Autocorrelation of electrical noise: An undergraduate experiment

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An undergraduate experiment is described in which the output of a noise generator is sampled by a computer equipped with a data acquisition board, after which the autocorrelation function is calculated. If the output of the noise generator passes through a low-pass filter before it is sampled by the computer, then the autocorrelation function is a decreasing exponential function with a decay constant equal to the time constant of the low-pass filter. The theory necessary to understand this phenomenon involves basic concepts in electrical noise, the analysis of linear systems, and Fourier transforms. In this paper the theoretical background for the experiment is discussed, typical data are presented, and a full analysis of these data is described.

I. INTRODUCTION

While the study of noise in electrical circuits has been a topic in the undergraduate physics curriculum for some time, the development of modern data acquisition systems has made laboratory experiments on noise much more feasible. For example, connecting an artificial source of electrical noise to the A/D input of a computer allows the student to measure the root-mean-square amplitude of the noise and to view a histogram of observed voltage levels. If a signal of some type is added to the noise, then repeated data acquisitions and signal averaging can reduce the signal-to-noise ratio to the point where a signal buried in the noise can be recovered. Papers describing undergraduate experiments on the use of noise as a source of counting statistics[1] and on noise thermometry[2] have appeared in this journal.

The use of Fourier transforms is also an important part of the undergraduate physics curriculum. An understanding of the mathematical properties of Fourier transforms is necessary in the study of such topics as optics and quantum mechanics, and laboratory experiments in optics and acoustics have been standard vehicles to illustrate the practical importance of Fourier transforms.

It is much less common to combine an investigation of both electrical noise and Fourier transforms as a way to better understand both. Yet modern data acquisition makes this quite easy. One way this can be done is to use the autocorrelation function as a way of investigating the properties of noise. Understanding how the autocorrelation function provides information about the noise requires some knowledge of Fourier transforms and their properties. Both a theoretical paper on the autocorrelation function[3] and experimental papers on autocorrelators[4] and light statistics[5] have appeared in this journal.

We describe an experiment which uses a homemade noise generator, some resistors and capacitors, and a computer equipped with an A/D board, to calculate the autocorrelation function of "white" noise after it has passed through a low-pass filter. This experiment grew out of a desire to place some rather abstract ideas of photon correlation spectroscopy into a more accessible format. In order to appreciate the results of the experiment, one must understand the basic characteristics of noise, the action of a low-pass filter, the properties of Fourier transforms, and the technique of computer-aided data acquisition. The combination of a variety of theoretical ideas with some very straightforward uses of modern instrumentation make it quite suitable for the undergraduate curriculum.

II. THEORY

The definition of white noise is that all frequencies are contained in the noise with equal amplitudes. The mathematical way of stating this is to say that if $V_N(t)$ is the time-dependent noise voltage, then $|\tilde{V}_N(\omega)|^2$, where $\tilde{V}_N(\omega)$ is the Fourier transform of $V_N(t)$, is independent of ω in the limit of a long sampling time for the Fourier transform calculation. An interesting question is this: what is the nature of the output, $V_0(t)$, if white noise passes through a low-pass filter composed of a resistor R and a capacitor C?

This question is a standard one in linear, time-independent networks.[6,7] One way to develop the answer is to realize that the time-varying input voltage, $V_N(t)$, can be represented by a series of impulses. If one knows how the circuit responds to an impulse, then its response to a time-varying input is the sum of a series of impulse responses of the proper magnitudes and delayed by the proper times. To be more specific, let $h(t)$ be the impulse response of the circuit and let $V_N(t)$ be the input to the circuit. The output at any time t from the impulse voltage appearing at the input at time τ, $V_N(\tau)$, is just $V_N(\tau)h(t-\tau)$. To find the output due the actual input voltage $V_N(t)$, one must integrate $V_N(\tau)h(t-\tau)$ over all τ. Thus the output from the circuit is

$$V_0(t)=\int_{-\infty}^{\infty} V_N(\tau)h(t-\tau)d\tau, \tag{1}$$

which is just the convolution of the two functions $V_N(t)$ and $h(t)$. One can therefore say that the response of a circuit to an input signal is just a convolution of the impulse response function of the circuit with the input signal.

A fundamental property of Fourier transforms is the convolution theorem,[8] which states that the Fourier transform of the convolution of the two functions is the product of the Fourier transforms of the two functions being convoluted. Mathematically, this can be written

$$\tilde{V}_0(\omega)=\tilde{V}_N(\omega)\tilde{h}(\omega), \tag{2}$$

where $\tilde{V}_0(\omega)$ is the Fourier transform of $V_0(t)$ and $\tilde{h}(\omega)$ is the Fourier transform of $h(t)$.

The autocorrelation function, $Z(t)$, of the output voltage is defined to be

$$Z(t)=\int_{-\infty}^{\infty} V_0(\tau)V_0(t+\tau)d\tau. \tag{3}$$

By the correlation theorem,[9] the Fourier transform of the autocorrelation function, $\tilde{Z}(\omega)$, is given by the modulus squared of the Fourier transform of $V_0(t)$,

$$\tilde{Z}(\omega)=\tilde{V}_0^*(\omega)\tilde{V}_0(\omega)=|\tilde{V}_0(\omega)|^2. \tag{4}$$

These general ideas can be applied to a low-pass filter with noise as the input. All one must realize is that the impulse response function for an RC low-pass filter is just a decreasing exponential with a time constant equal to RC.[10] That is,

$$h(t)=h(0)\exp\left(-\frac{t}{RC}\right), \tag{5}$$

where $h(0)$ is the value of the output voltage at $t=0$ due to an impulse at $t=0$. Since

$$\tilde{h}(\omega)=\frac{h(0)RC}{1+i\omega RC}, \tag{6}$$

then by Eqs. (2) and (4) the Fourier transform of the autocorrelation function is

$$\tilde{Z}(\omega)=|\tilde{V}_N(\omega)\tilde{h}(\omega)|^2=|\tilde{V}_N(\omega)|^2\frac{[h(0)RC]^2}{1+(\omega RC)^2}$$

$$=\frac{K}{1+(\omega RC)^2}, \tag{7}$$

where $K=|\tilde{V}_N(\omega)|^2[h(0)RC]^2$. Finding the inverse Fourier transform of $\tilde{Z}(\omega)$ gives

$$Z(t)=K'\ \exp\left(-\frac{t}{RC}\right), \tag{8}$$

where $K'=K/(2RC)=\frac{1}{2}|\tilde{V}_N(\omega)|^2h^2(0)RC$. Thus the autocorrelation function of the output of a low-pass filter is a decreasing exponential with a decay constant equal to RC in the limit of a long sampling time for the autocorrelation function calculation.

It is worthwhile to consider this result in physical terms. Theoretically, white noise is random with no correlation between the noise voltage at one time and the noise voltage at another time. The autocorrelation function for such noise is zero everywhere, except at $t=0$. For this to happen, the noise voltage must change rapidly so that voltage levels separated by short times are not correlated. In order to change rapidly, the noise must contain high frequency components. But the action of a low-pass filter is to remove the high frequency components of the noise, producing an output which no longer changes rapidly. By allowing only low frequency components to pass, the filter produces noise with an autocorrelation function which does show a correlation over short time intervals, i.e., below RC, the inverse cutoff frequency of the filter.

III. EXPERIMENT

The noise generator was based on digital devices and is described in Horowitz and Hill.[11] The clock frequency was 1 MHz and the output of the noise generator repeated itself after 16.8 s. While the data acquisitions were very short compared to this length of time, multiple data acquisitions were necessary to provide reasonable statistics. These multiple acquisitions were taken over times much longer than 16.8 s, but the start of each acquisition was not dependent on the clock signal in any way. Thus the data represented many small intervals in this 16.8 s sequence of voltages chosen at random.

The computer was a MacIntosh IIfx, equipped with a National Instruments NB-MIO-16 A/D board. The data acquisition program was written in QuickBASIC™, using a library of drivers supplied with the board. The computer measured 10 200 voltages, each separated by 0.15 ms. Since the noise generator output was about 1 V of rms noise voltage around a constant voltage of approximately 4 V, the computer first found the average of the 10 200 voltage measurements and subtracted the average from each measurement. This ensured that the data to be autocorrelated were centered around zero, producing an autocorrelation function which decreased toward zero at long times.

The computer then computed the autocorrelation function for times from 0 to 30 ms by (1) shifting the sequence of the first 10 000 voltages by a number between 0 and 200, (2) multiplying these shifted voltages one-by-one with the original sequence of voltages, and (3) adding these 10 000 products together. In order to obtain smooth autocorrelation functions, many acquisitions of 10 200 voltages were made, with the resulting autocorrelation functions added to one another. No attempt was made to write an efficient program and it was not compiled before running. Under these conditions, suitable autocorrelation functions could be obtained in a little over an hour.

The autocorrelation function of the output of the noise generator with no low-pass filter attached is shown in Fig. 1. The vertical axis is arbitrary so the final autocorrelation function is normalized so that $Z(0)=1$. Notice that the autocorrelation function is zero except for $Z(0)$, indicating that the noise is white up to at least 10 kHz. Figure 2 displays the autocorrelation functions for four different values of C and the same value for R in the low-pass filter following the noise generator. The lines in Fig. 2 are least-square fits to an exponential function using 30 ms of data. Table I lists the values for R and C along with the measurements of RC from both the autocorrelation function and individual measurements of R and C. The agreement is quite good, with three of the four measured time constants falling within one standard deviation of the time constants calculated from the measured values of R and C.

IV. DISCUSSION

Since only a small amount of time is needed to perform the actual acquisition of the data, this experiment can be speeded up significantly by careful programming and compilation of the program. Although slow, QuickBASIC™ was used so that the student could write the entire program, with a minimum amount of time devoted to the actual writing and debugging of the program.

There is a subtle feature of autocorrelation functions which becomes apparent from using various programs to perform the data acquisition and calculate the autocorrelation function. The subtraction of the average voltage is done so that the asymptotic limit of $Z(t)$ at long times is zero rather than a large number related to the nonzero offset voltage of the noise generator. When one subtracts the average voltage, however, it must be kept in mind that it is not the average voltage of an infinite number of voltages, but the average of a finite number of voltages. When the autocorrelation function is calculated using data from which the average of the finite number of voltages being autocorrelated has been sub-

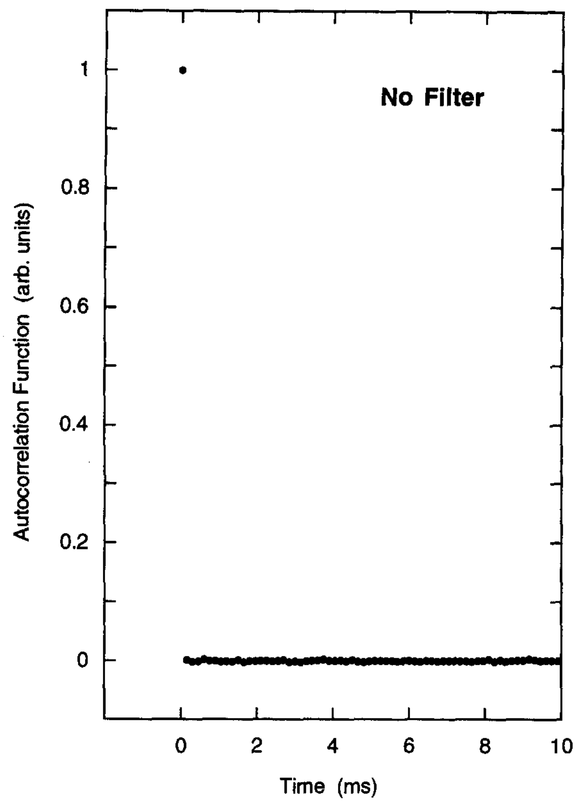

Fig. 1. Autocorrelation function of the output of the noise generator with no low-pass filter attached. The computer acquired 10 200 data points, 0.15 ms apart, calculated the autocorrelation function, and repeated this procedure 50 times, adding the autocorrelation functions together. The final autocorrelation function was normalized by dividing it by its value at $t=0$.

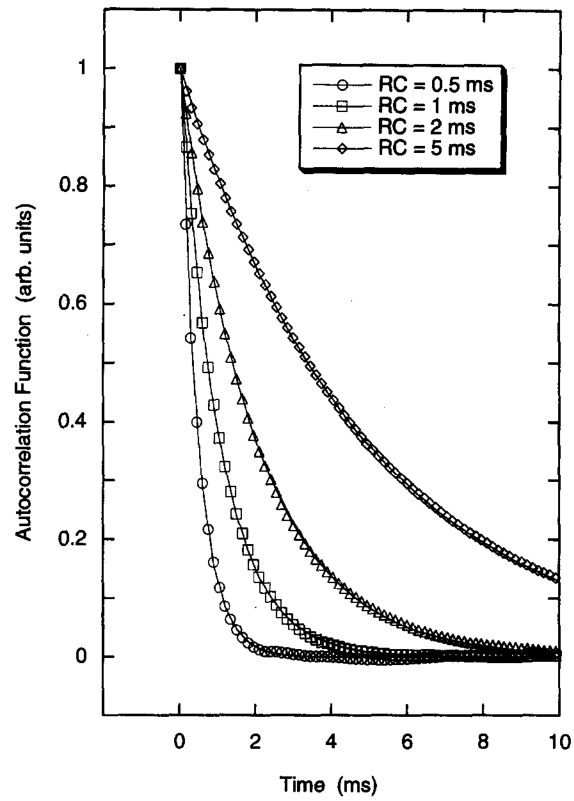

Fig. 2. Autocorrelation function of the output of the noise generator with different low-pass filters attached. The computer acquired 10 200 data points, 0.15 ms apart, calculated the autocorrelation function, and repeated this procedure 50 times, adding the autocorrelation functions together. The final autocorrelation function was normalized by dividing it by its value at $t=0$. The lines represent least-square fits to the data from 0 to 30 ms for a decreasing exponential function.

tracted, a bias is introduced. This bias can be understood in the following way. Imagine a histogram showing the number of times each voltage is measured for a very long sequence of voltages, say one million of them. Let us call this the parent distribution. Now imagine a histogram for a much shorter sequence of voltages, say 100. This distribution is a sample distribution taken from the parent distribution, and does not have the same average voltage as the parent distribution. If we subtract off the average voltage for the parent distribution from the sample distribution, then the resulting histogram, call it distribution 1, does not have an average voltage of zero. Yet this is the procedure which should be done if possible, since the average voltage for the parent distribution is more likely to be closer to the average voltage for an infinite number of voltages than the average voltage for the sample distribution. However, if we subtract off the average voltage for the smaller sample, then the resulting histogram, call it distribution 2, is likely to be more symmetric about zero and has an average voltage of exactly zero. Therefore, the computation of the autocorrelation function for distribution 2 involves more cases of opposite signed voltages multiplied together than a similar computation for distribution 1. This results in an autocorrelation function for distribution 2 (the method we used) which is slightly less than the autocorrelation function for distribution 1 (the method we would have preferred to use if possible). This negative bias is evident in our results by autocorrelation functions with asymptotic limits at long times which are slightly negative. As expected, the size of this effect decreases as the number of voltages used to compute the average voltage increases. It is therefore important to perform this experiment using data sets with large numbers of voltages, if subtraction of the average voltage is done. From our experience, 10 200 voltages are enough to ensure that this effect is insignificant.

If a programming language with a fast Fourier transform (FFT) algorithm is available, the time spent waiting while the autocorrelation function is acquired could be used to introduce the FFT algorithm for later use with the noise generator and low-pass filter. For example, the computer could be programmed to: (1) calculate the Fourier transform of 10 000 or so noise voltages using the FFT algorithm; (2) then calculate the modulus squared of the Fourier transform, which by Eq. (4) is just the Fourier transform of the autocor-

Table I. Values for the time constant of the different low-pass filters as computed from the measured values of R and C along with the results of the fitting procedure. The value of R is 10.14 ± 0.02 kΩ.

$C(\mu F)$	Computed time constant (ms)	Fitted time constant (ms)	Fitted normalization
0.048±0.001	0.49±0.01	0.491±0.002	0.999±0.002
0.105±0.002	1.06±0.02	1.054±0.003	1.001±0.002
0.204±0.005	2.07±0.04	2.026±0.006	0.994±0.002
0.515±0.016	5.22±0.16	4.909±0.011	0.997±0.002

relation function; and (3) finally calculate the inverse Fourier transform of the modulus squared of the Fourier transform, perhaps using an inverse fast Fourier transform algorithm. The result is the autocorrelation function of the noise, but this method is much faster than the direct calculation of the autocorrelation function described previously. Students will be impressed by the speed of this important numerical technique.

V. CONCLUSION

The experiment described here creates an opportunity for students to employ important concepts from electrical circuits and Fourier analysis, while gaining experience with digital electronics and modern computer-aided data acquisition. Although only one experiment is described, there are many possibilities as soon as a noise generator is connected to a data acquisition computer. Besides the noise experiments suggested in Sec. I, simple Fourier analysis of the noise, perhaps employing the FFT algorithm, would be a very illustrative exercise. Once this capacity is present, the introduction of filters would allow for further exploration of basic electric circuitry. At this point the computer would be functioning as a spectrum analyzer, which may not make sense if a spectrum analyzer is available. Still, one of the advantages in all of these experiments is that the student does all of the programming without devoting a large amount of valuable laboratory time to programming.

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[1]L. Basano and P. Ottonello, "Thermal Noise as a Source of Poisson Distributions," Am. J. Phys. **43**, 452–453 (1975).
[2]P. Kittel, W. R. Hackleman, and R. J. Donnelly, "Undergraduate Experiment on Noise Thermometry," Am. J. Phys. **46**, 94–100 (1978).
[3]P. P. Lottici, "Momentum Autocorrelation Functions from Classical Green's Functions," Am. J. Phys. **46**, 507–508 (1978).
[4]L. Basano and P. Ottonello, "New Correlator-Photon Counter to Illustrate the Fundamentals of Light Statistics," Am. J. Phys. **50**, 996–1000 (1982).
[5]S. R. Miller and M. A. Peterson, "Microcomputer as a Real-Time Digital Autocorrelator," Am J. Phys. **50**, 1129–1133 (1982).
[6]Clayton R. Paul, *Analysis of Linear Circuits* (McGraw-Hill, New York, 1989), pp. 575–586.
[7]Mischa Schwartz, *Information Transmission, Modulation, and Noise* (McGraw-Hill, New York, 1980), 3rd ed., pp. 349–354.
[8]Mary L. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, New York, 1983), 2nd ed., pp. 658–659.
[9]E. Oren Brigham, *The Fast Fourier Transform* (Prentice-Hall, Englewood Cliffs, NJ, 1974), pp. 66–69.
[10]Clayton R. Paul, *Analysis of Linear Circuits* (McGraw-Hill, New York, 1989), pp. 445–447.
[11]Paul Horowitz and Winfield Hill, *The Art of Electronics*, 2nd ed. (Cambridge University Press, Cambridge, U.K., 1989), pp. 655–664.

Heisenberg's lattice world: The 1930 theory sketch

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About 1930, physicists were increasingly frustrated about the infinities of quantum electrodynamics and the strange behavior of what were believed to be nuclear electrons. As a way out of the problems Heisenberg suggested that space be subdivided in cells of finite size, and indicated in a letter to Bohr the essence of his theory. In Heisenberg's lattice world, the electron could metamorphose into a proton, and the atomic nucleus consisted of protons and heavy "photons." We analyze Heisenberg's fascinating (but unpublished) theory in its historical context, and suggest a detailed reconstruction of the lattice world idea contained in the letter to Bohr. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

In a paper appearing in the fall of 1930, dealing with the infinite self-energy of the electron, Werner Heisenberg included the following remark[1]

> [It would seem] plausible to introduce the radius r_0 [of the electron] in such a way that space is divided into cells of finite magnitude r_0^3, and the previous differential equations are replaced with difference equations. In such a lattice world the self-energy will, at any rate, be finite. However, although such a lattice world possesses remarkable properties, one must also observe that it leads to deviations from the present theory which do not seem plausible from the point of view of experiment. In particular, the assumption that a minimal length exists is not relativistically invariant, and one can see no way to bring the demand for relativistic invariance into conformity with the introduction of a fundamental length.

Most readers of the issue of *Zeitschrift für Physik* probably found this comment rather cryptic, for Heisenberg did not give the slightest hint of either its context or how he had derived the results of a cellular space. What he had in mind was, in fact, a theory sketch which he had worked out earlier in the year and communicated to Bohr in a private letter. In the letter, Heisenberg suggested that the world is structured

