Tracking a Harmonic Oscillator using a webcam[*]

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Is there any thing that in this universe that does not oscillate? Look at things around you. From galaxies and stars in the skies to the inner workings of an atom, we observe never-ending motion. In this experiment we are going to study the simplest form of periodic motion— namely simple harmonic motion (SHM) using a webcam and image processing algorithms.

KEYWORDS Simple Harmonic Motion · Amplitude · Acceleration ·Angular Frequency · Damping

1 Conceptual Objectives

In this experiment, we will,

1. understand simple harmonic motion and its damping under various conditions;

2. start making simple measurements of lengths and understand the parallax error;

3. understand how errors propagate from an observed to an inferred quantity;

4. identify dimensions and units for simple physical quantities and transform between physical and logical coordinates;

5. fit experimentally observed curves with mathematically modelled solutions;

6. perform simple image processing and computational tasks on the personal computer; and last,

7. understand the formation and display of colour on TV and computer screens.

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2   Experimental Objectives

The aim of the present experiment is to examine the amplitude of a freely oscillating as well as an *underdamped harmonic oscillator*. The experiment requires measuring the damping constant and then make a quantitative comparison of theory with experimental results.

3   Theoretical Introduction

Stand against a light source. Tie a tennis ball to a string and whirl it over your head in a horizontal circle. Observe the shadow on the wall that is opposite to the light source. You will see linear motion of the shadow which is slowest on the edges (where it turns around) and fastest in the centre. This behaviour, slowing down when moving away from the centre and speeding up when approaching the center, is the signature of *simple harmonic motion* (SHM).

3.1   Characteristic Equation of SHM

Now let's analyze the above example in somewhat mathematical detail. The motion of the shadow of the ball cast on the wall can be described by,

$$x(t) = x_0 \cos(\omega t + \phi) \tag{1}$$

where $x(t)$ is the position of the shadow on the wall, x_0, called the amplitude, is the maximum distance of the shadow from the center, ω the rate at which you are rotating the ball, t is the time and ϕ, the phase, is the deviation of the wave from a reference. (Also see Fig. 1.)
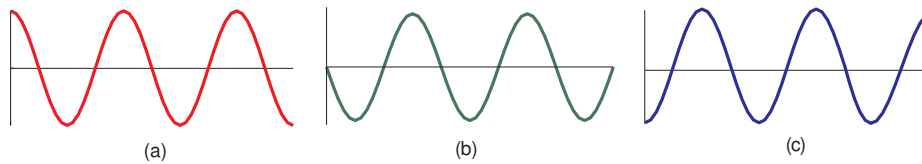

(a)   (b)   (c)

Figure 1: Position, velocity and acceleration in SHM. (a) The position x of the shadow, (b) the slope of x gives the velocity v and (c) the slope of v gives the acceleration a.

The gradient (slope) of equation (1) gives the velocity of the shadow and the sign shows the direction of the velocity, the negative means that as x increases, v decreases. So the shadow has maximum velocity at $x = 0$ and minimum velocity (in fact momentarily zero) at $x = x_0$.

$$v(t) = \frac{dx(t)}{dt} = -\omega x_0 \sin(\omega t + \phi). \tag{2}$$

This notion of changing velocity can also be expressed in terms of the acceleration, given by the gradient (slope) of equation (2).

$$a(t) = \frac{d^2x(t)}{dt^2} = \frac{d\,v(t)}{dt} = -\omega^2 x_0 \cos(\omega t + \phi). \tag{3}$$

We can replace $x_0 \cos(\omega t + \phi)$ in the above equation with equation (1) and as a result obtain,

$$a(t) = \frac{d^2x(t)}{dt^2} = -\omega^2 x(t). \tag{4}$$

This is the characteristic differential equation describing SHM.

Q 1. A particle is released from rest at a point P, which is 2 m from the origin in the positive direction. The particle next returns to P after 4 s. Write down the particle's equation of motion [1].

Q 2. A balloon performs SHM in a vertical line with a period of 40 s. Its height varies between 800 m and 850 m. Find the speed of the balloon when it is at 820 m [1].

3.2 Mass-Spring System

Consider a mass attached to a spring. Now combining *Newton's law*,

$$F = ma \tag{5}$$

and *Hooke's law*,

$$F = -kx \tag{6}$$

where k is the *spring constant*, we get

$$a = -\frac{k}{m}x. \tag{7}$$

This can also be written as,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \tag{8}$$

with the definition,

$$\omega_0^2 = \frac{k}{m}. \tag{9}$$

Equation (8) is identical to equation (4). Hence the periodic motion of a mass attached to a spring is simple harmonic. Now let's analyze the energy of this system. The total mechanical

energy (E_T) of the mass-spring system is the sum of *kinetic* and *potential* energies, E_K and E_P,

$$E_T = \frac{1}{2}mv^2 + \frac{1}{2}kx^2,$$ (10)

that can be re-written in light of equation (2) as,

$$E_T = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2.$$ (11)

The *conservation of energy* ensures that the total energy is unchanging. As a result, kinetic and potential energies inter-convert keeping the sum constant. At the point of maximum displacement, the energy is wholly potential ($E_K = 0, E_T = E_P$) and at the centre, all the energy is kinetic ($E_P = 0, E_T = E_K$).

Q 3. What are the units of ω, k, and m?

Q 4. Sketch a graph between E_T and time for one oscillation. Now on the same graph sketch curves for E_k and E_P.

Q 5. When the undamped spring is stretched and released, show that the total energy E_T at any given time is,

$$E_T = \frac{1}{2}mx_0^2\omega_0^2 = \frac{1}{2}kx_0^2.$$ (12)

HINT: Use $v = -x_0\omega sin(\omega t)$ and Equation (10).

Q 6. A spring of natural length 0.6 m is attached to a fixed point A on a smooth horizontal table. A force of $10\,x$ N is needed to keep the spring extended by x meters. A block of mass 0.1 Kg is attached to the other end of the spring. The block is pulled away from A until it is 0.75 m from A, and then let go. Describe the subsequent motion along with an equation [1].

3.3 The Decay of Vibrations

Equation (4) represents an ideal situation where there is no damping. A mass set in motion once will continue displaying its oscillations for all times to come. However, as we all very well know, this is never the case. The amplitude of the oscillating mass keeps on decreasing, till it eventually comes to a stop. We now investigate this behaviour, called the *damping effect*.

Every medium (such as air, water) exerts a frictional force on any object moving inside it. This resistive force slows down the object, eventually bringing it to rest. In our case, the medium is air, slowing down the oscillating mass-spring system. One form of Newton's second law that models this frictional effect is

$$F = -bv, \tag{13}$$

where v is the velocity of the particle and b is the *drag coefficient*.

Combining the two forces (-kx and -bv), we obtain

$$F = -bv - kx = ma \tag{14}$$

$$\implies a + \frac{b}{m}v + \frac{k}{m}x = 0. \tag{15}$$

This can be re-written as,

$$\frac{d^2x(t)}{dt^2} + \frac{b}{m}\frac{dx(t)}{dt} + \frac{k}{m}x(t) = 0, \tag{16}$$

substituting,

$$\frac{b}{m} = \gamma, \tag{17}$$

$$\frac{k}{m} = \omega_0^2, \tag{18}$$

the differential equation now becomes,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0. \tag{19}$$

Q 7. What is the difference between Equations (8) and (19)?

In this particular case, the damping is characterized by γ, while ω_0 represents the *natural angular frequency* of the system *if damping were absent.*

In the current experiment, we will consider only one kind of damping: *underdamped* SHM. As the term underdamped suggests, the oscillations die away, albeit slowly. We will not go into the details of the various kinds of damping. However Figure 2 gives a good intuitive feel of how these damped motions look like.

The solution to Equation (19), for the underdamped oscillations, is given by ($\phi = 0$),

$$x(t) = A\exp\left(-\frac{\gamma}{2}t\right)cos(\omega_1 t), \tag{20}$$

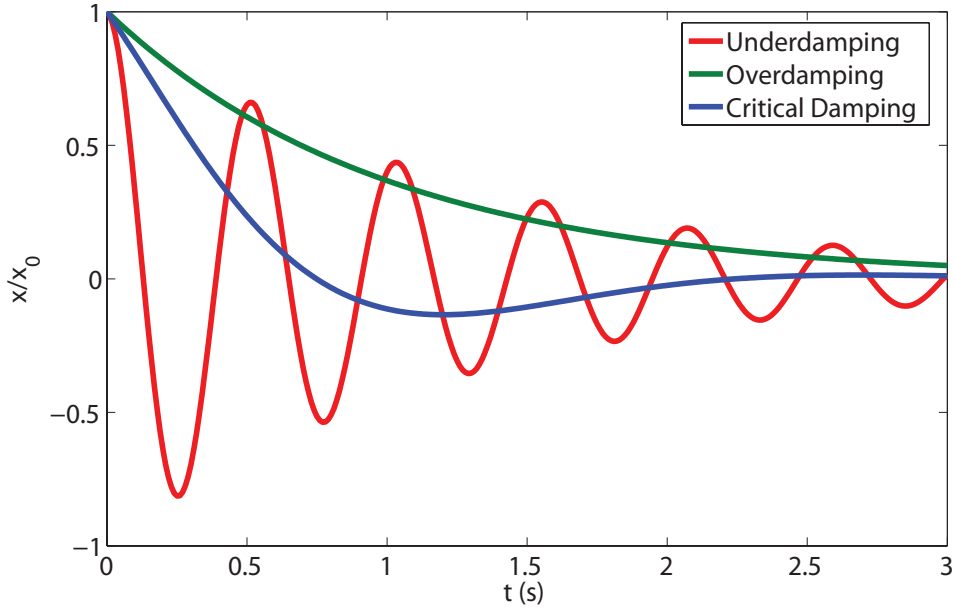where A is the maximum amplitude at $t = 0$ and,

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Figure 2: Decaying oscillations for underdamped, overdamped and critically damped harmonic oscillations. Our experiment is performed in the underdamped regime.

$$\omega_1 = \sqrt{\omega_0{}^2 - \frac{\gamma^2}{4}} \,. \tag{21}$$

The solution, Equation (20) is the mathematical representation of the underdamped oscillation of the mass-spring system, it shows how the position $x(t)$ varies as time t. This solution can be divided into two parts, $A\exp(-\frac{\gamma}{2}t)$ represents the damping of the amplitude (also called the envelope) whereas $\cos(\omega_1 t)$ represents the oscillations of the mass, with a frequency ω_1. Note that $\omega_1 \neq \omega_0$.

4 Apparatus

Our damped harmonic oscillator apparatus consists of a set of masses (locally fabricated) attached to the end of a helical spring. The mass is displaced slightly and the resulting motion is recorded using a web camera that is attached to a computer. The data processing is performed in Matlab that is equipped with the *Image Processing* toolbox. The stands holding the mass-spring system are manufactured in house. Fig. 3 shows the schematic setup of the experiment.
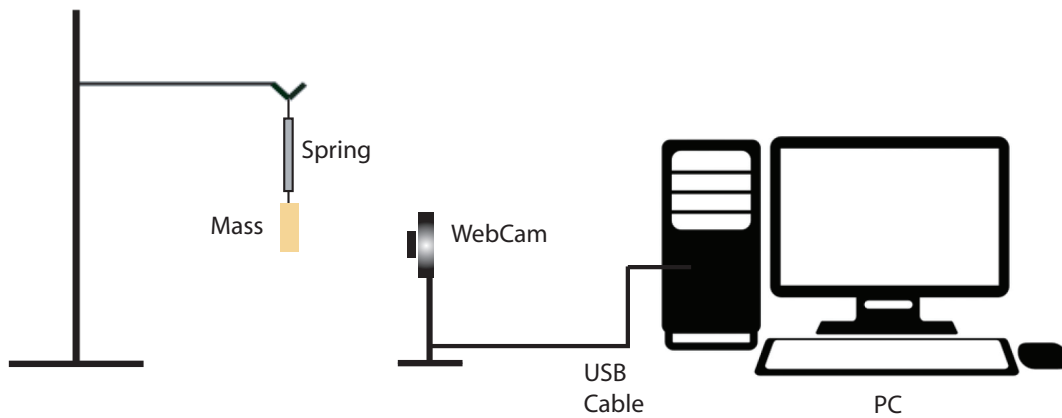
Figure 3: Schematic diagram of the experimental setup.

5 The Experiment

5.1 Determining the Spring Constant

You have been provided with the mass hanger $(20 \pm 1$ g$)$ along with different mass blocks.

Step 1 Attach a steel ruler and a mass hanger with a spring to the provided lab stand.

Step 2 Add weights of equal mass and measure the extension each time a weight is added.

Q 8. What are the units of extension?

Q 9. Use your experimental data to obtain the spring constant k (plot a graph!).

Q 10. What is the error in k?

Q 11. For a mass of 60 g on your spring, calculate an expected value of ω_0 using equation (18). Also predict the period of free oscillations. What is the propagated uncertainty in the time period?

5.2 Free Oscillations

In this section we would use MATLAB to capture a video of the oscillator using a webcam and then track its position and make a plot of the simple harmonic motion.

- Set up the apparatus as shown in the schematic diagram given in Figure 3. Make sure that the mass is in front of the black chart-paper hanging on your station, as we want it to be the background to your oscillatory motion.

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- Please make sure that the webcam isn't already being used by any other software.

- make sure that you have already downloaded the relevant **script** files from the Experiment 1.1 website[1] and have placed them in your working directory. See the pictorial procedure on the website for your reference.

As you will know by now, you can run MATLAB commands in several ways. One is by writing in some kind of script or function file (which MATLAB executes line-by-line once you click the 'Run' button). The other is by typing your commands in the command window. Figure 4 shows the scripting and command windows, respectively.
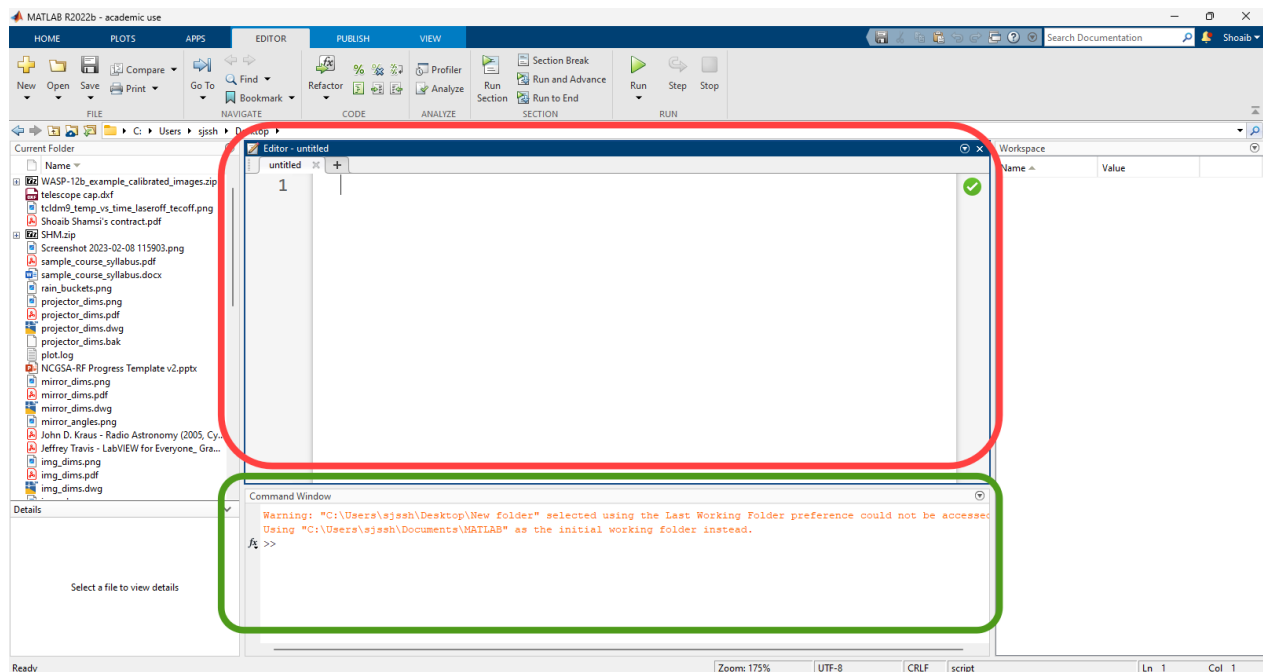

Figure 4: Your typical MATLAB workspace. The space encircled in red is the **scripting window**, while the space encircled in green is the **command window**.

You might know that if you have a script in your working directory, you can run it by just typing its name. The following instructions will have you run the scripts you downloaded one by one in the command window.

Step 1 Start the webcam using the following command:

≫ `webcam`

This activates a live preview from your webcam and you should be able to see the mass hanger on your computer screen. Adjust the apparatus to obtain a vertically oscillating mass appearing on the screen. Try to have the camera looking at the mass head-on, with the mass resting in the middle on the camera feed. See pictorial procedure on the website for reference.

Step 2 Type in the following command (don't hit enter yet!) and pull the mass (60 g in total) down and release it, then hit enter.

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[1]https://physlab.org/experiment/simple-harmonic-motion-is-observed-through-webcam/

```
≫ start(vid)
```

This will begin recording the video frames from the live preview. Observe the parameters updating themselves in the bottom of the window. Wait till 350 frames are recorded.

Step 3 Use the following command to store these frames in memory:

```
≫ savedata
```

This will create two arrays named **frames** and **time** which will store the acquired frames and their time stamped values. You can view them in your variables workspace.

Step 4 Check the size of **frames** using the following command:

```
≫ size(frames)
```

You should see 4 numbers that represent the resolution of the image, the number of colours in each pixel and the total number of recorded frames. Note that when we save data it removes the last 50 frames.

Step 5 View the first frame by typing in the following commands:

```
≫ firstframe = frames( : , : , : , 1)
```

```
≫ imtool(firstframe)
```

The first command creates a variable **firstframe** from **frames**, and the second command shows **firstframe** as an image.

Q 12. What is the size of the first frame?

Step 6 Try viewing the second, third and the fourth frame.

Step 7 So we have a bunch of frames from our camera showing the motion of the oscillating mass. But these frames might show more than just the mass and could have irrelevant information, like image areas outside the black chart-paper, or some other kind of obstruction. This is why we need to crop our acquired frames. Luckily, MATLAB has a built-in function called **imcrop** which can crop an image to a specified rectangle.

View the first frame again and select the **Crop Image** tool from the **Tools** dropdown menu in the imtool window. You should be able to draw a rectangle that encapsulates the area of motion of the oscillator. Right click on the area and select **Copy Position**. This will get you an array of four numbers, which are:

$$[x_{min} \quad y_{min} \quad \underbrace{(x_{max} - x_{min})}_{width} \quad \underbrace{(y_{max} - y_{min})}_{height}]$$

Where x_{min}, x_{max}, y_{min}, and y_{max} are the minimum and maximum coordinates in the x and

y axes respectively. You should save this array in a variable named **rect**. We'll pass along **rect** to the **imcrop** function so it knows to crop all frames within this rectangle.

≫ `rect = `*your copied array*

Use the following command to crop your region out of **firstframe** save the cropped image ina variable named **cropregion**:

≫ `cropregion = imcrop(firstframe, rect)`

Q 13. So now we have the cropped first frame in a nice variable called **cropregion**. How might you view **cropregion**?

We've successfully cropped one frame, only 299 to go! Although it'll be easier if we just run the **harmonic** script, which automatically crops all 300 frames and saves the cropped images as an array in the **regions** variable.

≫ `harmonic`

Once **harmonic** is done running, we can run the **imageprocess** script:

≫ `imageprocess`

This will process all the cropped images inside **regions** by calculating the x and y pixel coordinates of the centre of mass for the oscillating object in each frame.

To calculate the displacement coordinates for the centre of mass, use the **masscentre** script:

≫ `masscentre`

Look at the x and y coordinates of the centre of mass through the 300 frames by typing:

≫ `x = centre(:, 1)`

≫ `y = centre(:, 2)`

Q 14. Plot a graph of the x coordinates against time. What does this graph physically represent?

Q 15. What happens to the x coordinates and explain why?

Q 16. Now plot a graph of the y coordinates with time. Record the approximate time period of the oscillation in seconds.

Q 17. Compare this result with the time period from Question 11.

5.3   Damped Oscillations

We now move on to see how the energy changes with time in an underdamped system with $\omega_0 \gg \gamma$. The energy at any subsequent time is given by,

$$E(t) \approx E_0 e^{-\gamma t}. \tag{22}$$

Where E_0 is the energy of an initial oscillation. It is,

$$E_0 = \frac{1}{2} m x_0^2 \omega_0^2 = \frac{1}{2} k x_0^2. \tag{23}$$

Where x_0 is the amplitude of the initial oscillation.

Now place the beaker containing water under the mass-spring system. You might have to adjust your setup to ensure the mass is completely immersed in water throughout its oscillations. Repeat the experiment you performed in Section 5.2.

Q 18. What do you observe when you look at your new y vs. time graph?

Q 19. From this graph, obtain amplitudes for some (around 7 should be good) oscillations and their corresponding times.

Q 20. Plot a graph of $\ln\left[\dfrac{E(t)}{E_0}\right]$ versus time, where E_0 represents the energy of the first oscillation that you took into consideration and $E(t) = \frac{1}{2} k x_t^2$ represents the energy at any given time t thereafter.

Q 21. Why do you think plotting a graph of $\ln\left[\dfrac{E(t)}{E_0}\right]$ against time is more appropriate than a graph of $\ln[E(t)]$ against time?

Q 22. Fit the plot and calculate the best estimated value of γ. Also calculate the values of ω_1 and b.

6   Idea Experiments

1. Use glycerine instead of water for damping. Can you determine the viscosity of glycerine? HINT: Use Stokes Law.

2. Solve the differential equation (16) using the quadratic formula and analyze the discriminant for less than, equal to and greater than zero.

3. Try horizontal mass-spring system instead of gravity driven. Discuss friction.

4. Try to find out if friction is surface area dependent or not. HINT:Slide blocks down a plank and observe using a webcam.

5. Study the diffusion of ink in water using webcam [2].

6. Demonstrate the damping of a pendulum [3].

References

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