Solution Assignment 2: Modern Physics

1. **Answer 1:** We are given that,

$$\begin{aligned} \text{Rate of sun's entire emission} = P &= \frac{E}{t} = 3.9 \times 10^{26}\ \text{W} \\ \text{Wave length} = \lambda &= 550\ \text{nm} = 5.5 \times 10^{-7}\ \text{m} \\ \text{Rate of emission of photons} = n/t &= ?, \end{aligned}$$

where n represents the number of photons emitted. Energy of a photon is given by,

$$\begin{aligned} E &= hf = \frac{hc}{\lambda} \\ \text{For } n \text{ photins} \qquad E &= \frac{nhc}{\lambda} \\ \Rightarrow \quad \frac{E}{t} &= \frac{nhc}{t\lambda} \\ \Rightarrow \quad P &= \frac{nhc}{t\lambda} \\ \Rightarrow \quad \frac{n}{t} &= \frac{P\lambda}{hc}. \end{aligned}$$

Substitute values from given data, we get'

$$\begin{aligned} \frac{n}{t} &= \frac{(3.9 \times 10^{26}\ \text{W}) \times (5.5 \times 10^{-7}\ \text{m})}{(6.63 \times 10^{-34}\ \text{j.s}) \times (3 \times 10^{8}\ ms^{-1})} \\ &= 1.08 \times 10^{45}\ \text{photones/s.} \end{aligned}$$

2. **Answer 2:**

(a) According to normalization condition,

$$\begin{aligned} \int_0^L |\psi(x)|^2\, dx &= 1 \\ \int_0^L |A|^2 \sin^2\left(\frac{2\pi x}{L}\right) dx &= 1 \\ |A|^2 \int_0^L \left[\frac{1-\cos\left(\frac{4\pi x}{L}\right)}{2}\right] dx &= 1 \\ \frac{|A|^2}{2}\left[\int_0^L dx - \int_0^L \cos\left(\frac{4\pi x}{L}\right) dx\right] &= 1 \\ \frac{|A|^2}{2}\left[L - \frac{L}{4\pi}\left|\sin\left(\frac{4\pi x}{L}\right)\right|_0^L\right] &= 1 \\ \frac{|A|^2}{2}\left[L - \frac{L}{4\pi}\left(\sin 4\pi - sin0\right)\right] &= 1 \\ \frac{|A|^2}{2}(L-0) &= 1 \end{aligned}$$

$$
\begin{aligned}
\frac{L|A|^2}{2} &= 1 \\
|A|^2 &= \frac{2}{L} \\
A &= \sqrt{\frac{2}{L}}.
\end{aligned}
$$

(b) Probability is given by,

$$P = \int_0^{L/4} |\psi(x)|^2 \, dx.$$

The wave function is,

$$
\begin{aligned}
\psi(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \\
\Rightarrow \quad P &= \frac{2}{L} \int_0^{L/4} \sin^2\left(\frac{2\pi x}{L}\right) \, dx \\
&= \frac{2}{L} \int_0^{L/4} \left[\frac{1 - \cos\left(\frac{4\pi x}{L}\right)}{2}\right] \, dx \\
&= \frac{1}{L}\left[\int_0^{L/4} dx - \int_0^{L/4} \cos\left(\frac{4\pi x}{L}\right) \, dx\right] \\
&= \frac{1}{L}\left[\frac{L}{4} - \frac{L}{4\pi}\left|\sin\left(\frac{4\pi x}{L}\right)\right|_0^{L/4}\right] \\
&= \frac{1}{L}\left[\frac{L}{4} - \frac{L}{4\pi}(\sin\pi - \sin 0)\right] \\
&= \frac{1}{L}\left[\frac{L}{4} - 0\right] \\
&= \frac{1}{4}.
\end{aligned}
$$

3. **Answer 3:**

(a) According to normalization condition,

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx &= 1 \\
\int_{-\infty}^{\infty} |Ae^{-ax^2}|^2 \, dx &= 1 \\
A^2 \int_{-\infty}^{\infty} e^{-2ax^2} \, dx &= 1 \\
A^2 I &= 1, \qquad (1)
\end{aligned}
$$

where $I = \int\limits_{-\infty}^{\infty} e^{-2ax^2}\ dx$ is the Gaussian integral. Let's at first solve Gaussian integral.

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-2ax^2}\ dx \\ \Rightarrow\quad I^2 &= \int_{-\infty}^{\infty} e^{-2ax^2}\ dx \cdot \int_{-\infty}^{\infty} e^{-2ay^2}\ dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2a(x^2+y^2)}\ dx\ dy. \end{aligned}$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$ and area $dx\ dy = r\ dr\ d\theta$. Therefore,

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-2ar2}\ r\ dr\ d\theta \\ &= \left[\int_0^{2\pi} d\theta\right]\cdot\left[\left(\frac{-1}{4a}\right)\int_0^{\infty} e^{-2ar2}(-4ar)\ dr\right] \\ &= (2\pi)\left(\frac{-1}{4a}|e^{-2ar2}|_0^{\infty}\right) = \frac{-\pi}{2a}(e^{-\infty} - e^0) \\ &= \frac{-\pi}{2a}(0-1) = \frac{\pi}{2a} \\ \Rightarrow\quad I &= \sqrt{\frac{\pi}{2a}} \end{aligned}$$

Thus equation (**??**) will become,

$$A^2\sqrt{\frac{\pi}{2a}} = 1 \quad\Rightarrow\quad A^2 = \sqrt{\frac{2a}{\pi}} \quad\text{and}\quad A = \left[\frac{2a}{\pi}\right]^{1/4}.$$

(b) Expectation value of x is given by,

$$\begin{aligned} \langle c\rangle &= \int_{-\infty}^{\infty} \psi^*(x)\ x\ \psi(x)\ dx \\ &= \int_{-\infty}^{\infty} x|\psi(x)|^2\ dx \\ &= \int_{-\infty}^{\infty} x|Ae^{-ax^2}|^2\ dx \\ &= A^2\int_{-\infty}^{\infty} xe^{-2ax^2}\ dx \\ &= \frac{-A^2}{4a}\int_{-\infty}^{\infty} e^{-2ax^2}\ (-4ax)\ dx \\ &= \frac{-A^2}{4a}|e^{-2ax^2}|_{-\infty}^{\infty} \\ &= \frac{-A^2}{4a}(e^{-\infty} - e^{\infty}) \\ &= 0. \end{aligned}$$

Expectation value of x is the average value of an infinite number of measurements of the position of the particles all prepared in the same state described by the wave function $\psi(x)$.

4. **Answer 4:** we are given that,

$$\begin{aligned} \text{Mass of ball} = m &= 0.145 \text{ kg} \\ \text{Uncertainty in position} = \Delta x &= 1\ \mu\text{m} = 10^{-6} \text{ m} \\ \text{Minimum uncertainty in speed} = \Delta v &= \ ? \end{aligned}$$

According to uncertainty principle,

$$\begin{aligned} \Delta x\ \Delta p &\geq \frac{\hbar}{2} \\ \Delta x\ m\ \Delta v &\geq \frac{\hbar}{2} \\ \Delta v &\geq \frac{\hbar}{2m\Delta x}. \end{aligned}$$

Substitute values from given data, we get,

$$\begin{aligned} \Delta v &= \frac{1.05 \times 10^{-34} \text{ j.s}}{2 \times 0.145 \text{ kg} \times 10^{-6} \text{ m}} \\ &= 3.6 \times 10^{-28} \text{ m/s}. \end{aligned}$$

5. **Answer 5:**

we are given that,

$$\begin{aligned} \text{Mass of mosquito=m} &= 0.15 \text{ mg} = 0.15 \times 10^{-6} \text{ kg} \\ \text{Speed of mosquito} = v &= 50 \text{ cm/s} = 0.5 \text{ m/s} \\ \text{Uncertainty in speed} = \Delta v &= 0.5 \text{ mm/s} = 0.5 \times 10^{-3} \text{ m/s}. \end{aligned}$$

(a) According to uncertainty principle,

$$\begin{aligned} \Delta x \Delta P &\geq \frac{\hbar}{2} \\ \Delta x\ m\ \Delta v &\geq \frac{\hbar}{2} \\ \Rightarrow \quad \Delta x &\geq \frac{\hbar}{2m\Delta v}. \end{aligned}$$

Substitute values from given data we get,

$$\begin{aligned}\Delta x &= \frac{1.05 \times 10^{-34} \text{ j.s}}{2 \times 0.15 \times 10^{-6} \text{ kg} \times 0.5 \times 10^{-3} \text{ m/s}} \\ &= 7 \times 10^{-25} \text{ m.}\end{aligned}$$

(b) Since uncertainty is very small, it will never present any hindrance to the application of classical mechanics. Such a small uncertainty can be neglected.

6. **Answer 6:**

We are given that,

$$\begin{aligned}\text{Uncertainty in position of proton} = \Delta x &= 5 \times 10^{-15} \text{ m} \\ \text{Minimum speed needed to move proton} = \Delta v &= ?.\end{aligned}$$

According to uncertainty principle,

$$\begin{aligned}\Delta x \Delta P &\geq \frac{\hbar}{2} \\ \Delta x \; m \; \Delta v &\geq \frac{\hbar}{2} \\ \Rightarrow \quad \Delta v &\geq \frac{\hbar}{2m\Delta x}.\end{aligned}$$

Substitute values from given data we get,

$$\begin{aligned}\Delta x &= \frac{1.05 \times 10^{-34} \text{ j.s}}{2 \times 1.67 \times 10^{-27} \text{ kg} \times 5 \times 10^{-15} \text{ m}} \\ &= 6.28 \times 10^{6} \text{ m/s.}\end{aligned}$$

7. **Answer 7:**

(a) The electrons have a quantum field ψ as they are present in the region between the screens. This field is a superposition of two fields, ψ_1 and ψ_2 corresponding to two electrons ejecting from either of the slits. The phase difference between ψ_1 and ψ_2 results in these field interfering causing an interference pattern.

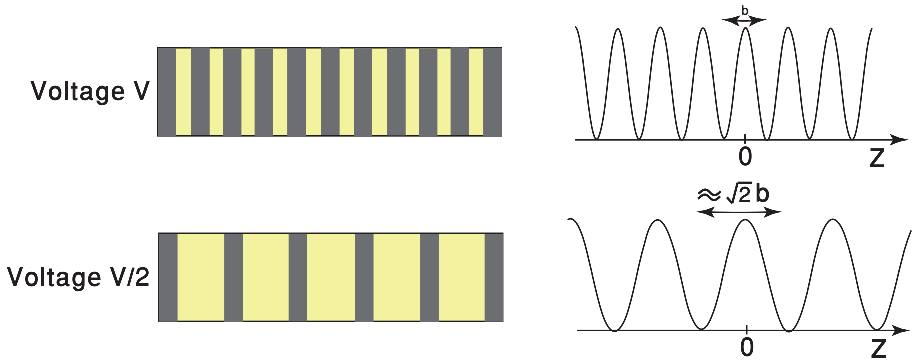
N.B. I don't like the expression "wavefunction splitting" or "electron splitting". "Electrons interfering with themselves" is fine.

(b) V is halved, so energy $eV = p^2/2m$ is halved. Hence p^2 is halved or p is reduced $\sqrt{2}$ times. Since $p = h/\lambda$, λ increases $\sqrt{2}$ times and $k = 2\pi/\lambda$ decreases $\sqrt{2}$ times.

If the voltage were V gave a wavenumber k, then a voltage $V/2$ gives a wavenumber $k/\sqrt{2}$. The interference pattern is proportional to $\cos^2(kd\sin\theta/2)$. Hence after the halving, the pattern is proportional to $\cos^2((kd\sin\theta)/2\sqrt{2})$.

Let $(kd\sin\theta)/2 = \alpha$. With voltage V, intensity pattern $\propto \cos^2(\alpha)$. With voltage $V/2$, intensity pattern $\propto \cos^2(\alpha/\sqrt{2}) = \cos^2(0.707\alpha)$.

Now a minimum is observed when $\cos^2(\alpha) = 0 \quad \Rightarrow \alpha = \pi/2$, when voltage is V. Whereas, if the voltage is $V/2$, the minimum appears at $\cos^2(0.707\alpha) = 0 \quad \Rightarrow 0.707\alpha = \pi/2 \quad \Rightarrow \alpha = \pi/2 \times 0.707$, which is larger than the previous case. Hence the fringe width increases $\sqrt{2} = 1.411$ times.



(c)

(i) $E = p^2/2m = eV$, if V is halved E is halved.

(ii) If V is halved p^2 is halved, so p is reduced $\sqrt{2}$ times.

(d) The protons gain the same energy as electrons, because they carry the same charge. But protons are heavier than electrons, so their momentum is larger ($p^2 = 2mE$), hence λ is shorter and $k = 2\pi/\lambda$ is larger. Since energy pattern $\propto \cos^2((kd\sin\theta)/2)$, a large k would result in rapid spatial variation in bright and dark fringes which will therefore be squeezed close together.