Solution Assignment 6: Quantum Field Theory

1. Show that equal time commutator for a scalar field and its time like component of the canonical momentum is given by,

$$\left[\hat{\phi}(t,\mathbf{x}),\hat{\Pi}^{\circ}(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

Answer 1

$$\begin{aligned}
\hat{\Pi}^{\circ} &= \partial^{\circ}\hat{\phi} \\
&= \partial^{t}\int \frac{d^3p}{(2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}}\left(\hat{a}_{\mathbf{p}}e^{-i(E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})}+\hat{a}^{\dagger}_{\mathbf{p}}e^{+i(E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})}\right) \\
&= \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}}(-iE_{\mathbf{p}})\left(\hat{a}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}-\hat{a}^{\dagger}_{\mathbf{p}}e^{+i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= -i\int \frac{d^3p}{(2\pi)^{3/2}(2)^{1/2}}E_{\mathbf{p}}^{1/2}\left(\hat{a}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}-\hat{a}^{\dagger}_{\mathbf{p}}e^{+i\mathbf{p}\cdot\mathbf{x}}\right). \\
\left[\hat{\phi}(t,\mathbf{x}),\hat{\Pi}^{\circ}(t,\mathbf{y})\right] &= -i\Bigg[\int \frac{d^3p}{(2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}}\left(\hat{a}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}+\hat{a}^{\dagger}_{\mathbf{p}}e^{+i\mathbf{p}\cdot\mathbf{x}}\right), \\
&\qquad \int \frac{d^3q}{(2\pi)^{3/2}(2)^{1/2}}E_{\mathbf{q}}^{1/2}\left(\hat{a}_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}}-\hat{a}^{\dagger}_{\mathbf{q}}e^{+i\mathbf{q}\cdot\mathbf{y}}\right)\Bigg] \\
&= -i\int \frac{d^3p\ d^3q}{(2\pi)^3(2)}\ \cdot\ \frac{E_{\mathbf{q}}^{1/2}}{E_{\mathbf{p}}^{1/2}}\Big(-[\hat{a}_{\mathbf{p}},\hat{a}^{\dagger}_{\mathbf{q}}]e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&+ [\hat{a}_{\mathbf{p}},\hat{a}_{\mathbf{q}}]e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\Big) \\
&= -\frac{i}{2}\int \frac{d^3p\ d^3q}{(2\pi)^3}\Bigg(\left(\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}\right)^{1/2}(-\delta^{(3)}(\mathbf{p}-\mathbf{q}))e^{+i(\mathbf{q}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&+ \left(\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}\right)^{1/2}(-\delta^{(3)}(\mathbf{p}-\mathbf{q}))e^{+i(\mathbf{q}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\Bigg) \\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\left(e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) \\
&= i\delta^{(3)}(\mathbf{x}-\mathbf{y})
\end{aligned}$$

using $\int \frac{d^3p}{(2\pi)^3}e^{\pm i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$.

2. Show the complete step-by-step working for the canonical quantization of the complex scalar field,

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$
$$\psi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2).$$

Answer 2

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2$$

(a)

$$\psi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$
$$\psi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$
$$\mathscr{L} = \partial^\mu\psi\ \partial_\mu\psi^\dagger - m^2\psi\psi^\dagger$$

(b)

$$\Pi^\mu_\sigma = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\sigma)}$$
$$\Pi^\mu_\psi = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)} \quad \text{and} \quad \Pi^\mu_{\psi^\dagger} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^\dagger)}$$
$$\text{e.g} \quad \Pi^\mu_\psi = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)} = \partial^\mu\psi^\dagger$$
$$\Pi^\mu_{\psi^\dagger} = \partial^\mu\psi$$
$$\Pi^o_\psi = \partial^o\psi^\dagger$$
$$\Pi^o_{\psi^\dagger} = \partial^o\psi$$

(c)

$$\left[\hat{\psi}(\mathbf{x},t), \hat{\Pi}^o_\psi(\mathbf{y},t)\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

$$\left[\hat{\psi}(\mathbf{x},t),\partial^o\hat{\psi}^\dagger(\mathbf{y},t)\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \quad \text{etc.}$$
$$\left[\hat{\psi}^\dagger(\mathbf{x},t),\hat{\Pi}^o_{\psi^\dagger}(\mathbf{y},t)\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

(d)

$$\begin{aligned}
\mathcal{H} &= \sum_\alpha \pi_\sigma\partial^o\sigma(x) - \mathscr{L} \\
&= \Pi^o_\psi\partial_o\psi(x) + \Pi^o_{\psi^\dagger}\partial_o\psi^\dagger(x) - \partial^\mu\psi\partial_\mu\psi^\dagger + m^2\psi^\dagger\psi \\
&= \partial^o\psi^\dagger\partial_o\psi(x) + \partial^o\psi\partial_o\psi^\dagger(x) - \partial^\mu\psi\partial_\mu\psi^\dagger + m^2\psi^\dagger\psi \\
&= \partial^o\psi^\dagger\partial_o\psi - \partial^i\psi\partial_i\psi^\dagger + m^2\psi^\dagger\psi \\
&= \partial^o\psi^\dagger\partial_o\psi + \nabla\psi\cdot\nabla\psi^\dagger + m^2\psi^\dagger\psi
\end{aligned}$$

(e) **Mode Expansion:**

$$\begin{aligned}
\hat{\psi} &= \int \frac{d^3p}{(2\pi)^{3/2}(2E_\mathbf{p})^{1/2}}\left(\hat{a}_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{b}^\dagger_\mathbf{p}e^{+i\mathbf{p}\cdot\mathbf{x}}\right) \\
\hat{\psi}^\dagger &= \int \frac{d^3q}{(2\pi)^{3/2}(2E_\mathbf{q})^{1/2}}\left(\hat{a}^\dagger_\mathbf{q}e^{+i\mathbf{q}\cdot\mathbf{x}} + \hat{b}_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right) \\
\partial_o\hat{\psi} &= \int \frac{d^3p}{(2\pi)^{3/2}(2E_\mathbf{p})^{1/2}}\left(\hat{a}_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}(-iE_\mathbf{p}) + \hat{b}^\dagger_\mathbf{p}e^{+i\mathbf{p}\cdot\mathbf{x}}(+iE_\mathbf{p})\right) \\
\partial_o\hat{\psi} &= \int \frac{d^3p}{(2\pi)^{3/2}(2)^{1/2}(E_\mathbf{p})^{1/2}}(-iE_\mathbf{p})\left(\hat{a}_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}} - \hat{b}^\dagger_\mathbf{p}e^{+i\mathbf{p}\cdot\mathbf{x}}\right) \\
\partial_o\hat{\psi}^\dagger &= \int \frac{d^3q}{(2\pi)^{3/2}(2E_\mathbf{q})^{1/2}}\left(\hat{a}^\dagger_\mathbf{q}e^{+i\mathbf{q}\cdot\mathbf{x}}(+iE_\mathbf{q}) + \hat{b}_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}(-iE_\mathbf{q})\right) \\
&= \int \frac{d^3q}{(2\pi)^{3/2}(2E_\mathbf{q})^{1/2}}(iE_\mathbf{q})\left(\hat{a}^\dagger_\mathbf{q}e^{+i\mathbf{q}\cdot\mathbf{x}} - \hat{b}_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right) \\
\Delta\hat{\psi} &= \int \frac{d^3p}{(2\pi)^{3/2}(2E_\mathbf{p})^{1/2}}\left(\hat{a}_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}(+i\mathbf{p}) + \hat{b}^\dagger_\mathbf{p}e^{+i\mathbf{p}\cdot\mathbf{x}}(-i\mathbf{p})\right) \\
&= \int \frac{d^3p}{(2\pi)^{3/2}(2E_\mathbf{p})^{1/2}}(i\mathbf{p})\left(\hat{a}_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}} - \hat{b}^\dagger_\mathbf{p}e^{+i\mathbf{p}\cdot\mathbf{x}}\right)
\end{aligned}$$

$$\begin{aligned}
\Delta\hat{\psi}^\dagger &= \int \frac{d^3q}{(2\pi)^{3/2}(2E_{\mathbf{q}})^{1/2}}\left(\hat{a}^\dagger_{\mathbf{q}} e^{+i\mathbf{q}\cdot\mathbf{x}}(-i\mathbf{q}) + \hat{b}_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}}(i\mathbf{q})\right) \\
&= \int \frac{d^3q}{(2\pi)^{3/2}(2E_{\mathbf{q}})^{1/2}}(-i\mathbf{q})\left(\hat{a}^\dagger_{\mathbf{q}} e^{+i\mathbf{q}\cdot\mathbf{x}} - \hat{b}_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}}\right) \\
\hat{\psi}^\dagger\hat{\psi} &= \int \frac{d^3q\ d^3p}{(2\pi)^3(2E_{\mathbf{q}})^{1/2}(2E_{\mathbf{p}})^{1/2}}\left(\hat{a}^\dagger_{\mathbf{q}} e^{+i\mathbf{q}\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}}\right)\left(\hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{b}^\dagger_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= \int \frac{d^3q\ d^3p}{(2\pi)^3(2)(E_{\mathbf{q}})^{1/2}(E_{\mathbf{p}})^{1/2}}\Bigg(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{+i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}} e^{+i(\mathbf{q}+\mathbf{q})\cdot\mathbf{x}} \\
&+ \hat{b}_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{-i(\mathbf{q}+\mathbf{q})\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}} e^{-i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}\Bigg) \\
&= \int \frac{d^3x\ d^3q\ d^3p}{(2\pi)^3(2)(E_{\mathbf{q}})^{1/2}(E_{\mathbf{p}})^{1/2}}(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q})\Bigg(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
&- \hat{a}_{\mathbf{p}}\hat{b}_{\mathbf{q}} e^{-i(\mathbf{q}+\mathbf{q})\cdot\mathbf{x}} - \hat{b}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}} e^{+i(\mathbf{q}+\mathbf{q})\cdot\mathbf{x}} + \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} e^{+i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\Bigg) \\
&+ \int \frac{d^3x\ d^3q\ d^3p}{(2\pi)^3(2)(E_{\mathbf{q}})^{1/2}(E_{\mathbf{p}})^{1/2}} m^2\Bigg(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{-i(\mathbf{q}+\mathbf{q})}\cdot\mathbf{x} \\
&+ \hat{a}^\dagger_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}} e^{+i(\mathbf{q}+\mathbf{q})\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}} e^{+i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\Bigg) \\
H &= \int \frac{d^3q\ d^3p}{2(E_{\mathbf{q}})^{1/2}(E_{\mathbf{p}})^{1/2}}(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q})\Bigg(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}} e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{q}-\mathbf{p}) \\
&- \hat{a}_{\mathbf{p}}\hat{b}_{\mathbf{q}} e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) - \hat{b}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}} e^{+i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&- \hat{b}^\dagger_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{q}} e^{+i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q})\Bigg) + m^2\Bigg(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&+ \hat{b}_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) + \hat{a}^\dagger_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}} e^{+i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&+ \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}} e^{+i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q})\Bigg) \\
&= \frac{1}{2}\int d^3p\ \frac{(E^2_{\mathbf{p}}+p^2)\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}}{E_{\mathbf{p}}} + \frac{m^2\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}}{E_{\mathbf{p}}} + \frac{(E^2_{\mathbf{p}}+p^2)\hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}}{E_{\mathbf{p}}} \\
&+ \frac{m^2\hat{b}_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{p}}}{E_{\mathbf{p}}} - \frac{(E^2_{\mathbf{p}}-p^2)\hat{a}_{\mathbf{p}}\hat{b}_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t}}{E_{\mathbf{p}}} + \frac{m^2\hat{b}_{-\mathbf{p}}\hat{a}_{\mathbf{p}}e^{-2iE_{\mathbf{p}}t}}{E_{\mathbf{p}}}
\end{aligned}$$

3. In the class we discussed that the non-relativistic Lagrangian density

for a complex scalar field can be written as,

$$\mathscr{L} = -\Psi^\dagger \partial_0 \Psi - \frac{1}{2m} \nabla \Psi \cdot \nabla \Psi^\dagger .$$

Apply the Euler-Lagrange equation with respect to the field $\psi^\dagger$ and derive the Schrödinger equation. Also show that the mode expansion

$$\hat{\Psi} = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}} e^{-ip\cdot x}$$

yields the conventional energy dispersion for a free particle

$$E_{\mathbf{p}} = \frac{|\mathbf{p}|^2}{2m} .$$

Answer 3

$$\mathscr{L} \;=\; i\Psi^\dagger \partial_o \Psi - \frac{1}{2m} \nabla \Psi \cdot \nabla \Psi^\dagger$$

The Euler-Lagrangian equation is,

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Psi^\dagger)} \right) &= \frac{\partial \mathscr{L}}{\partial \Psi^\dagger} \\ \frac{\partial \mathscr{L}}{\partial \Psi^\dagger} &= i \partial_o \Psi \end{aligned}$$

Likewise,

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Psi^\dagger)} &= -\frac{1}{2m} \partial^i \Psi \frac{\partial}{\partial(\partial_\mu \Psi^\dagger)} (\partial_i \Psi^\dagger) \\ &= -\frac{1}{2m} \partial^i \Psi \delta_{i,\mu} \\ \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Psi^\dagger)} \right) &= -\frac{1}{2m} \delta_{i,\mu} \partial_\mu \partial^i \Psi \\ &= -\frac{1}{2m} \partial_i \partial^i \Psi \\ &= -\frac{1}{2m} \nabla^2 \Psi \\ \text{Hence} \qquad -\frac{1}{2m} \nabla^2 \Psi &= o \frac{\partial}{\partial t} \Psi \end{aligned}$$

which is the schrodinger equation.

$$\begin{aligned}
\text{if} \qquad \hat{\Psi} &= \sum_{\mathbf{p}} a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \\
&= \sum_{\mathbf{p}} a_{\mathbf{p}} e^{-i(E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})} \\
\frac{\partial \Psi}{\partial t} &= -i \sum_{\mathbf{p}} E_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \\
\nabla^2 \Psi &= -\sum_{\mathbf{p}} |p|^2 a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + \frac{1}{2m}\sum_{\mathbf{p}} |p|^2 a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \\
&= \sum_{\mathbf{p}} E_{\mathbf{p}} a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}
\end{aligned}$$

Comparing the to sides yields $E_{\mathbf{p}} = \frac{|p|^2}{2m}$.

4. Starting with the non-relativistic lagrangian given in Q3, and the definition of the amplitude $\rho(x)$ and phase $\theta(x)$ fields, show that the lagrangian density can also be written as,

$$\mathscr{L} = \frac{i}{2}\partial_0 \rho - \rho \partial_0 \theta - \frac{1}{2m}\left[\frac{1}{4\rho}(\nabla \rho)^2 + \rho(\nabla\theta)^2\right].$$

Answer 4

$$\begin{aligned}
\mathscr{L} &= i\sqrt{\rho}e^{-i\Theta}\partial_o(\sqrt{\rho}e^{i\Theta}) - \frac{1}{2m}\nabla(\sqrt{\rho}e^{i\Theta}) \cdot \nabla(\sqrt{\rho}e^{-i\Theta}) \\
&= i\sqrt{\rho}e^{-i\Theta}(\partial_o\sqrt{\rho})e^{i\Theta} + i\sqrt{\rho}e^{-i\Theta}\sqrt{\rho}\partial_o(e^{i\Theta}) \\
&- \frac{1}{2m}(\nabla\sqrt{\rho}e^{i\Theta} + \sqrt{\rho}\nabla e^{i\Theta}) \cdot (\nabla\sqrt{\rho}e^{-i\Theta} + \sqrt{\rho}\nabla e^{-i\Theta}) \\
&= i\sqrt{\rho}\partial_o\sqrt{\rho} + i\rho e^{-i\Theta}\partial_o(e^{i\Theta}) - \frac{1}{2m}\Bigg((\sqrt{\rho}\nabla)^2 + \sqrt{\rho}\nabla\sqrt{\rho}e^{i\Theta} \cdot \nabla(e^{-i\Theta}) \\
&+ \sqrt{\rho}\nabla e^{+i\Theta} \cdot \nabla\sqrt{\rho}e^{-i\Theta} + \rho\nabla e^{i\Theta} \cdot \nabla e^{-i\Theta}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
\text{Now} \qquad \partial_o\sqrt{\rho} &= \partial_o \rho^{1/2} = \frac{1}{2\sqrt{\rho}}\partial_o\rho \\
\partial_o(e^{\pm i\Theta}) &= \pm i(\partial_o\Theta)e^{\pm i\Theta} \\
\nabla\sqrt{\rho} &= \frac{1}{2\sqrt{\rho}}\nabla\rho \quad , \quad \nabla(e^{\pm i\Theta}) = \pm i(\nabla e^{\pm i\Theta}) \\
\therefore \qquad \mathscr{L} &= \frac{i\sqrt{\rho}}{i\sqrt{\rho}}\partial_o\rho + i\rho e^{-i\Theta}(+i\partial_o\Theta)e^{+i\Theta} - \frac{1}{2m}\bigg[\frac{1}{4\rho}(\nabla\rho)^2 \\
&+ \sqrt{\rho}e^{i\Theta}\frac{1}{2\sqrt{\rho}}\nabla\rho\cdot(-ie^{-i\Theta})\nabla\Theta + \sqrt{\rho}(i\nabla\Theta)e^{+i\Theta}\cdot\frac{1}{2\sqrt{\rho}}\nabla\rho e^{-i\Theta} \\
&+ \rho i(\nabla\Theta)\cdot(-i\nabla\Theta)\bigg] \\
&= \frac{i}{2}\partial_o\rho - \rho\partial_o\Theta - \frac{1}{2m}\bigg[\frac{1}{4\rho}(\nabla\rho)^2 - \frac{i}{2}\nabla\rho\cdot\nabla\Theta + \frac{i}{2}\nabla\rho\cdot\nabla\Theta \\
&+ \rho(\nabla\Theta)^2\bigg] \\
&= \frac{i}{2}\partial_o\rho - \rho\partial_o\Theta - \frac{1}{2m}\left[\frac{1}{4\rho}(\nabla\rho)^2 + \rho(\nabla\Theta)^2\right]
\end{aligned}
$$