Solution Mid-Term: Quantum Field Theory

Attempt all questions.

1. Given the complex scalar field,

$$\mathscr{L} = (\partial^\mu \psi^\dagger)(\partial_\mu \psi) - m^2 \psi \psi^\dagger$$

Identify the symmetry transformation and find an expression for the conserved charge. (10 **marks)**

Answer 1

$$\begin{aligned} \mathscr{L} &= (\partial^\mu \psi^\dagger)(\partial_\mu \psi) - m^2 \psi \psi^\dagger \\ \Pi^\mu_\psi(x) &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)} = \partial^\mu \psi^\dagger \\ \text{and} \qquad \Pi^\mu_{\psi^\dagger}(x) &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi^\dagger)} = \partial^\mu \psi \end{aligned}$$

Under a $u(1)$ transformation,

$$\begin{aligned} \phi(x) \longmapsto \phi'(x') &= \phi(x) e^{i\alpha} = \psi(1 + i\alpha) \approx \psi + i\psi\alpha \\ \Rightarrow \qquad D\psi &= i\psi. \\ \phi(x)^\dagger \longmapsto \phi'^\dagger(x') &= \phi(x) e^{i\alpha} = \psi^\dagger(1 - i\alpha) \approx \psi^\dagger - i\psi^\dagger\alpha \\ \Rightarrow \qquad D\psi^\dagger &= -i\psi^\dagger. \\ \text{So,} \qquad J^\mu_N(x) &= \sum_\sigma \Pi^\mu_\sigma(x) D\sigma = i(\partial^\mu \psi^\dagger)\psi - i(\partial^\mu \psi)\psi^\dagger. \end{aligned}$$

Conserved charge is,

$$\begin{aligned} Q_N &= \int d^3x \; J^o_N(x) \\ &= i \int d^3x \Big[(\partial^o \psi^\dagger)\psi - (\partial^o \psi)\psi^\dagger \Big]. \end{aligned}$$

2. Consider a massless scalar field with Lagrangian density;

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi).$$

Consider the transformation;

$$x^\mu \longmapsto (x')^\mu = x^\mu e^\alpha$$
$$\phi(x) \longmapsto \phi'(x') = \phi(x) e^{-\alpha}.$$

Show that under this transformation, the action is invariant.

$$S = \int d^4x \ \mathscr{L} = \frac{1}{2}\int d^4x (\partial^\mu \phi)(\partial_\mu \phi). \ \textbf{(10 marks)}$$

<u>Answer</u> 2

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) \\
S &= \int d^4x \ \mathscr{L} = \int dt \ d^3x \ \ \mathscr{L} = \int d^4x' \ \mathscr{L}' \\
x^\mu \longmapsto (x')^\mu &= x^\mu e^\alpha \\
\phi(x) \longmapsto \phi'(x') &= \phi(x) e^{-\alpha} \\
\partial_\mu \phi &= \frac{\partial \phi}{\partial x^\mu} \longmapsto \frac{\partial \phi'}{\partial x'^\mu} = \frac{\partial \phi'}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\mu} \\
\text{Now} \quad x'^\mu &= x^\mu e^{+\alpha} \\
x'^\mu e^{-\alpha} &= x^\mu \\
\frac{x^\mu}{x'^\mu} &= e^{-\alpha} \\
\phi' &= \phi e^{-\alpha} \\
\therefore \quad \partial_{\mu'} \phi' &= \partial_\mu \phi e^{-\alpha} e^{-\alpha} = \partial_\mu \phi e^{-2\alpha}. \\
\text{So,} \quad S &= \int d^4x' \mathscr{L}' = \int d^4x e^{+4\alpha} (\partial^\mu \phi)(\partial_\mu \phi) e^{-4\alpha} = \int d^4x \mathscr{L}.
\end{aligned}$$

3. (a) We are given the relativistic theory;

$$\mathscr{L} = i\Psi^\dagger(x)\partial_o \Psi - \frac{1}{2m}\nabla\Psi^\dagger \cdot \nabla \Psi$$

use the mode expansion

$$\hat{\Psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\ \hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}$$

to canonically quantize the theory and determine the Hamiltonian. **(15 marks)**

(b) Show that using positive *and* negative frequencies in the mode expansion

$$\hat{\Psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}}\ \left(\hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^{\dagger}_{\mathbf{p}} e^{+i\mathbf{p}\cdot\mathbf{x}}\right)$$

does not satisfy the canonical commutation relations derived in part (a) above.

(c) Find the Noether current $(J^o, \mathbf{J})$. Show that $\mathbf{J}$ is the non relativistic probability current. What is the conserved charge for this system? **(10 marks)**

<u>**Answer** 3</u>

(a) The canonical quantization machinery proceeds through the following logical steps.

$$
\begin{aligned}
\Pi^{\mu}_{\psi}(x) &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)} \\
\Pi^{o}_{\Psi}(x) &= i\Psi^{\dagger} \\
\Pi^{o}_{\Psi^{\dagger}}(x) &= 0 \\
\mathcal{H} &= i\Psi^{\dagger}\partial_o\Psi - i\Psi^{\dagger}\partial_o\Psi + \frac{1}{2m}\nabla\Psi^{\dagger}\cdot\nabla\Psi \\
&= \frac{1}{2m}\nabla\Psi^{\dagger}\cdot\nabla\Psi \\
i\Big[\hat{\Psi}(t,\mathbf{x}), \hat{\Psi}^{\dagger}(t,\mathbf{y})\Big] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
\Rightarrow \qquad \Big[\hat{\Psi}(t,\mathbf{x}), \hat{\Psi}^{\dagger}(t,\mathbf{y})\Big] &= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
\hat{\Psi} &= \int \frac{d^3p}{(2\pi)^{3/2}}\ \hat{a}_{\mathbf{p}} e^{-ip\cdot x} \\
\nabla\hat{\Psi} &= \int \frac{d^3p}{(2\pi)^{3/2}}\ (+i\mathbf{p})\hat{a}_{\mathbf{p}} e^{-ip\cdot x} \\
\hat{\Psi}^{\dagger} &= \int \frac{d^3q}{(2\pi)^{3/2}}\ \hat{a}^{\dagger}_{\mathbf{q}} e^{+iq\cdot x} \\
\nabla\hat{\Psi}^{\dagger} &= \int \frac{d^3q}{(2\pi)^{3/2}}\ (-i\mathbf{q})\hat{a}^{\dagger}_{\mathbf{q}} e^{+iq\cdot x} \\
\hat{H} &= \int d^3x \mathscr{H} \\
&= \frac{1}{2m}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^3}(-i\mathbf{q})\cdot(+i\mathbf{p})\hat{a}^{\dagger}_{\mathbf{q}} e^{+iq\cdot x}\hat{a}_{\mathbf{p}} e^{+ip\cdot x} \\
&= \frac{1}{2m}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^3}(\mathbf{q}\cdot\mathbf{p})\hat{a}^{\dagger}_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{+i(q-p)\cdot x} \\
\text{Since} \qquad \int \frac{d^3x}{(2\pi)^3} e^{+i(q-p)\cdot x} &= e^{+i(E_{\mathbf{p}-E_{\mathbf{q}}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \ \ \text{etc.} \\
\hat{H} &= \frac{1}{2m}\int d^3p\ (\mathbf{p}\cdot\mathbf{q})\Big(\hat{a}^{\dagger}_{\mathbf{q}}\hat{a}_{\mathbf{p}} e^{+i(E_{\mathbf{p}-E_{\mathbf{q}}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big) \\
&= \frac{1}{2m}\int d^3p|\mathbf{p}|^2\hat{a}^{\dagger}_{\mathbf{q}}\hat{a}_{\mathbf{p}}
\end{aligned}
$$

$\hat{H} = \int d^3p \frac{|\mathbf{p}|^2}{2m} \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}}$ is the desired QFT in the normal ordered form.

(b)

$$\begin{aligned}
\left[\hat{\Psi}(t,\mathbf{x}), \hat{\Psi}^\dagger(t,\mathbf{x})\right] &= \int \frac{d^3pd^3q}{(2\pi)^3} \left[\hat{a}_{\mathbf{p}} e^{-ip\cdot x}, \hat{a}_{\mathbf{q}} e^{+iq\cdot y}\right] \\
&= \int \frac{d^3pd^3q}{(2\pi)^3} e^{-ip\cdot x} e^{+iq\cdot y} \left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\right] \\
&= \int \frac{d^3pd^3q}{(2\pi)^3} e^{-ip\cdot x} e^{+iq\cdot y} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \text{ as desired.}
\end{aligned}$$

Suppose I incorporate +ve and −ve frequencies in the mode expansion.

$$\left[\hat{\Psi}(t,\mathbf{x}), \hat{\Psi}^\dagger(t,\mathbf{x})\right] = \int \frac{d^3pd^3q}{(2\pi)^3(2E_{\mathbf{p}})^{1/2}(2E_{\mathbf{q}})^{1/2}} \Big[\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}} e^{+ip\cdot x}, \hat{a}^\dagger_{\mathbf{q}} e^{+iq\cdot y} + \hat{a}_{\mathbf{q}} e^{-iq\cdot y}\Big]$$

The only surviving terms in the commutator are:

$$\begin{aligned}
&= \int \frac{d^3pd^3q}{(2\pi)^3(2)(E_{\mathbf{p}})^{1/2}(E_{\mathbf{q}})^{1/2}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] e^{-ip\cdot x} e^{+iq\cdot y} + \left[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\right] e^{+ip\cdot x} e^{-iq\cdot y} \right) \\
&= \int \frac{d^3pd^3q}{(2\pi)^3(2)(E_{\mathbf{p}})^{1/2}(E_{\mathbf{q}})^{1/2}} \left(\delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-ip\cdot x} e^{+iq\cdot y} + \delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{+ip\cdot x} e^{-iq\cdot y} \right) \\
&= \int \frac{d^3p}{2(2\pi)^3 E_{\mathbf{p}}} \left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right)
\end{aligned}$$

Since p can take positive and negative values symmetrically, one can swap $x \longleftrightarrow y$ in the second term yielding 0. Hence the desired commutation relations are not fulfilled.

(c)

$$\begin{aligned}
\mathscr{L} &= i\Psi^\dagger\partial_o\Psi - \frac{1}{2m}\partial_i\Psi^\dagger\partial^i\Psi \\
\Pi^o_\Psi(x) &= i\Psi^\dagger \\
\Pi^o_{\Psi^\dagger}(x) &= 0 \\
\Pi^i_\Psi(x) &= -\frac{1}{2m}\partial^i\Psi^\dagger \\
\Pi^i_{\Psi^\dagger}(x) &= -\frac{1}{2m}\partial^i\Psi \\
D\Psi &= i\Psi \\
D\Psi^\dagger &= -i\Psi^\dagger \\
J^i_N &= -\frac{1}{2m}\partial^i\Psi^\dagger\Psi + i\frac{1}{2m}\partial^i\Psi\Psi^\dagger \\
\mathbf{J}_N &= \frac{i}{2m}(\partial^i\Psi\Psi^\dagger - \partial^i\Psi^\dagger\Psi) \\
&= \frac{i}{2m}(\Psi^\dagger\nabla\Psi - \nabla\Psi^\dagger\Psi) \\
J^o_N &= i\Psi^\dagger(i\Psi) = -\Psi^\dagger\Psi, \quad J_{NC} = +\Psi^\dagger\Psi \\
Q_{NC} &= \int d^3x\ \Psi^\dagger\Psi.
\end{aligned}$$

The total probability is conserved.