Solution HW 3: Electrostatics

1. **Answer:**

$$\begin{aligned}\phi(r=0) &= \frac{1}{4\pi\varepsilon_0}\int\frac{d^2r'\sigma(\vec{r'})}{r'} = \frac{1}{4\pi\varepsilon_0}\int_a^R\frac{\sigma r d\phi}{r}dr \\ &= \frac{\sigma 2\pi}{4\pi\varepsilon_0}(R-a) = \frac{\sigma}{2\varepsilon_0}(R-a).\end{aligned}$$

2. **Answer:**

   (a) The building up process results in an energy

$$\int_0^Q\frac{qdq}{4\pi\varepsilon_0 R} = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

   (b) Doubling the charge results in the energy

$$\frac{(2Q)^2}{8\pi\varepsilon_0 R} = \frac{Q^2}{2\pi\varepsilon_0 R} \tag{1}$$

   which is the correct result from a shell carrying twice the charge.

   (c) Consider two spheres, each carrying a charge Q and radius R.

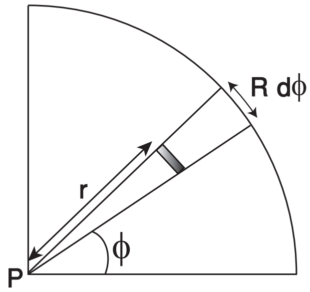
$$\text{Energy from two spheres} = \frac{Q^2}{8\pi\varepsilon_0 R}+\frac{Q^2}{8\pi\varepsilon_0 R} = \frac{Q^2}{4\pi\varepsilon_0 R}.$$

   This result, however, does not conform to equation (1). The reason is that we've ignored the interaction term which can be written as

$$\begin{aligned}\int d^3r\rho_1(\mathbf{r})\phi_2(\mathbf{r}) &= \int d^3r\rho_1(\mathbf{r})\frac{Q}{4\pi\varepsilon_0 R} \\ &= \frac{Q}{4\pi\varepsilon_0 R}\int d^3r\rho_1(\mathbf{r}) \\ &= \frac{Q^2}{4\pi\varepsilon_0 R} \\ \text{Hence total energy} &= \frac{Q^2}{8\pi\varepsilon_0 R}+\frac{Q^2}{8\pi\varepsilon_0 R}+\frac{Q^2}{4\pi\varepsilon_0 R} = \frac{Q^2}{2\pi\varepsilon_0 R}.\end{aligned}$$

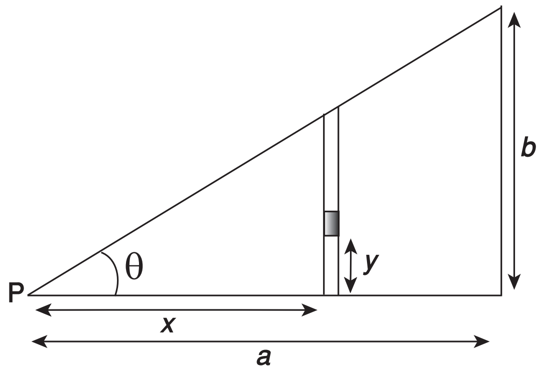
3. **Answer:**

4. **Answer:**



For the charged quadrant,

$$\phi \text{ at } P = \int_{\phi=0}^{\pi/2}\int_{r=0}^{R} \frac{\sigma r d\phi dr}{4\pi\varepsilon_0 r} = \frac{\sigma}{4\pi\varepsilon_0}\frac{\pi}{2}R = \frac{\sigma R}{8\varepsilon_0}.$$



For the charged triangle follow the geometry shown here.

$$\frac{b}{a} = \tan\theta \quad \Rightarrow \quad b = a\tan\theta.$$

The potential at P due to the tiny shaded element is,

$$d\phi = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, dx\, dy}{\sqrt{x^2+y^2}}$$

$$\phi \text{ due to the vertical strip} = \frac{1}{4\pi\varepsilon_0}\sigma dx \int_0^{x\tan\theta} \frac{dy}{\sqrt{x^2+y^2}}$$

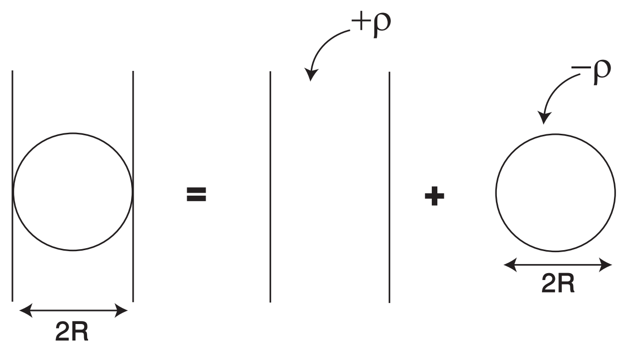
$$\text{Finally, } \phi \text{ due to the triangle} = \int_{x=0}^{a} dx \int_{y=0}^{(xb/a)} \frac{dy}{\sqrt{x^2+y^2}}$$

$$= \int_{x=0}^{a} dx \left[-\frac{1}{2}\ln(x^2) + \ln\left(\frac{b}{a}x + \sqrt{\frac{(a^2+b^2)x^2}{a^2}} \right) \right]$$
$$= a - \frac{1}{2}\, a\ln(a^2) + a\big(-1 + \ln(b + \sqrt{a^2+b^2})\big)$$
$$= a\ln(b + \sqrt{a^2+b^2}) - a\ln a$$
$$= a\ln\left(\frac{b + \sqrt{a^2+b^2}}{a} \right)$$
$$= a\ln(\tan\theta + \sec\theta).$$

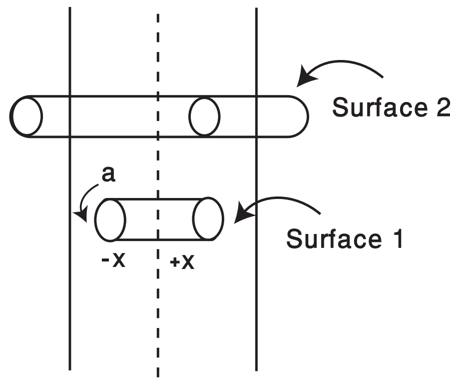
5. **Answer:** Do it yourself.

6. **Answer:**

(a) We consider that the charge distribution is a superposition of a continuously uniformly charged slab and a sphere carrying the opposite charge. This conceptualization is shown below.



Let's first consider the electric **field due to the slab**. For inside the slab, we draw a Gaussian surface labeled 1 and for outside we draw the protruding cylinder 2. See the accompanying diagram.



For inside the slab, we have,

$$\begin{aligned}\frac{\rho(2x)a}{\varepsilon_0} &= 2E_1 a\\ E_1 &= \frac{\rho x}{\varepsilon_0}\\ \mathbf{E}_1 &= \frac{\rho x}{\varepsilon_0}\hat{\mathbf{e}}_x,\end{aligned}$$

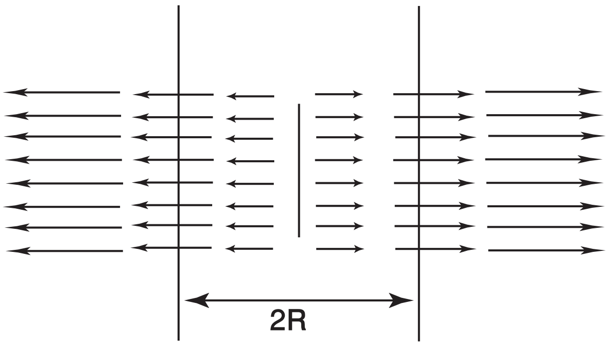
while outside the slab, we obtain

$$\begin{aligned}2E_2 a &= \frac{1}{\varepsilon_0}\rho 2Ra\\ E_2 &= \frac{\rho R}{\varepsilon_0}\\ \mathbf{E_2} &= \pm\frac{\rho R}{\varepsilon_0}\hat{\mathbf{e}}_x = sgn(x)\frac{\rho R}{\varepsilon_0}\hat{\mathbf{e}}_x,\end{aligned}$$

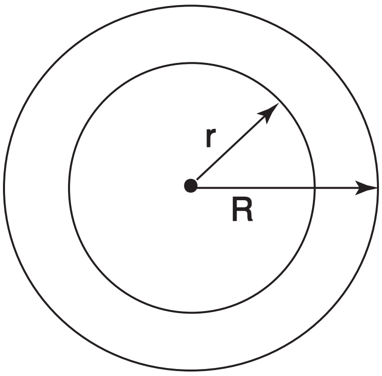
where

$$sgn(x) = \left\{\begin{array}{l} +1,\ x > 0\\ -1,\ x < 0.\end{array}\right\}$$

The diagram below shows a visualization of the electric field only due to the positively charged slab.



We now turn to the electric field produced by **only the charged sphere** (representing the cavity)



Inside the cavity, we obtain

$$\mathbf{E} = -\frac{\rho \frac{4}{3}\pi r^3}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = -\frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}}\cdot$$

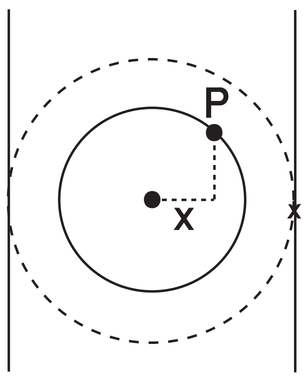
and outside, we have

$$\mathbf{E} = -\frac{\rho \frac{4}{3}\pi R^3}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = -\frac{\rho R^3}{3\varepsilon_0 r^2}\hat{\mathbf{r}}\cdot$$

Therefore the electric fields are:

$$\begin{aligned} \mathbf{E}(\text{inside}) &= \frac{\rho x}{\varepsilon_0}\hat{\mathbf{e}}_x - \frac{\rho r}{3\varepsilon}\hat{\mathbf{r}} \\ \mathbf{E}(\text{outside}) &= sgn(x)\frac{\rho R}{\varepsilon_0}\hat{\mathbf{e}}_x - \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{\mathbf{r}}. \end{aligned}$$

(b)



We first calculate the potential inside the cavity and consider only the field due to the slab. Point P, variable x are defined in the figure. The point P is inside the cavity. Hence, due to only the slab, we have, for $x < R$,

$$\phi(x) - \phi(0) = -\int_0^x \frac{\rho x}{\varepsilon_0}\hat{\mathbf{e}}_x \cdot dx\hat{\mathbf{e}}_x = -\frac{\rho x^2}{2\varepsilon_0}\cdot$$

Furthermore for $x > R$

$$\begin{aligned} \phi(x) - \phi(R) &= -\int_R^x \frac{\rho R}{\varepsilon_0}\ \hat{\mathbf{e}}_x \cdot dx\hat{\mathbf{e}}_x \\ &= -\frac{\rho R}{\varepsilon_0}(x - R)\cdot \end{aligned}$$

Hence potential inside the slab due to slab alone $(x < R) = -\dfrac{\rho x^2}{2\varepsilon_0}$.

Potential outside the slab due to the slab alone $(x > R) = -\dfrac{\rho R^2}{2\varepsilon_0} - -\dfrac{\rho R}{\varepsilon_0}(x - R)$.

Note that these are calculated with respect to center of the cavity.

Let's now find the potential due to the negatively charged cavity alone.

For $r < R$, we have (defining $\phi(0) = 0$)

$$\begin{aligned} \phi(r) - \phi(0) &= -\int_0^r \frac{-\rho r}{3\varepsilon_0}\ \hat{\mathbf{r}} \cdot dr\hat{\mathbf{r}} \\ &= -\frac{\rho}{3\varepsilon_0}\frac{r^2}{2} = \frac{\rho r^2}{6\varepsilon_0} \\ \Rightarrow\ \phi(R) &= \frac{\rho R^2}{6\varepsilon_0}\cdot \end{aligned}$$

For $r > R$,

$$\begin{aligned} \phi(r) - \phi(R) &= -\int_r^R \frac{-\rho R^3}{3\varepsilon_0 r^2}\ \hat{\mathbf{r}} \cdot dr\hat{\mathbf{r}} \\ &= \frac{\rho R^3}{3\varepsilon_0}\int_r^R \frac{dr}{r^2} \\ &= -\frac{\rho R^3}{3\varepsilon_0}\frac{1}{r}\bigg|_r^R \\ &= -\frac{\rho R^3}{3\varepsilon_0}\left(\frac{1}{R} - \frac{1}{r}\right). \end{aligned}$$

Hence potential inside the cavity due to the cavity alone $(r < R) = \dfrac{\rho r^2}{6\varepsilon_0}$, and potential outside the cavity due to only the cavity $(r > R) = \dfrac{\rho R^2}{6\varepsilon_0} - \dfrac{\rho R^3}{3\varepsilon_0}\left(\dfrac{1}{R} - \dfrac{1}{r}\right)$.

$$\text{Potential at } W = \underbrace{-\frac{\rho R^2}{2\varepsilon_0}}_{\text{from slab}} + \underbrace{\frac{\rho R^2}{6\varepsilon_0}}_{\text{from cavity}} = -\frac{\rho R^2}{3\varepsilon_0} \cdot$$

$$\text{Potential at } P \text{ (inside the cavity)} = -\frac{\rho x^2}{2\varepsilon_0} + \frac{\rho r^2}{6\varepsilon_0} \cdot$$

(c) As we move from W to T, the change in potential is only due to the sphere, since the slab alone produces a potential that is invariant with vertical displacement. Therefore

$$\begin{aligned}
\triangle\phi &= -\int_R^\infty \left(\frac{-\rho R^3}{3\varepsilon_0 r^2}\right) dr \\
&= \frac{\rho R^3}{3\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = -\frac{\rho R^3}{3\varepsilon_0} \left.\frac{1}{r}\right|_R^\infty \\
&= \frac{\rho R^3}{3\varepsilon_0 R} = \frac{\rho R^2}{3\varepsilon_0}.
\end{aligned}$$

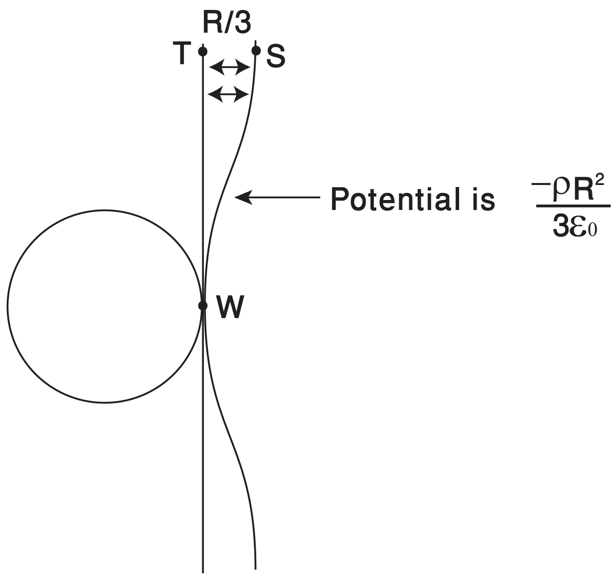
Hence potential rises as we approach infinity. Very far off, the potential becomes $-\dfrac{\rho R^2}{3\varepsilon_0} + \dfrac{\rho R^2}{3\varepsilon_0} = 0$. Hence ϕ at T is zero.

(d)

$$\begin{aligned}
\text{Potential at } W &= -\frac{\rho R^2}{3\varepsilon_0} \\
\text{Potential at } T &= 0.
\end{aligned}$$

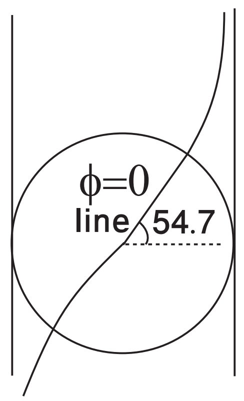
Point T is far away. Hence the field due to negative cavity is zero and only the field due to the slab is appreciable. Hence in the vicinity of T, the field (outside the slab) is $\dfrac{\rho R}{\varepsilon_0}\hat{\mathbf{e}}_x$.

$$\begin{aligned}
\text{At a distance } x' \text{ from the point } T\ \phi(x') &= -\frac{\rho R x'}{\varepsilon_0} \\
-\frac{\rho R x'}{\varepsilon_0} &= -\frac{\rho R^3}{3\varepsilon_0} \\
x' &= \frac{R}{3}
\end{aligned}$$



The diagram shows a rough sketch of the equipotential line.

(e)



Inside the cavity the potential is $-\frac{\rho x^2}{2\varepsilon_0}+\frac{\rho r^2}{6\varepsilon_0}$. For the $\phi=0$ line passing through the center, we obtain

$$\begin{aligned}\frac{r}{x} &= \sqrt{3}\\ \frac{x}{r} &= \frac{1}{\sqrt{3}}\\ \cos\theta &= \frac{1}{\sqrt{3}}\\ \theta &= \cos^{-1}\frac{1}{\sqrt{3}}\approx 54.7^\circ.\end{aligned}$$

Along this line potential is zero, and the line is shown in the accompanying figure.

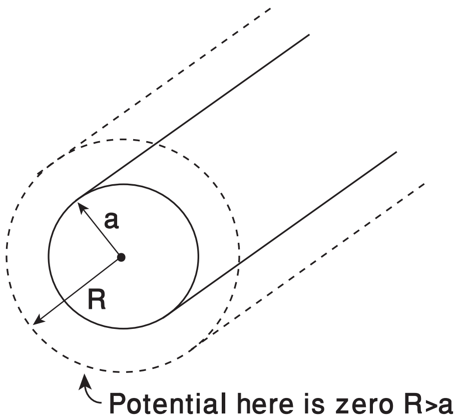
The potential ourside the cavity but inside the slab is

$$\frac{\rho R^2}{6\varepsilon_0} - \frac{\rho R^3}{3\varepsilon_0}\left(\frac{1}{R} - \frac{1}{r}\right) - \frac{\rho x^2}{2\varepsilon_0}.$$

Equating this to zero, one can find the contour whose equation will be

$$\frac{3R^3}{(x^2+y^2)^{1/2}} = 3x^2 + R^2.$$

7. **Answer:**



(a) E **inside:** $r < a$

$$\begin{aligned} E(2\pi r \ell) &= \frac{\rho \pi r^2 \ell}{\varepsilon_0} \\ \mathbf{E} &= \frac{\rho r}{2\varepsilon_0}\hat{\mathbf{r}}\cdot \end{aligned}$$

E **outside:** $r > a$

$$\begin{aligned} E(2\pi r \ell) &= \frac{\rho \pi a^2 \ell}{\varepsilon_0} \\ \mathbf{E} &= \frac{\rho a^2}{2\varepsilon_0 r}\hat{\mathbf{r}}\cdot \end{aligned}$$

Let's find the potential inside the cylinder at $r < a$. We know that for $R > a$,

$$\begin{aligned} \phi(R) - \phi(r) &= -\int_r^a \frac{\rho r}{2\varepsilon_0} dr - \int_a^R \frac{\rho a^2}{2\varepsilon_0 r} dr \\ &= -\frac{\rho}{2\varepsilon_0}\frac{1}{2}(a^2 - r^2) - \frac{\rho a^2}{2\varepsilon_0}\ln\left(\frac{R}{a}\right) \\ &= -\frac{\rho}{4\varepsilon_0}(a^2 - r^2) - \frac{\rho a^2}{2\varepsilon_0}\ln\left(\frac{R}{a}\right)\cdot \end{aligned}$$

Hence the potential inside the cylinder ($r < a$) with respect to R, is

$$\phi(r) - \phi(R) = \frac{\rho}{4\varepsilon_0}(a^2 - r^2) + \frac{\rho a^2}{2\varepsilon_0}\ln\left(\frac{R}{a}\right).$$

(b)

$$\begin{aligned}
\frac{U_E}{\ell} &= \frac{1}{2}\cdot\frac{\rho}{\varepsilon_0}\int\limits_{r=0}^{a}\left[\left(\frac{a^2 - r^2}{4}\right) + \frac{a^2}{2}\ln\left(\frac{R}{a}\right)\right]2\pi r dr \\
&= \frac{\rho^2\pi}{\varepsilon_0}\int_{r=0}^{a}\left[\frac{a^2}{4}r - \frac{r^3}{4} + \frac{a^2}{2}\ln\left(\frac{R}{a}\right)\right]dr \\
&= \frac{\rho^2\pi}{\varepsilon}\left[\left(\frac{a^2}{4} + \frac{a^2}{2}r\ln\left(\frac{R}{a}\right)\right)\frac{a^2}{2} - \frac{a^4}{16}\right] \\
&= \frac{\rho^2\pi}{\varepsilon}\left(\frac{a^4}{8} + \frac{a^4}{4}\ln\left(\frac{R}{a}\right) - \frac{a^4}{16}\right) \\
&= \frac{\rho^2\pi a^4}{\varepsilon}\left(\frac{1}{16} + \frac{1}{4}\ln\frac{R}{a}\right) \\
&= \frac{\rho^2\pi a^4}{4\varepsilon}\left(\frac{1}{4} + \ln\frac{R}{a}\right).
\end{aligned}$$

8. **Answer:**

   Attached at the end.

9. **Answer:**

   Since the hemisphere is at $R \to \infty$, there is zero stress tensor on the dome and we need to consider only the xy plane.

   Inside the Disk:

$$F_z\big|_{\text{bottom}} = \frac{Q^2}{64\pi\varepsilon_0 a^2} \qquad \text{(derived in the class).}$$

Outside the Disk:

$$
\begin{aligned}
\vec{E} &= \frac{Q}{4\pi\varepsilon_0 R^2}(\cos\phi\ \hat{e}_x + \sin\phi\ \hat{e}_y) \\
T_{zz} &= \frac{\varepsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = -\frac{\varepsilon_0}{2}\frac{Q^2}{(4\pi\varepsilon_0 R^2)^2}\ . \\
T_{xz} &= \varepsilon_0 E_x E_z = 0 = T_{yz} \\
(d\vec{S}\cdot\vec{T}\ )_z &= dS_x T_{xz} + dS_y T_{yz} + dS_z T_{zz} \\
d\vec{S} &= -dS\ \hat{e}_z \\
df_z = (d\vec{S}\cdot\vec{T})_z &= -dS\hat{e}_z\cdot\left(-\frac{\varepsilon_0}{2}\right)\frac{Q^2}{(4\pi\varepsilon_0 R^2)^2}\hat{e}_z = dS\ \frac{\varepsilon_0}{2}\ \frac{Q^2}{(4\pi\varepsilon_0 R^2)^2}\ . \\
F_z &= df_z = \frac{Q^2}{64\pi\varepsilon_0 a^2} + \frac{\varepsilon_0}{2}\ \frac{Q^2}{(4\pi\varepsilon_0)^2}\int\limits_{R=a}^{\infty}\frac{dS}{R^4} \\
dS &= R\ d\phi\ dR \\
\int\frac{dS}{R^4} &= \int\limits_{\phi=0}^{2\pi}\int\limits_{R=a}^{\infty}\frac{R}{R^4}\ d\phi\ dR = 2\pi\int\limits_{a}^{\infty}\frac{1}{R^3}dR \\
&= -\frac{2\pi}{2}\ \frac{1}{R^2}\bigg|_a^{\infty} = -\pi\left(0-\frac{1}{a^2}\right) = \frac{\pi}{a^2} \\
F_z &= \frac{Q^2}{64\pi\varepsilon_0 a^2} + \frac{Q^2}{2(4)^2\pi\varepsilon_0}\left(\frac{\pi}{a^2}\right) = \frac{Q^2}{\pi\varepsilon_0 a^2}\left(\frac{1}{64}+\frac{1}{32}\right) \\
&= \frac{Q^2}{\pi\varepsilon_0 a^2}\ \frac{3}{64} = \frac{3}{16}\ \frac{Q^2}{4\pi\varepsilon_0 a^2},
\end{aligned}
$$

which is identical to what is given in Eq.(3).

10. **Answer:**

Attached at the end.