

for the ten cars for which a minimum in the standard deviation for the average optimum P/M was found, as indicated in Table I. The P/M point for the Ferrari is also shown as a special case. For modest performance requirements, that is, accelerating to 60 mph in 7 s or longer, both of the optimum power-use curves are very close to the ideal, indicating that a high initial acceleration is not important for minimizing engine size if performance requirements are modest. For more demanding performance, the two optimum curves climb and diverge rapidly, until at 4.2 s (the acceleration time for the 1991 Ferrari F40), the P_{opt}/M for 0.65 g maximum acceleration is about 170 W/kg compared with 100 W/kg for 0.9 g maximum acceleration.

In conclusion, we have demonstrated that the minimum engine size required for an automobile can be approximately determined from acceleration performance requirements based on an optimal acceleration strategy beginning with maximum acceleration without slipping, followed by acceleration determined by application of constant maxi-

imum power. Comparison of performance data with optimal performance for 18 automobiles shows that nearly all modern automobiles achieve nearly optimal acceleration.

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⁵W. Hamilton, *Electric Automobiles* (McGraw-Hill, New York, 1980), pp. 88-134.

A simple method of determining Debye temperatures

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A simple undergraduate laboratory experiment on the determination of Debye temperatures of solids is described. A sample of the solid is immersed in liquid nitrogen; from the mass of nitrogen boiled off in cooling the solid, the change in its internal energy is determined. The quantum corrections to the classical law of Dulong and Petit are calculated and the Debye temperature is obtained using a simple graphical technique.

I. INTRODUCTION

The heat capacity of a constant volume crystal lattice C_v is

$$C_v = \left. \frac{\partial U}{\partial T} \right|_v, \quad (1)$$

where U is the internal energy of the lattice, T is the temperature, and v is the molar volume. C_v can be calculated starting from the fact that a lattice of N atoms has $3N$ independent vibrational modes, each with two degrees of freedom. If all of the modes are assumed to be thermally active then, by the equipartition of energy, the average vibrational energy associated with each mode is $\langle E_j \rangle = k_B T$. The vibrational contribution to the internal energy is then just

$$U = \sum_j \langle E_j \rangle = 3nRT, \quad (2)$$

where $n = N/N_A$ is the number of moles in the crystal. This leads to the well-known result of Dulong and Petit^{1,2} for the high temperature molar heat capacity of a lattice at constant volume,

$$c_v = C_v/n = 3R. \quad (3)$$

Equation (3) is a classical result. At low temperatures the quantum mechanical nature of the lattice vibrations

becomes important and not all vibrational modes are thermally active. In this case the average vibrational energy of the j th mode must be calculated from the more exact expression

$$\langle E_j \rangle = \frac{h\nu_j}{\exp(h\nu_j/k_B T) - 1} + \frac{h\nu_j}{2}, \quad (4)$$

where ν_j is the vibrational frequency of the j th mode and depends on the crystal structure of the lattice. The second term on the right-hand side of Eq. (4) is due to the zero-point energy of the quantum oscillators. Since it is independent of temperature, it does not contribute to c_v and will be neglected below. The internal energy of the lattice cannot now be calculated analytically and the determination of c_v requires some form of approximation. One very useful approximation scheme, which is exact in the low-temperature limit, is due to Debye.^{1,2} In the Debye approximation the internal energy is

$$U = 3nRTF(\Theta_D/T), \quad (5)$$

where the function $F(t)$ is given by

$$F(t) = \frac{3}{t^3} \int_0^t dy \frac{y^3}{e^y - 1}. \quad (6)$$

Θ_D is the Debye temperature of the lattice, defined as³

$$\Theta_D = \gamma hc/ak_B, \quad (7)$$

where c is the speed of sound, appropriately averaged over all directions in the lattice, and a is the crystal lattice spacing. The coefficient γ is a factor of order 1 which depends on the structure of the lattice. The quantity $k_B\Theta_D$ can be thought of as the energy scale of the highest frequency vibrations in the lattice, which have a wavelength of order $1/2a$ and energy $E_j \approx hc/2a$. Assuming that the thermal expansion of the lattice is small, i.e., that changes in the lattice constant a do not contribute significantly to changes in the internal energy, c_v is in the Debye model

$$c_v = 3R \left(\frac{T}{\Theta_D} \right)^3 \int_0^{\Theta_D/T} \frac{y^4 e^y dy}{(e^y - 1)^2}. \quad (8)$$

At high temperatures, when $\Theta_D/T \ll 1$, $F(\Theta_D/T) = 1$ and we recover the Dulong–Petit result, while at low temperatures the lattice specific heat is proportional to T^3 , in agreement with observed behavior.

The Debye model provides a quantitative description of the thermodynamics of a lattice at low temperatures, and a qualitative description over a large temperature range; it is thus of considerable importance in understanding the properties of real solids. The crystal structure and other properties of a particular solid are absorbed into the single parameter Θ_D . The Debye temperatures of many metals are comparable with room temperature,^{1,2} indicating that the quantum mechanical nature of the lattice vibrations manifests itself at relatively high temperatures.

In the remainder of this paper we describe an experiment, the results of which can be analyzed graphically to provide a reasonably accurate determination of the Debye temperature of metals. This experiment has been used successfully in a second year undergraduate physics laboratory. A rather different technique for determining Θ_D is described in Ref. 4.

II. EXPERIMENT

The experiment is very simple. A sample of metal is cooled by immersing it in liquid nitrogen, and the amount of nitrogen which boils off is measured. This quantity, and the known initial and final temperatures of the metal, are used in the manner described below to calculate the change in internal energy of the metal. The Debye temperature is determined graphically from the intersection of a straight line, determined from the experimental data, with the function $F(t)$ as illustrated below.

The apparatus is similar to that described by Thomson and White, and used by them to determine the latent heat of vaporization of nitrogen and the “average specific heat” of metals.⁵ A small amount of liquid nitrogen, which boils at 77 K, is poured into a “dewar” consisting of a stack of two or three styrofoam coffee cups. An isothermal shield, consisting of a length of copper tubing approximately 5 cm in diameter and 5 cm long, was placed inside the innermost cup to minimize the amount of nitrogen boiled off due to splashing. The cups, along with the experimental sample, were placed on the pan of a lab balance with 0.1-g resolution. The mass was measured for a few minutes to determine the background nitrogen boil-off rate; in our set-up it was about 3 g/min. The metal sample was then placed into the nitrogen.

Boil-off of excess nitrogen, due to splashing when the metal was dropped into the dewar, was found to be a source

of systematic error in earlier versions of this experiment. We minimized this problem by putting the sample—which consisted of several cylindrical pieces of metal about 0.5 cm long by 0.5 cm in diameter—into a small nylon bag, whose heat capacity was negligible. This bag was then lowered slowly into the nitrogen.

After 1 or 2 min, the rapid boiling of the nitrogen stopped as the metal cooled to its final temperature, and the background boil-off rate was again measured for a few minutes. The mass of nitrogen m boiled off in cooling the sample is determined from a plot of the mass as a function of time. The difference between extrapolations of the data from before and after the sample was immersed gives m directly. Typical experimental data for aluminum are shown in Fig. 1.

III. ANALYSIS

The heat given up by the metal to the nitrogen as it cools is

$$\Delta Q = mL, \quad (9)$$

where m is the mass of gas boiled off and L is the latent heat of vaporization of nitrogen.⁶ Neglecting any volume change in the solid accompanying the drop in temperature, ΔQ is equal to the change in internal energy of the metal, so from Eq. (5) we have

$$\begin{aligned} \Delta Q &= U_i - U_f \\ &= 3nRT_i F(\Theta_D/T_i) - 3nRT_f F(\Theta_D/T_f), \end{aligned} \quad (10)$$

where T_i and T_f are the initial temperature (room temperature in our case) and the final temperature (77 K, the boiling point of the liquid nitrogen) of the metal. Equation (10) could be solved numerically for Θ_D . However, if $T_f \ll \Theta_D \lesssim T_i$, Θ_D can be determined approximately using a simple graphical method. If t is small, $F(t)$ can be expanded about $t = 0$ to give

$$F(t) = 1 - 3t/8 + \dots \quad (11)$$

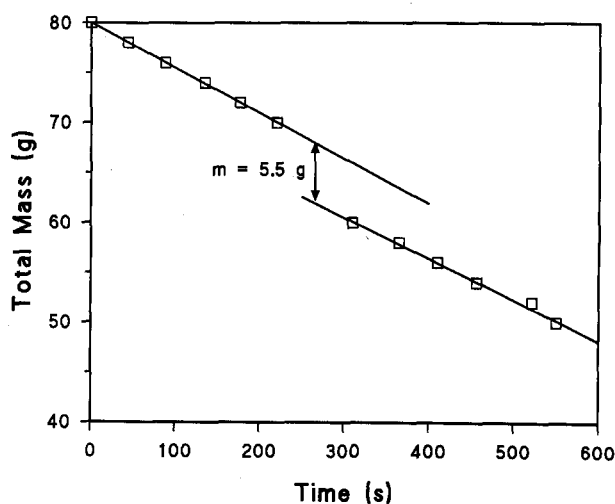


Fig. 1. Mass of the nitrogen dewar and a 7.9-g aluminum sample as a function of time. The sample was immersed in the nitrogen at about 240 s. The mass of nitrogen boiled off in cooling the sample was 5.5 g, determined from the distance between the two extrapolated background boil-off rates.

If $\Theta_D \lesssim T_i$ we can use this expansion in the first term on the right-hand side of Eq. (10) to get

$$\begin{aligned} \Delta Q &= 3nRT_i \left(1 - \frac{3\Theta_D}{8T_i}\right) - 3nRT_f F\left(\frac{\Theta_D}{T_f}\right) \\ &= 3nR(T_i - T_f) + 3nRT_f \left(1 - \frac{3\Theta_D}{8T_f} - F\left(\frac{\Theta_D}{T_f}\right)\right). \end{aligned} \quad (12)$$

In fact $\Theta_D \sim T_i$ for the metals used in our experiments, so this approximation may not be strictly valid; this point is discussed briefly below. The first term on the right-hand side of Eq. (12) can be identified with the classical contribution to the internal energy of the lattice vibrations,

$$\Delta U_{cl} = 3nR(T_i - T_f). \quad (13)$$

The remaining terms on the right-hand side of Eq. (12) are quantum corrections to the classical result. We therefore write

$$\Delta U_q = 3nRT_f \left(1 - \frac{3\Theta_D}{8T_f} - F(\Theta_D/T_f)\right). \quad (14)$$

Defining $t = \Theta_D/T_f$, Eq. (12) can be rewritten as

$$\alpha - \frac{3}{8}t = F(t), \quad (15)$$

where the quantity α is defined as

$$\alpha = 1 + (\Delta U_{cl} - \Delta Q)/3nRT_f \quad (16)$$

and may be calculated from the measured value of ΔQ and the known temperatures T_i and T_f . The solution of Eq. (15) is the intersection of the curve $F(t)$, which was calculated numerically, with the line $\alpha - 3t/8$. Thus Θ_D can be determined by drawing a straight line of slope $-3/8$ and intercept determined from the experimental measurements using Eq. (16), on a plot of $F(t)$.

IV. RESULTS

We performed experiments on three metals: aluminum (99.999% pure), copper (99.9%), and (white) tin (99.75%).⁷ The Debye temperatures of these metals, taken from Ref. 2, are given in Table I in the column labeled Θ_D^* . Table I also shows our experimental results. The quoted uncertainties in ΔQ , α and the experimental values of Θ_D reflect a (random) uncertainty of about 2% in the measurement of m , the mass of nitrogen boiled off in cooling the metal. We give two experimental values of Θ_D in the table. $\Theta_D^{(1)}$ was found from a numerical solution of Eq. (10), with no approximations. $\Theta_D^{(2)}$ was found using the approximate graphical procedure described above and illustrated in Fig. 2, which shows a plot of $F(t)$ and $\alpha - 3t/8$ for our three samples. As can be seen from the data in the table, our values of $\Theta_D^{(1)}$ are in good agreement, within experimental error, with the textbook values Θ_D^* .¹⁻³ The ap-

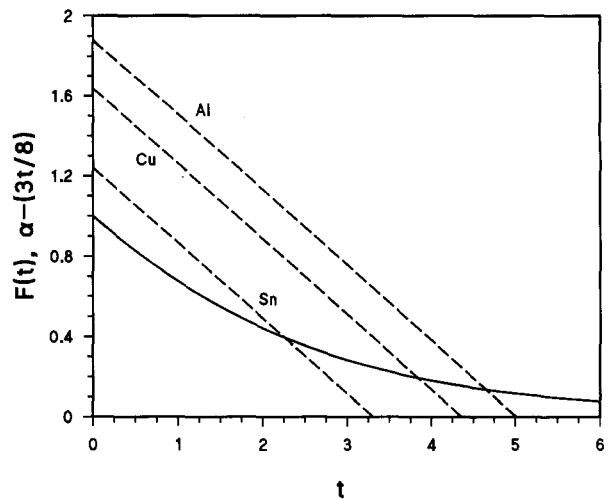


Fig. 2. A plot of $F(t)$ and $\alpha - 3t/8$ vs t , where $t = \Theta_D/T_f$, for samples of aluminum, copper, and tin. α is determined from the experimental data as described in the text. $\Theta_D^{(2)}$ is determined from the intersection of the straight line with the $F(t)$ curve.

proximation of Eq. (11), which allows the simple graphical determination of $\Theta_D^{(2)}$, gives results which are roughly 20% lower than both $\Theta_D^{(1)}$ and Θ_D^* .

We note that, strictly speaking, the approximation of Eq. (11) is valid for $\Theta_D/T_i \ll 1$, while $\Theta_D/T_i \sim 1$ for both aluminum and copper. It is possible to improve the accuracy of the results obtained from the graphical determination of the Debye temperature by replacing this approximation with a linear interpolation that gives a more accurate approximation to the function $F(t)$ in the temperature range of interest. However, the overall accuracy of the experimental technique and the inherent uncertainty involved in the definition of Θ_D (Ref. 3) do not justify such a procedure.

V. CONCLUSIONS

We have described a simple procedure for determining the Debye temperatures of solids, suitable for use in an undergraduate laboratory. The theory, and the analysis of the experimental results, involve the use of several important concepts from a first course in thermodynamics. These include heat capacity and latent heat, the equipartition of energy, lattice vibrations, and the notion that quantum mechanical effects can have a measurable influence on the properties of materials even near room temperature. The experiment, although relatively simple, must be carried out with some care if meaningful results are to be obtained, and the graphical solution of Eq. (15) is a nice illustration of how some careful thinking can eliminate the need for lots of

Table I. Experimental results.

Material	Θ_D^* (K)	Mass (g)	ΔQ (J)	ΔQ (J/mol)	ΔU_q (J/mol)	α	$\Theta_D^{(1)}$ (K)	$\Theta_D^{(2)}$ (K)
Aluminum	428	7.91	1100(20)	3750(75)	1696	1.88(4)	456(10)	345(10)
Copper	343	22.64	1500(30)	4250(85)	1225	1.64(3)	354(10)	272(10)
Tin	200	21.42	900(20)	4990(100)	463	1.24(2)	191(10)	162(10)

messy and uninteresting calculation, at the expense, in this case, of some experimental accuracy. Our results for $\Theta_D^{(1)}$ agree within our experimental error with values given in the literature,² while the approximate results $\Theta_D^{(2)}$ are about 20% lower.

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¹N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976), pp. 426–430 and 452–461.

²M. A. Omar, *Elementary Solid State Physics* (Addison-Wesley, Reading, MA, 1975), pp. 75–86.

³There is a certain amount of arbitrariness in the definition of Θ_D . The values we use, taken from Ref. 2, were determined from fits of experimental heat capacity data covering a wide range of temperature to the

Debye specific heat, Eq. (8). One could also define Θ_D from a fit to specific heat data in the low-temperature regime, where $c_v \sim T^3$, as is done in Ref. 1. Some authors (see the discussion in Ref. 1) allow Θ_D to vary with T . The values in Ref. 2 are most appropriate to the temperature range of our experiment. We emphasize that the Debye model, despite its utility, gives only an approximation to the true specific heat.

⁴T. W. Listerman and X.-L. Zhou, "Determining the temperature-dependent characteristic temperature of beryllium from electrical resistance measurements," *Am. J. Phys.* **53**, 460–463 (1985).

⁵C. W. Thomson and H. W. White, "Latent heat and low-temperature heat capacity experiment for the general physics laboratory," *Am. J. Phys.* **51**, 362–364 (1983).

⁶Energy is also required to heat the nitrogen gas from its boiling point to room temperature. Most of this energy comes from the surroundings and not from the metal sample. Although it may be possible for some bubbles of nitrogen gas to nucleate on the metal surface and boil off at temperatures greater than 77 K, we do not believe this effect to be significant, and for the purposes of an undergraduate laboratory it may be disregarded. To a good approximation, the gas is boiled off at 77 K, after which it is no longer in thermal contact with the metal, which is completely immersed in the liquid nitrogen in the dewar.

⁷The metals were obtained from Goodfellow Metals, Ltd., Cambridge, England.

Fourier series and the Gibbs phenomenon

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The occurrence in Fourier series of an overshoot effect near function discontinuities, called the Gibbs phenomenon, is discussed from a pedagogical viewpoint. The reader is led along a path to discover why the phenomenon depends only upon the existence of the discontinuity, but not on other properties of the function that is Fourier analyzed.

I. INTRODUCTION

An understanding of Fourier series and their generalizations is important for physics and engineering students, as much for mathematical and physical insight as for applications. Students are usually confused by the so-called Gibbs phenomenon—the persistent discrepancy, an “overshoot,” between a discontinuous function and its approximation by a Fourier series as the number of terms in the series becomes indefinitely large. Although the phenomenon is mentioned under Fourier series in almost every textbook of mathematical physics,¹ the treatment is often confined to the square-pulse example, so that students are often left wondering what aspect of this pulse gives rise to the phenomenon and whether it depends upon the function investigated. The subject has also been discussed in this Journal,^{2–4} but only from a limited perspective. The aim of this paper is to lead the reader along the steps to solving the mystery of the Gibbs phenomenon.

Historically, the explanation of the Gibbs phenomenon is usually attributed to one of the first American theoretical physicists, J. Willard Gibbs, in two notes published in 1898 and 1899 (Ref. 5). Gibbs was motivated to make an excursion into the theory of Fourier series by an observation of Albert Michelson that his harmonic analyzer (one of the

first mechanical analog computers) produced persistent oscillations near discontinuities of functions that it Fourier analyzed, even up to the maximum harmonic (80) that the machine could handle. Examples of these oscillations are shown in the 1898 paper⁶ by Michelson and Stratton. The phenomenon had, however, already been observed and essentially explained by the English mathematician Henry Wilbraham 50 years earlier⁷ in correcting a remark by Fourier on the convergence of Fourier series. It might be more appropriate to call it the Wilbraham–Gibbs phenomenon than the Gibbs phenomenon.

The first extensive generalization of the Gibbs phenomenon, including the conditions for its existence, was provided by the mathematician Bôcher in 1906 (Ref. 8). Both this treatment and those in subsequent mathematical treatises on Fourier series⁹ are at an advanced level, usually unsuitable for physics students. In the following we consider by a method accessible to physics and engineering students the problem of Fourier series for functions with discontinuities. The method is rigorous; it contains the essence of the mathematical treatments without their complexity; and it discusses how to estimate the overshoot numerically.

The presentation is organized as follows. In Sec. II a generalization of the sawtooth function is made, to include