[14]See, e.g., J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 225.
[15]See, Ref. 14, p. 188.
[16]If a field satisfies a hyperbolical equation, like the wave equation, then one says that it propagates hyperbolically. It can be shown that the field **E** in Eq. (5), after specifying its sources, satisfies a wave equation.
[17]An *alternate* demonstration of this theorem is found in Ref. 5. It should be noted that Eq. (13) of Ref. 5 is equivalent to Eq. (17) of the present paper.
[18]J. A. Heras, "Comment on 'alternate derivation of Maxwell's source equations from gauge invariance of classical mechanics' by James S. Marsh," Am. J. Phys. **62**, 949–950 (1994).
[19]This other extension of Helmholtz's theorem has been discussed by E. Kapuscik, "Generalized Helmholtz theorem and gauge invariance of classical field theories," Lett. N. Cim. **42**, 263–264 (1985); and also recently by the present author in Ref. 8. For a version of this theorem in relativistic notation see D. H. Kobe, "Helmholtz's theorem for antisymmetric second-rank tensor fields and electromagnetism with magnetic monopoles," Am. J. Phys. **52**, 354–358 (1984); and also J. A. Heras, "A short proof of the generalized Helmholtz theorem," *ibid.* **58**, 154–155 (1990).
[20]It should be noted that the primes of the time derivatives in Eq. (25) [and also in Eq. (33)] have been dropped. From the definition of retarded time $t'=t-R/c$, it follows that $\partial f(t')/\partial t=\{\partial f(t')/\partial t'\}\{\partial t'/\partial t\}=\partial f(t')/\partial t'$ for an arbitrary function $f(t')$, i.e., a change with respect to t' is equal to a change with respect to t at any point.

Two student experiments on electrical fluctuations

Yaakov Kraftmakher
Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

(Received 5 October 1994; accepted 24 January 1995)

Variants of student experiments on electrical fluctuations are described. Thermal noise is measured over a wide temperature range using incandescent lamps. A common resonant circuit serves as the load of a thermionic diode when measuring shot noise. The quality factor of this resonant circuit must be measured while the diode current passes through it. Boltzmann's constant and the electron charge are evaluated from the data obtained. Finally, a simple circuit is recommended as a rms voltmeter for measuring noise voltages. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

Electrical fluctuations are an example of fluctuation phenomena which play an important role in modern physics and technology. At present, measurements of electrical fluctuations are quite accessible to undergraduate students.

To study quantitatively the electrical fluctuations, one has (i) to employ an amplifier of proper sensitivity and frequency band; (ii) to determine the frequency response of the amplifier; (iii) to measure correctly the mean square of the fluctuation voltage. The amplifier's gain has to be of the order of 10^5 for measuring thermal noise and of 10^4 for studying shot noise. The appropriate frequency band is 10–100 kHz. Low frequencies are excluded to avoid $1/f$ noise and low-frequency interference. High frequencies are undesirable since the frequency band of the measured noise is restricted by the time constant governed by the resistance of the noise source and the capacitance of the wiring and amplifier's input that is of the order of 10^{-10} F. The best tool for measuring the noise is a "true rms" voltmeter. This term means that rms values of a voltage are measured, regardless of its wave form.

When measuring a sum of two or more independent (uncorrelated) noise voltages, the mean square of the sum is equal to the sum of the mean squares of the voltages. This evident rule is important for all measurements of electrical fluctuations.

The aim of this paper is to show how electrical fluctuations can be observed in a simple manner compatible with good results for Boltzmann's constant and the electron charge. Measurements of thermal noise over a wide temperature range are performed by using incandescent lamps. For the study of shot noise, a setup employing a resonant circuit as the load of a vacuum diode and an oscilloscope as the amplifier is described. In this case, the quality factor of the resonant circuit must be measured while the current passes through it. Finally, a simple method is proposed for true rms measurements. The paper can be regarded as an extension of previous work.[1–4]

II. THERMAL NOISE

The measurements employ two sources of thermal noise. First, filaments of vacuum incandescent lamps (28 V, 40 mA) are utilized. To demonstrate the nature of thermal noise, it is very useful to vary the temperature of the resistor in a wide range. When heating the filaments with a current, there is no equilibrium between the sample and its surroundings but Nyquist's theorem is still valid. To avoid superfluous calculations, two similar lamps are connected in series relative to a dc source and in parallel relative to the input of an amplifier (Fig. 1). Hence, the noise source is to be considered as a resistor of a quarter of the resistance of two filaments connected in series. An amplifier, PAR model 124A, in the selective mode (the range 10 μV, f_0=50 kHz, Q=2 or 5) is used to amplify the noise. Values of Q correspond to selectivity of the amplifier set by a switch.

The output voltage of the amplifier proceeds to the input of an oscilloscope, Kenwood CS-4025, through a low pass filter (R=1 kΩ, C=2200 pF). The filter reduces the gain at high frequencies. The oscilloscope has an output terminal of one amplification channel, with gain 0.02 (the range 5 V/cm) to 100 (the range 1 mV/cm). The input of a true rms voltmeter, Keithley 196 digital multimeter, is connected to this out-

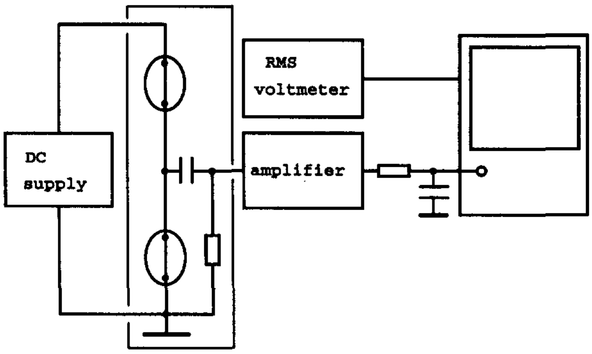

Fig. 1. Circuit to measure thermal noise.

put terminal. The gain of the oscilloscope is set as 10 (the range 10 mV/cm). The mean square of the amplified thermal noise is

$$\langle V^2\rangle = 4k_B RTG_0^2 \int_0^\infty (G/G_0)^2 df = 4k_B RTG_0^2 B, \quad (1)$$

where k_B is Boltzmann's constant, R is the resistance, T is the absolute temperature, G is the total gain depending on the frequency, G_0 is the maximum total gain, and $B=\int_0^\infty (G/G_0)^2 df$ is called the noise equivalent bandwidth. The frequency response of the amplifier is measured using a function generator (Newtronix 200P) and a digital frequency meter. These measurements are performed at low gain of the amplifier (the range 100 mV corresponds to nominal gain equal to unity). From numerical integration, the noise equivalent bandwidth appeared to be 33.3 kHz for $Q=2$ and 15.2 kHz for $Q=5$ (Fig. 2). To determine the maximum total gain, the internal attenuators of the generator are employed (−20 and −40 dB). The gain of the oscilloscope can be changed in a wide range, and the total gain is determined with proper steps. In our case, G_0^2 is 4.85×10^9 (the nominal value 10^{10} is reduced by the RC filter).

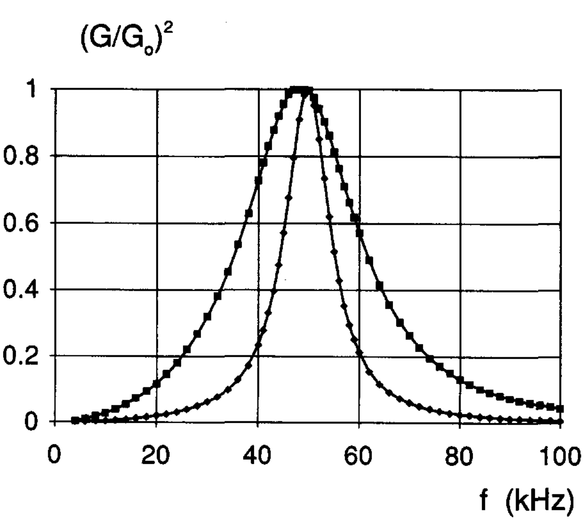

Fig. 2. Frequency response of the setup: (■) $Q=2$, (♦) $Q=5$.

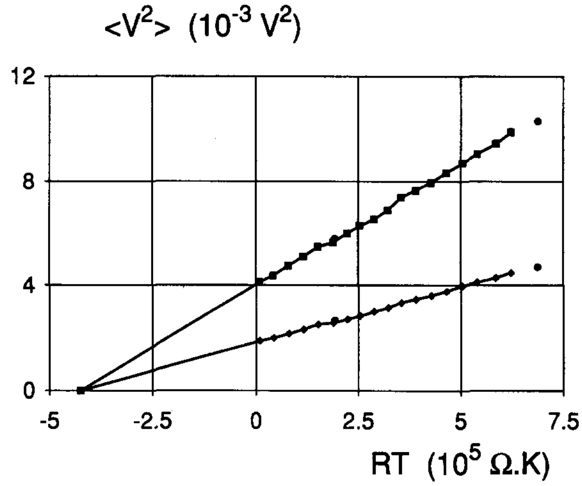

Fig. 3. Thermal noise of the incandescent lamps: (■) $Q=2$, (♦) $Q=5$; (●) noise of the resistor.

The resistance of the filaments is evaluated from the voltage drop across them and the heating current. The absolute temperature of the filaments can be deduced from the relation

$$T = 103 + 207(R/R_{293}) - 1.8(R/R_{293})^2, \quad (2)$$

where R is the resistance of the filaments at temperature T, and R_{293} is the resistance at 293 K. This expression fits data recommended by Roeser and Wensel[5] (1500–3600 K) and by Kohl[6] (293–1800 K) and is valid in the range 400–2500 K.

The $\langle V^2\rangle$ values obtained with both frequency bands are plotted as a function of RT (Fig. 3). The plots are straight lines with slopes equal to $4k_B G_0^2 B$. The values of Boltzmann's constant obtained with the least-squares method are 1.46×10^{-23} J K^{-1} ($Q=2$) and 1.44×10^{-23} J K^{-1} ($Q=5$). Due to the inherent noise of the amplifier, the lines cross the X axis at a negative value of RT. The inherent noise is specified by so-called equivalent noise resistance. This term refers to the resistance which at room temperature would generate thermal noise equal to the inherent noise of the amplifier. The equivalent noise resistance of the amplifier can be easily calculated from the data.

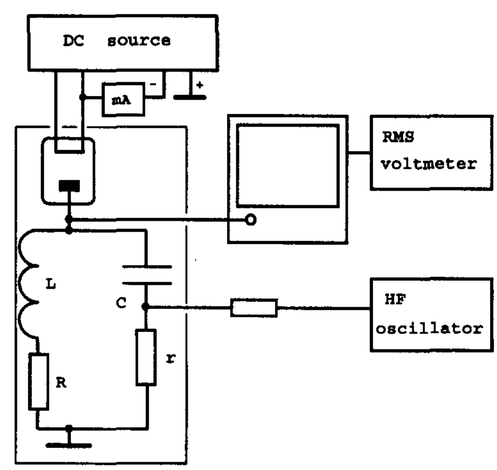

Fig. 4. Circuit to measure shot noise across a resonance circuit.

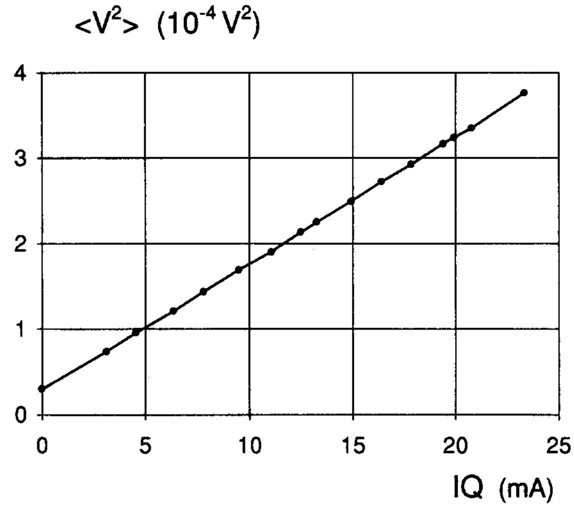

Fig. 5. Mean square of the noise voltage across the resonance circuit.

Second, thermal noise of a common resistor (2.4 kΩ) is measured at room temperature and at liquid nitrogen temperature. A metal box containing the resistor can be immersed in a Dewar filled by liquid nitrogen. The measurements are performed only at two temperatures but those are well known. The resistance is measured at both temperatures. The gain and the frequency band of the amplifier are the same as when measuring the noise of the incandescent lamps, so the data can be presented together. The values of Boltzmann's constant calculated from the noise of the resistor at the two temperatures are 1.41×10^{-23} J K^{-1} ($Q=2$) and 1.40×10^{-23} J K^{-1} ($Q=5$). The results show that employment of an amplifier with relatively high inherent noise causes no troubles in measurements of the thermal noise.

The data confirm the validity of Nyquist's theorem regardless of the nature of the resistor. This theorem is the basis of the noise thermometry: one compares noise voltages of two resistors one of which is kept at the temperature to be measured and the other has a known temperature.

III. SHOT NOISE

A vacuum diode 1B3GT is used for the measurements of shot noise (however, other types are also acceptable). A parallel LCR circuit is employed as the load of the diode (Fig. 4). The noise voltage across the circuit is amplified with the oscilloscope employing gain $G=100$ (range 1 mV/cm). The mean square of the amplified noise across a load with an impedance $Z(f)$ is

$$\langle V^2\rangle=2eI\int_0^\infty G^2|Z(f)|^2df, \tag{3}$$

where e is the electron charge, and I is the mean value of the current. The impedance of a parallel LCR circuit is

$$Z(\omega)=(R+j\omega L)/(1-\omega^2LC+j\omega RC), \tag{4}$$

where $\omega=2\pi f$ is the angular frequency. Let us denote $\omega/\omega_0=x$, $1/LC=\omega_0^2$, and $\omega_0L/R=Q$. Taking into account that $Q\gg1$, only values of x close to unity are significant, and one obtains

$$|Z(x)|^2=R^2Q^2x^2/[(1-x^2)^2+x^2/Q^2]. \tag{5}$$

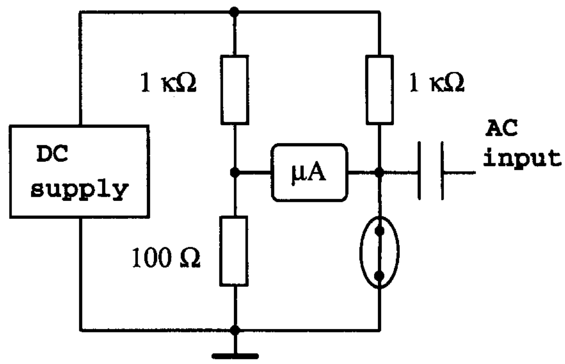

Fig. 6. Bridge circuit to measure true rms values.

It is now useful to introduce a new variable, $y=Q(1/x-x)$. Then, $dy=-Q(1/x^2+1)dx\simeq-2Q\,dx$, so

$$\langle V^2\rangle=(eIQG^2/2\pi\omega_0C^2)\int_{-\infty}^{\infty}dy/(y^2+1)$$
$$=eIQG^2/2\omega_0C^2. \tag{6}$$

The total capacitance C is evaluated from the inductance L and the resonance frequency f_0 ($f_0=67.05$ kHz). The coil has no magnetic core, so the inductance does not depend on a dc current passing through it and can be measured with any accessible method. The simplest way is to determine resonance frequencies of a circuit formed by the coil and known capacitors connected in series. In our case, $L=15.6$ mH.

Even in a temperature-limited regime, the current of a thermionic diode increases along with the applied voltage because of field dependence of the work function (Schottky's effect). This means a finite value of the internal resistance of the diode which shunts the resonance circuit. For this reason, the quality factor, Q, has to be measured when the current of the diode passes through the circuit. The measurements are based on the frequency response of the circuit: $Q=f_0/(f_2-f_1)$ where f_2 and f_1 are frequencies for which the response is $\sqrt{2}$ times smaller than at the resonance frequency f_0. A voltage drop across the resistor $r=1$ Ω serves

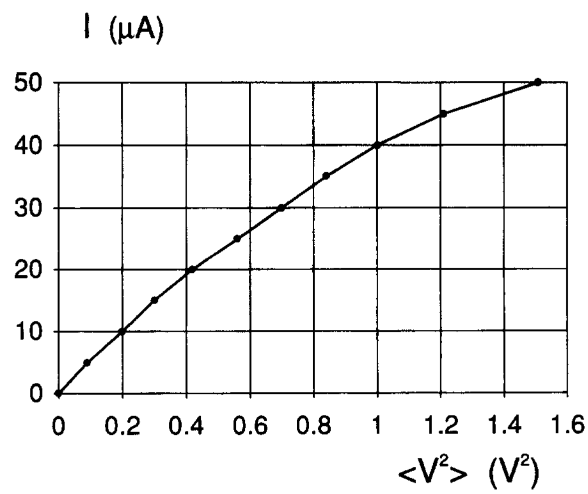

Fig. 7. Calibration curve for determinations of rms values.

here as the driving voltage. It is set with a generator Newtronix 200P. Under these measurements, the voltage is much larger than the noise. Fine adjustments of the frequency are achieved by electrical control of the generator. For this purpose, a small regulated dc voltage is fed to the sweep terminal of the generator. The dependence of the quality factor Q upon the diode current is necessary for further calculations. The plot $\langle V^2 \rangle$ vs IQ is a straight line with a slope equal to $eG^2/2\omega_0 C^2$ (Fig. 5). The value of the electron charge obtained with the least-squares method is 1.63×10^{-19} C.

IV. MEASUREMENT NOTES

Measurements of electrical fluctuations are accompanied by noticeable errors. Determinations of Boltzmann's constant and of the electron charge require measurements of several quantities, and the total error includes all the corresponding contributions. It is difficult to obtain an error in the determination of Boltzmann's constant by these methods of less than 5%. A somewhat smaller error may be claimed for the electron charge. Such errors in these fundamental constants are quite acceptable for student works.

To measure noise voltages, a simple and inexpensive device can be recommended. Three resistors and a low-power incandescent lamp (6 V, 50 mA) form a bridge circuit fed by a regulated dc current (Fig. 6). A dc microammeter (50 μA, 4 kΩ) serves as an indicator. Before the measurements, the feeding current is adjusted to balance the bridge. The temperature of the filament is of about 1000 K. Then the ac voltage to be measured is applied to the lamp. Due to changes in the temperature and resistance of the filament, a dc voltage appears at the output terminals of the bridge. The current measured by the microammeter is a function of the mean square of the ac voltage, regardless of its wave form. The circuit can be calibrated using a sine voltage and a usual ac voltmeter, and this calibration will be valid for any other ac voltage including noise. The curvature in the calibration line (Fig. 7) does not mean that the tool responds to anything other than the mean square of the applied voltage. A disadvantage of this device is that the sensitivity is not sufficient to measure the voltages obtainable at the output terminals of the employed PAR amplifier or of the Kenwood oscilloscope. An additional amplifier allowing for output voltages up to 1 V is therefore necessary.

ACKNOWLEDGMENT

The author gratefully acknowledges the support by the Ministry of Absorption of Israel.

[1]J. A. Earl, "Undergraduate experiment on thermal and shot noise," Am. J. Phys. **34**, 575–579 (1966).

[2]D. L. Livesey and D. L. McLeod, "An experiment on electronic noise in the freshman laboratory," Am. J. Phys. **41**, 1364–1367 (1973).

[3]P. Kittel, W. R. Hackleman, and R. J. Donnelly, "Undergraduate experiment on noise thermometry," Am. J. Phys. **46**, 94–100 (1978).

[4]W. T. Vetterling and M. Andelman, "Comments on: Undergraduate experiment on noise thermometry," Am. J. Phys. **47**, 382–384 (1979).

[5]W. F. Roeser and H. T. Wensel, in *Temperature, Its Measurement and Control in Science and Industry* (Reinhold, New York, 1941), pp. 1293–1323.

[6]W. H. Kohl, *Materials and Techniques for Electron Tubes* (Chapman and Hall, London, 1962), p. 277.

δ well with a reflecting barrier

Claude Aslangul[a)]

Groupe de Physique des Solides, Tour 23, 2 Place Jussieu 75251 Paris, Cedex 05 France

(Received 1 September 1994; accepted 2 January 1995)

The influence of a reflecting barrier on the eigenstates of a Dirac potential well is analyzed in detail. This simple case provides a good example for discussing in a very simple way several basic features of quantum mechanics. First, it is shown that the presence of the barrier entails a destabilization of the bound state and can even suppress it; this is expected on physical grounds since the barrier acts in a repulsive way. Second, an image method is used to recover the results obtained in the conventional way. As regards the unbound states, it is seen that the presence of the barrier suppresses the twofold degeneracy always present in one dimension. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

The Dirac potential $\delta(x-x_0)$ is a convenient representation of a localized (short-ranged) potential when the energy scale of the latter is much larger than any other energy and when its spatial extension is much smaller than any other length, as the de Broglie wavelength of the particle subject to this potential. Besides its physical relevance, the δ-potential well is a part of any elementary course on quantum mechanics since it provides a simple illustration of basic tools and methods.

I consider a potential which consists of a Dirac well and a perfectly reflecting barrier at a distance L apart (see Fig. 1). Indeed, the potential energy $V(x)$ has the form:

$$V(x)=g\delta(x-L) \quad \text{if } x>0 \quad V(x)=+\infty \quad \text{if } x<0, \tag{1}$$

where g is a negative quantity, an obvious necessary (but not sufficient, see below) condition for having bound states.