Assignment 6: Quantum Field Theory
Due Date: 13 March. 10 am

1. Show that equal time commutator for a scalar field and its time like component of the canonical momentum is given by,

$$\left[\hat{\phi}(t,\mathbf{x}), \hat{\Pi}^{\circ}(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

2. Show the complete step-by-step working for the canonical quantization of the complex scalar field,

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \psi^\dagger &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \end{aligned}$$

3. In the class we discussed that the non-relativistic Lagrangian density for a complex scalar field can be written as,

$$\mathscr{L} = -\Psi^\dagger\partial_0\Psi - \frac{1}{2m}\nabla\Psi\cdot\nabla\Psi^\dagger.$$

Apply the Euler-Lagrange equation with respect to the field $\psi^\dagger$ and derive the Schrödinger equation. Also show that the mode expansion

$$\hat{\Psi} = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}} e^{-ip\cdot x}$$

yields the conventional energy dispersion for a free particle

$$E_{\mathbf{p}} = \frac{|\mathbf{p}|^2}{2m}.$$

4. Starting with the non-relativistic lagrangian given in Q3, and the definition of the amplitude $\rho(x)$ and phase $\theta(x)$ fields, show that the lagrangian density can also be written as,

$$\mathscr{L} = \frac{i}{2}\partial_0\rho - \rho\partial_0\theta - \frac{1}{2m}\left[\frac{1}{4\rho}(\nabla\rho)^2 + \rho(\nabla\theta)^2\right].$$