$$

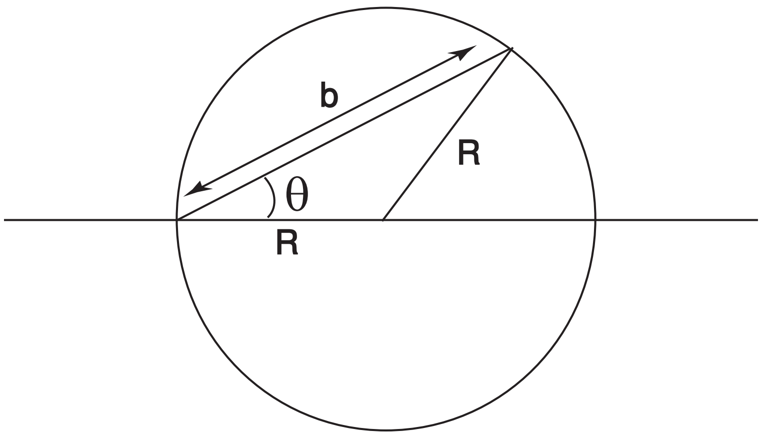
4. **Answer:**

We first find the potential at the point P. Originating at P, we make wedges of opening angle $d\theta$.



$$\text{Now } \phi_p \text{ due to the shaded element} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma r d\theta dr}{r}$$
$$= \frac{\sigma d\theta dr}{4\pi\varepsilon_0}$$
$$\text{and } \phi_p \text{ due to a single wedge} = \frac{\sigma d\theta}{4\pi\varepsilon_0}\int_{r=0}^{b} dr = \frac{\sigma b d\theta}{4\pi\varepsilon_0}$$

Now we need to see how b depends on θ. Applying cosine rule to the triangular formation shown below.



$$
\begin{aligned}
R^2 &= b^2 + R^2 - 2bR\cos\theta \\
b^2 &= 2bR\cos\theta \\
b &= 2R\cos\theta. \\
\text{Hence } \phi_p \text{ due to wedge} &= \frac{2\sigma R\cos\theta d\theta}{4\pi\varepsilon_0}\cdot \\
\phi_p \text{ due to the disk of radius } R &= \frac{2R\sigma}{4\pi\varepsilon_0}\int\limits_{\Theta=-\pi/2}^{\pi/2}\cos\theta d\theta \\
&= \frac{2R}{4\pi\varepsilon_0}\cdot 2\sigma \\
&= \frac{\sigma}{\pi\varepsilon_0}R.
\end{aligned}
$$

5. **Answer:**

We use the result from the previous question to compute

$$
\begin{aligned}
dU_E &= \frac{\sigma}{\pi\varepsilon_0}\ r\ dq \\
U_E &= \int \frac{\sigma}{\pi\varepsilon_0}\ r\ \sigma(2\pi r dr) \\
&= \int\limits_0^a \frac{2\sigma^2 r^2}{\varepsilon_0}dr \\
&= \frac{2\sigma^2 a^3}{3\varepsilon_0}.
\end{aligned}
$$

6. **Answer:** Label points on the curve by their distance r from the origin, and by the angle θ that the line of this distance subtends with the y-axis. Then a point charge q on the curve provides a y component of the electric field at the origin equal to

$$E_y = \frac{q}{4\pi\varepsilon_0 r^2}\cos\theta.$$

If we want this to be independent of the charge's location on the curve, we must have $r^2 \propto \cos\theta$. The curve is therefore be described by the equation,

$$r^2 = a^2\cos\theta \quad \Rightarrow \quad r = a\sqrt{\cos\theta},$$

where the constant a is the value of r at $\theta = 0$, which is, the height of the curve along the y-axis. We therefore have a family of curves indexed by a.

7. **Answer:**

For a surface charge density $\sigma = \sigma(\theta) = \sigma_0 \cos\theta$ on a spherical shell, the volume charge density is $\rho(r) = \sigma(\theta)\delta(r-R)$. Hence

$$\begin{aligned}\mathbf{P} &= \int d^3r\mathbf{r}\rho(\mathbf{r}) \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} drr^2[r\cos\theta\hat{\mathbf{e}}_z + r\sin\theta\sin\phi\hat{\mathbf{e}}_y + r\sin\theta\cos\phi\hat{\mathbf{e}}_x][\sigma_0\cos\theta\delta(r-R)].\end{aligned}$$

Only the integral along the z-axis is non-zero. We can compute it as follows,

$$P_z = 2\pi\sigma_0 R^3 \int_0^{\pi} d\theta \sin\theta\cos^2\theta = -\frac{2\pi\sigma_0 R^3}{3}\cos^3\theta\Big|_0^{\pi} = \frac{4\pi R^3}{3}\sigma_0$$

Hence $\mathbf{P} = \frac{4}{3}\pi R^3\sigma_0\hat{\mathbf{e}}_z.$