Solution Assignment 1: Modern Physics

1. **Answer 1**

(a) $z^2 = 8(1 + \sqrt{3}\, i)$

i. **Rectangular Form:** We are given that

$$\begin{aligned} z^2 &= 8(1 + \sqrt{3}\, i) \\ z^2 &= 8 + 8\sqrt{3}\, i \qquad (1) \\ \text{Let} \qquad z &= a + ib \\ z^2 &= (a + ib)^2 \\ &= a^2 - b^2 + i2ab \qquad (2) \end{aligned}$$

Compare equation 1 and 2 we get,

$$\begin{aligned} a^2 - b^2 + i2ab &= 8 + 8\sqrt{3}\, i \\ (a^2 - b^2) + i(2ab) &= 8 + (8\sqrt{3})\, i \end{aligned}$$

Compare Real and imaginary parts on both sides we get,

$$\begin{aligned} a^2 - b^2 &= 8 \qquad (3) \\ 2ab &= 8\sqrt{3} \\ \Rightarrow \quad a &= \frac{4\sqrt{3}}{b} \qquad (4) \end{aligned}$$

Substitute value of a from equation 4 to equation 3 we get,

$$\begin{aligned} \left(\frac{4\sqrt{3}}{b}\right)^2 - b^2 &= 8 \\ 48 - b^4 &= 8b^2 \\ b^4 + 8b^2 - 48 &= 0 \end{aligned}$$

Solution of above quadratic equation will be,

$$\begin{aligned} \text{either} \quad b &= \pm 2 \quad \text{or} \quad b = \pm 2\sqrt{3}\, i \text{ (not possible)} \\ \Rightarrow \quad a &= \frac{4\sqrt{3}}{\pm 2} = \pm 2\sqrt{3} \end{aligned}$$

Therefore rectangular form of given complex number is,

$$z = \pm 2(\sqrt{3} + i)$$

ii. **Polar Form:** We are given that

$$
\begin{aligned}
z^2 &= 8(1+\sqrt{3}\ i) \\
\text{If} \quad z^2 &= r\text{Cis}(\theta+2k\pi) \quad \text{where } k \in z \\
\text{then} \quad z &= \big(r\text{Cis}(\theta+2k\pi)\big)^{1/2} \\
z &= \sqrt{r}\big(\text{Cis}(\theta+2k\pi)\big)^{1/2}.
\end{aligned}
$$

By De-Moivre's theorem

$$
z = \sqrt{r}\bigg(\text{Cis}\bigg(\frac{\theta}{2}+\frac{2k\pi}{2}\bigg)\bigg) = \sqrt{r}\bigg(\text{Cis}\bigg(\frac{\theta}{2}+k\pi\bigg)\bigg). \tag{5}
$$

Now let's calculate r and θ.

$$
\begin{aligned}
r &= \sqrt{a^2+b^2} = \sqrt{8^2+(8\sqrt{3})^2} = 16, \\
\text{and} \quad \theta &= \tan^{-1}\bigg(\frac{b}{a}\bigg) = \tan^{-1}\bigg(\frac{8\sqrt{3}}{8}\bigg) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}.
\end{aligned}
$$

Thus given complex number in polar form will be,

$$
\begin{aligned}
z &= \sqrt{16}\bigg(\text{Cis}\bigg(\frac{\pi/3}{2}+k\pi\bigg)\bigg) \\
z &= 4\bigg(\text{Cis}\bigg(\frac{\pi}{6}+k\pi\bigg)\bigg). \\
(\text{For} \quad k=o) \quad z &= 4\text{Cis}\bigg(\frac{\pi}{6}\bigg) = 4\bigg(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\bigg). \\
(\text{For} \quad k=1) \quad z &= 4\text{Cis}\bigg(\frac{\pi}{6}+\pi\bigg) = -4\bigg(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\bigg).
\end{aligned}
$$

(b) $z^2 = -5(1-\sqrt{7}\ i)$

i. **Rectangular Form:** We are given that

$$
\begin{aligned}
z^2 &= -5(1-\sqrt{7}\ i) \\
z^2 &= -5+5\sqrt{7}\ i \qquad (6) \\
\text{Let} \quad z &= a+ib \\
z^2 &= (a+ib)^2 \\
&= a^2-b^2+i2ab \qquad (7)
\end{aligned}
$$

Compare equation 6 and 7 we get,

$$
\begin{aligned}
a^2-b^2+i2ab &= -5+5\sqrt{7}\ i \\
(a^2-b^2)+i(2ab) &= -5+(5\sqrt{7})\ i
\end{aligned}
$$

Compare Real and imaginary parts on both sides we get,

$$\begin{aligned} a^2 - b^2 &= -5 \qquad (8) \\ 2ab &= 5\sqrt{7} \\ \Rightarrow \quad a &= \frac{5\sqrt{7}}{2b} \qquad (9) \end{aligned}$$

Substitute value of a from equation 9 to equation 8 we get,

$$\begin{aligned} \left(\frac{5\sqrt{7}}{2b}\right)^2 - b^2 &= -5 \\ 175 - 4b^4 &= -20b^2 \\ 4b^4 - 20b^2 - 175 &= 0 \end{aligned}$$

Solution of above quadratic equation will be,

$$\begin{aligned} \text{either} \quad b &= \pm 3.09 \quad \text{or} \quad b = 2.14i \text{ (not possible)} \\ \Rightarrow \quad a &= \frac{5\sqrt{7}}{2 \times 3.09} \\ a &= \pm 2.14 \end{aligned}$$

Therefore rectangular form of given complex number is,

$$z = \pm(2.14 + 3.09i).$$

ii. **Polar Form:** We are given that

$$\begin{aligned} z^2 &= -5(1 - \sqrt{7}\, i) \\ \text{If} \quad z^2 &= r \text{ Cis}(\theta + 2k\pi) \quad \text{where } k \in z \\ \text{then} \quad z &= \big(r\text{Cis}(\theta + 2k\pi)\big)^{1/2} \\ z &= \sqrt{r}\big(\text{Cis}(\theta + 2k\pi)\big)^{1/2}. \end{aligned}$$

By De-Moivre's theorem

$$z = \sqrt{r}\left(\text{Cis}\left(\frac{\theta}{2} + \frac{2k\pi}{2}\right)\right) = \sqrt{r}\left(\text{Cis}\left(\frac{\theta}{2} + k\pi\right)\right). \qquad (10)$$

Now let's calculate r and θ.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} = \sqrt{(-5)^2 + (5\sqrt{7})^2} = 10\sqrt{2} = 14.14, \\ \text{and} \quad \theta &= \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{5\sqrt{7}}{-5}\right) = \tan^{-1}(-\sqrt{7}) = -69.2°. \end{aligned}$$

Thus given complex number in polar form will be,

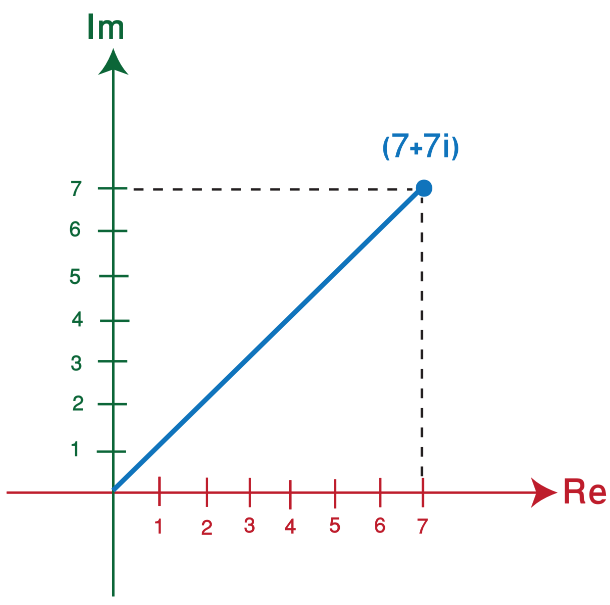
$$
\begin{aligned}
z &= \sqrt{10\sqrt{2}}\left(\text{Cis}\left(\frac{\theta}{2} + k\pi\right)\right) \\
z &= \sqrt{10\sqrt{2}}\left(\text{Cis}\left(\frac{-69.2^\circ}{2} + k\pi\right)\right). \\
(\text{For}\quad k = o)\quad z &= \sqrt{14.14}\ \text{Cis}(-34.6^\circ) \\
&= \sqrt{14.14}(\cos(-34.6^\circ) + i\sin(-34.6^\circ)) \\
&= \sqrt{14.14}(\cos(34.6^\circ) - i\sin(34.6^\circ)). \\
(\text{For}\quad k = 1)\quad z &= \sqrt{14.14}\ \text{Cis}(-34.6^\circ + \pi) \\
&= \sqrt{14.14}\big(\cos(-34.6^\circ + \pi) + i\sin(-34.6^\circ + \pi)\big) \\
&= -\sqrt{14.14}\big(\cos(34.6^\circ) - i\sin(34.6^\circ)\big).
\end{aligned}
$$

2. **Answer 2**

We are given three complex numbers $z_1 = 3 + 5\ i$, $z_2 = 4 + 2\ i$, and $z_3 = 6 -\ i$.

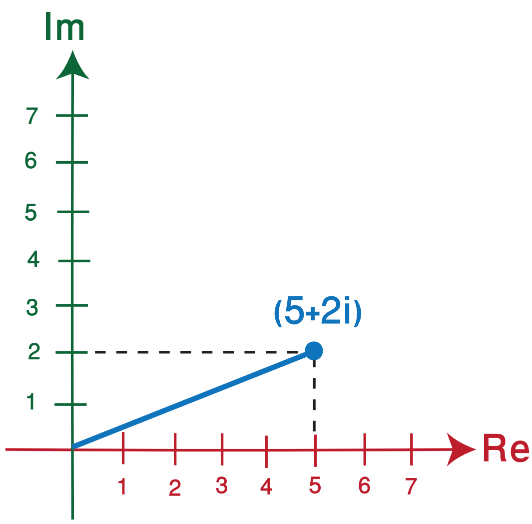
(a)

$$z_1 + z_2 = (3 + 5\ i) + (4 + 2\ i) = (3 + 4) + (5 + 2)i = 7 + 7i$$



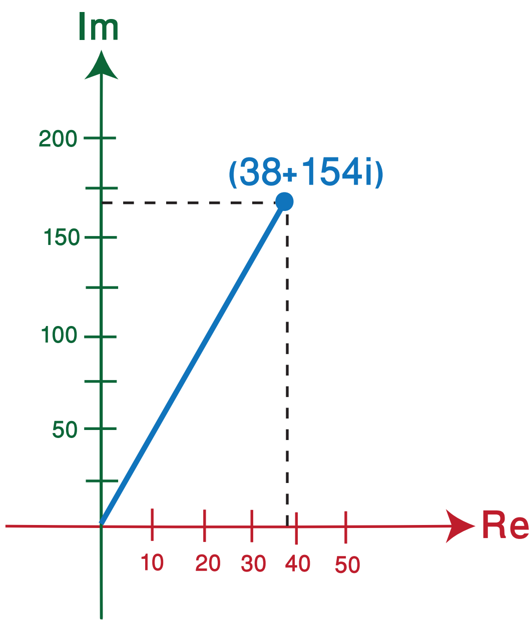
(b)

$$
\begin{aligned}
z_1 - z_2 + z_3 &= (3 + 5\ i) - (4 + 2\ i) + (6 -\ i) \\
&= (3 - 4 + 6) + (5 - 2 - 1)i = 5 + 2i
\end{aligned}
$$



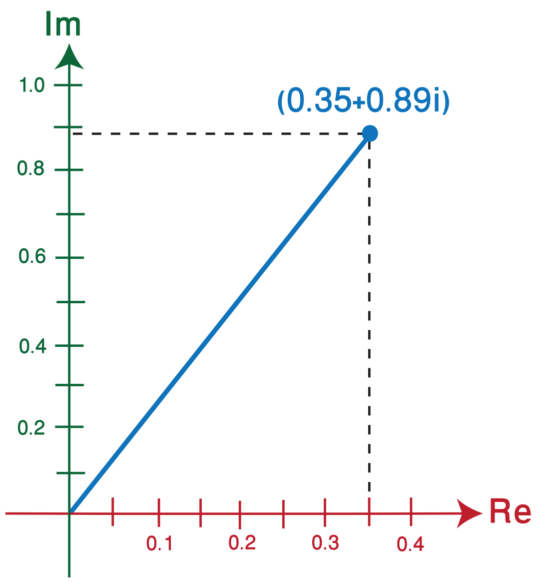
(c)

$$
\begin{aligned}
z_1 z_2 z_3 &= (3+5\,i)(4+2\,i)(6-\,i) = (3+5\,i)(24-4i+12i+2) \\
&= (3+5\,i)(26+8i) = (78+24i+130i-40) = 38+154i
\end{aligned}
$$



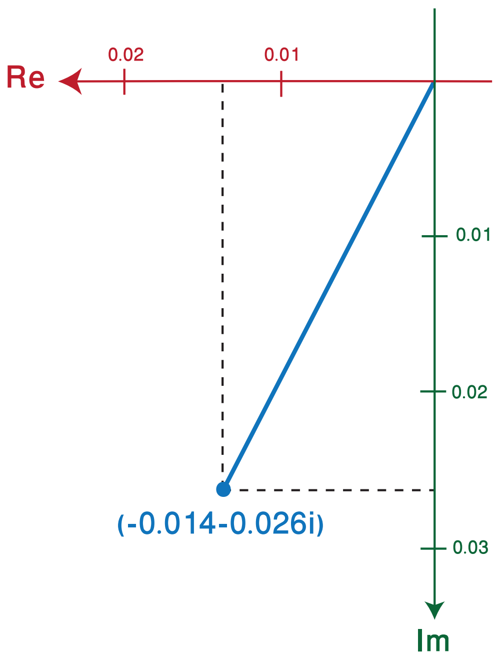
(d)

$$
\begin{aligned}
\frac{z_1}{z_3} &= \frac{3+5\,i}{6-\,i} = \frac{3+5\,i}{6-\,i} \times \frac{6+\,i}{6+\,i} \quad \text{(taking congugate)} \\
&= \frac{18+3i+30i-5}{36+1} = \frac{13+33i}{37} = \frac{13}{37} + \frac{33}{37}i
\end{aligned}
$$



(e) i.

$$\begin{aligned}
\frac{1}{z_1^2} &= \frac{1}{(3+5\ i)^2} = \frac{1}{(3+5\ i)(3+5\ i)} \\
&= \frac{1}{(9+15i+15i-25)} = \frac{1}{(-16+30i)} \\
&= \frac{1}{(-16+30i)} \times \frac{(-16-30i)}{(-16-30i)} = \frac{(-16-30i)}{(256+900)} \\
&= \frac{(-16-30i)}{1156} = \frac{(-8-15i)}{578} = -\frac{8}{578} - \frac{15}{578}i
\end{aligned}$$

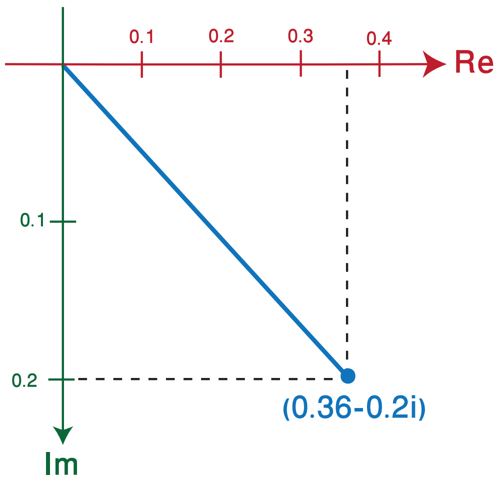


ii.

$$\frac{1}{\sqrt{z_1}} = \frac{1}{\sqrt{3+5\,i}}$$

Let's at first solve $\sqrt{z_1}$.

$$\begin{aligned}
\text{Let} \quad z_1 &= r\ \text{Cis}(\theta + 2k\pi) \quad \text{where } k \in z \\
\sqrt{z_1} &= \sqrt{r\ \text{Cis}(\theta+2k\pi)} = \sqrt{r}\ \left[\text{Cis}(\theta+2k\pi)\right]^{1/2} = \sqrt{r}\ \text{Cis}\left(\frac{\theta}{2} + k\pi\right) \\
\text{where} \quad r &= \sqrt{a^2+b^2} = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34} = 5.8 \\
\text{and} \quad \theta &= \tan^{-1/2}\left(\frac{b}{a}\right) = \tan^{-1/2}\left(\frac{5}{3}\right) = 59^\circ \\
\Rightarrow \quad \sqrt{z_1} &= \sqrt{5.8}\ \text{Cis}\left(\frac{59^\circ}{2} + k\pi\right) = \sqrt{5.8}\ \text{Cis}(29.5^\circ) \\
&= \sqrt{5.8}(\cos 29.5^\circ + i \sin 29.5^\circ) = 2.09 + 1.2i \\
\Rightarrow \quad \frac{1}{\sqrt{z_1}} &= \frac{1}{2.09+1.2i} = \frac{1}{2.09+1.2i} \times \frac{2.09-1.2i}{2.09-1.2i} \\
&= \frac{2.09-1.2i}{5.8} = 0.36 - 0.2i
\end{aligned}$$



(f) i.

$$|z_1 + z_2| = |7+7i| = \sqrt{(7)^2+(7)^2} = \sqrt{49+49} = \sqrt{98} = 9.89$$

ii.

$$\begin{aligned}
|z_2 - z_3| &= \left|(4+2\,i) - (6-\ i)\right| = \left|(4-6)+(2-1)i\right| = \left|-2+i\right| \\
&= \sqrt{(-2)^2+(1)^2} = \sqrt{4+1} = \sqrt{5} = 2.24
\end{aligned}$$

iii.

$$|z_1| = \big|3 + 5\ i\big| = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34} = 5.8$$

3. **Answer 3**

   (a) Since $i = \sqrt{-1}, \quad i^2 = -1.$

   (b) $i^3 = i^2\ i = (-1)i = -i.$

   (c) $i^4 = (i^2)^2 = (-1)^2 = 1.$

   (d) $\dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{i^2} = \frac{i}{-1} = -i.$

   (e) $\dfrac{1}{i^2} = \dfrac{1}{-1} = -1.$

   (f) $\dfrac{1}{i^3} = \dfrac{1}{i^2} \times \dfrac{1}{i} = \dfrac{1}{-1}(-i) = i.$

4. **Answer 4**

   We are given $z_1 = e^{ik_1x}$ and $z_2 = e^{ik_2x}$

   (a)

$$\begin{aligned} z_1 + z_2 &= e^{ik_1x} + e^{ik_2x} \\ &= \big(\cos(k_1x) + i\sin(k_1x)\big) + \big(\cos(k_2x) + i\sin(k_2x)\big) \\ &= \big(\cos(k_1x) + \cos(k_2x)\big) + i\big(\sin(k_1x) + \sin(k_2x)\big). \end{aligned}$$

   (b)

$$\begin{aligned} z_1 - z_2 &= e^{ik_1x} - e^{ik_2x} \\ &= \big(\cos(k_1x) + i\sin(k_1x)\big) - \big(\cos(k_2x) + i\sin(k_2x)\big) \\ &= \big(\cos(k_1x) - \cos(k_2x)\big) + i\big(\sin(k_1x) - \sin(k_2x)\big). \end{aligned}$$

   (c)

$$\begin{aligned} \mathrm{Re}\, z_1 + z_2 &= \big(\cos(k_1x) + \cos(k_2x)\big) \\ \mathrm{Re}\, z_1 - z_2 &= \big(\cos(k_1x) - \cos(k_2x)\big). \end{aligned}$$

(d)

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{e^{ik_1x}}{e^{ik_2x}} \\
&= \frac{\big(\cos(k_1x) + i\sin(k_1x)\big)}{\big(\cos(k_2x) + i\sin(k_2x)\big)} \\
&= \frac{\big(\cos(k_1x) + i\sin(k_1x)\big)}{\big(\cos(k_2x) + i\sin(k_2x)\big)} \times \frac{\big(\cos(k_2x) - i\sin(k_2x)\big)}{\big(\cos(k_2x) - i\sin(k_2x)\big)} \\
&= \frac{\cos(k_1x)\cos(k_2x) - i(\cos(k_1x)\sin(k_2x) + i\sin(k_1x)\cos(k_2x) + \sin(k_1x)\sin(k_2x)}{\cos(k_2x)^2 + \sin(k_2x)^2} \\
&= \frac{\big(\cos(k_1x)\cos(k_2x) + \sin(k_1x)\sin(k_2x)\big) + i\big(\sin(k_1x)\cos(k_2x) - \cos(k_1x)\sin(k_2x)\big)}{1} \\
&= \cos(k_1 - k_2)x + i\sin(k_1 - k_2)x \\
&= e^{i(k_1-k_2)x}.
\end{aligned}
$$

(e)

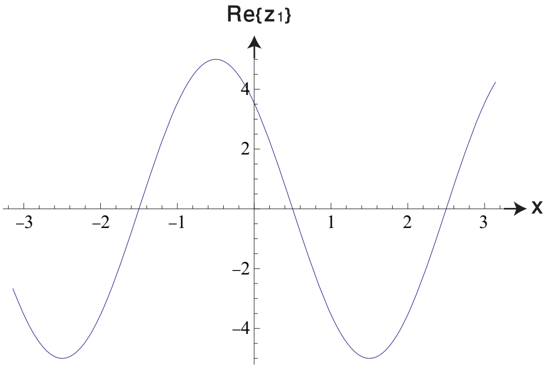
$$
\begin{aligned}
z_1z_2 &= e^{ik_1x} \cdot e^{ik_2x} \\
&= \big(\cos(k_1x) + i\sin(k_1x)\big) \cdot \big(\cos(k_2x) + i\sin(k_2x)\big) \\
&= \cos(k_1x)\cos(k_2x) + i(\cos(k_1x)\sin(k_2x) + i\sin(k_1x)\cos(k_2x) - \sin(k_1x)\sin(k_2x) \\
&= \big(\cos(k_1x)\cos(k_2x) - \sin(k_1x)\sin(k_2x)\big) + i\big(\sin(k_1x)\cos(k_2x) + \cos(k_1x)\sin(k_2x)\big) \\
&= \cos(k_1 + k_2)x + i\sin(k_1 + k_2)x \\
&= e^{i(k_1+k_2)x}.
\end{aligned}
$$

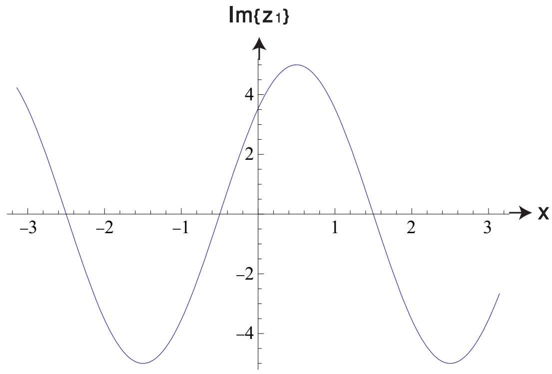
5. **Answer 5**

We are given $z_1 = 5e^{i(kx+\frac{\pi}{4})}$ and $z_2 = 6e^{i(kx-\frac{\pi}{3})}$.

(a)

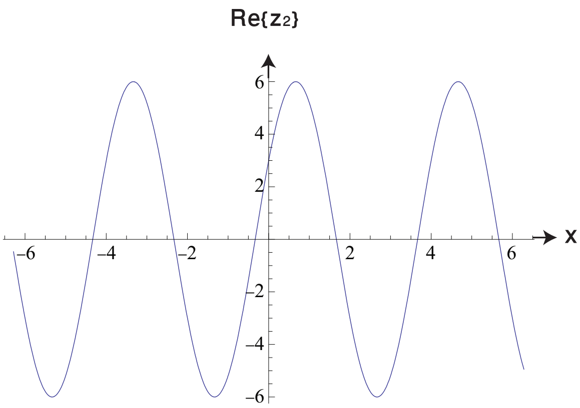
$$
\begin{aligned}
\text{Since } \; z_1 &= 5e^{i(kx+\frac{\pi}{4})} \\
&= 5\left[\cos\left(kx + \frac{\pi}{4}\right) + i\sin\left(kx + \frac{\pi}{4}\right)\right] \\
&= 5\cos\left(kx + \frac{\pi}{4}\right) + i5\sin\left(kx + \frac{\pi}{4}\right) \\
\Rightarrow \quad \mathrm{Re}\{z_1\} &= 5\cos\left(kx + \frac{\pi}{4}\right) \\
\mathrm{Im}\{z_1\} &= 5\sin\left(kx + \frac{\pi}{4}\right)
\end{aligned}
$$

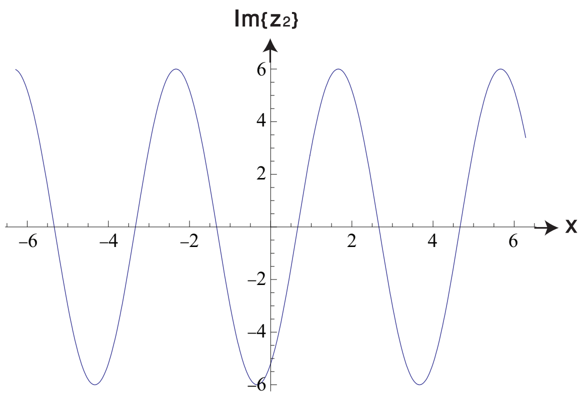




(b)

$$
\begin{aligned}
z_2 &= 6e^{i(kx-\frac{\pi}{3})} \\
&= 6\left[\cos\left(kx-\frac{\pi}{3}\right)+i\sin\left(kx-\frac{\pi}{3}\right)\right] \\
&= 6\cos\left(kx-\frac{\pi}{3}\right)+i5\sin\left(kx-\frac{\pi}{3}\right) \\
\Rightarrow \quad \mathrm{Re}\{z_2\} &= 6\cos\left(kx-\frac{\pi}{3}\right) \\
\mathrm{Im}\{z_2\} &= 6\sin\left(kx-\frac{\pi}{3}\right)
\end{aligned}
$$





(c) i.

$$
\begin{aligned}
z_1 &= 5e^{i(kx+\frac{\pi}{4})} \\
&= 5\left[\cos\left(kx+\frac{\pi}{4}\right)+i\sin\left(kx+\frac{\pi}{4}\right)\right] \\
&= 5\cos\left(kx+\frac{\pi}{4}\right)+i5\sin\left(kx+\frac{\pi}{4}\right) \\
|z_1| &= 5\sqrt{\cos\left(kx+\frac{\pi}{4}\right)^2+\sin\left(kx+\frac{\pi}{4}\right)^2} \\
&= 5\sqrt{1} \\
&= 5
\end{aligned}
$$

ii.

$$
\begin{aligned}
z_1+z_2 &= 5e^{i(kx+\frac{\pi}{4})}+6e^{i(kx-\frac{\pi}{3})} \\
&= 5\left[\cos\left(kx+\frac{\pi}{4}\right)+i\sin\left(kx+\frac{\pi}{4}\right)\right]+6\left[\cos\left(kx-\frac{\pi}{3}\right)+i\sin\left(kx-\frac{\pi}{3}\right)\right] \\
&= \left[5\cos\left(kx+\frac{\pi}{4}\right)+6\cos\left(kx-\frac{\pi}{3}\right)\right]+i\left[5\sin\left(kx+\frac{\pi}{4}\right)+6\sin\left(kx-\frac{\pi}{3}\right)\right] \\
|z_1+z_2| &= \sqrt{\left[5\cos\left(kx+\frac{\pi}{4}\right)+6\cos\left(kx-\frac{\pi}{3}\right)\right]^2+\left[5\sin\left(kx+\frac{\pi}{4}\right)+6\sin\left(kx-\frac{\pi}{3}\right)\right]^2} \\
&= \sqrt{25+36+60\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} = \sqrt{25+36+60\cos\frac{7\pi}{12}} \\
&= \sqrt{45.47} \\
&= 6.7.
\end{aligned}
$$