This is in agreement with several other authors' work[3,4,6,7] based on harmonic oscillations, such as Eq. (11). However, as may be seen from Eqs. (9) and (10), it is possible that with different initial conditions, the lower harmonics may not be excited, and a much shorter period could occur. For a single mode, the period is given by

$$T_n = (4l_0/c)/(2n+1) = 4(m/k)^{1/2}/(2n+1).$$

Then, if we pursue this notion to logical extremes, one is tempted to define the effective mass of a single mode as

$$m_{\text{eff}} = (4/\pi^2)m/(2n+1)^2. \qquad (13)$$

This course leads to many difficulties. If a general oscillation includes many harmonics, which effective mass should be used? Since all the higher frequencies are odd multiples of $f_0 = c/4l_0$ the entire motion will be periodic with the period of the *lowest* excited harmonic. One might then argue that the value of n in Eq. (13) for general oscillations should be that value for which the first nonzero coefficient appears in the Fourier series, Eq. (9).

This leaves us in an impossible situation. The effective mass defined this way depends on initial conditions! Worse yet, two motions virtually indistinguishable (one with $A_0 = B_0 = 0$, the other with $A_0 = 0$, $B_0 = \epsilon$ = small nonzero quantity) have very different effective masses.

If the idea is to be useful at all, I would suggest that the notion of effective mass be tied only to properties of the Slinky. The "natural" or fundamental motion of the spring has period $T = 4(m/k)^{1/2}$; it seems likely that all reasonable mechanisms do in fact excite the lowest mode. So let us adopt $(4/\pi^2)m$ as the effective mass of an unloaded Slinky, regardless of its actual motion.

ACKNOWLEDGMENTS

I wish to thank Stephen Becker for helpful conversations and Bruce Holenstein for the drafting of the figures.

[1]Robert Weinstock, Am. J. Phys. **32**, 370 (1964).
[2]Michael Parkinson, Am. J. Phys. **33**, 341 (1965).
[3]Ernesto E. Galloni and Mario Kohen, Am. J. Phys. **47**, 1076 (1979).
[4]H. L. Armstrong, Am. J. Phys. **37**, 447 (1969).
[5]T. W. Edwards and R. A. Hultsch, Am. J. Phys. **40**, 445 (1972).
[6]Francis W. Sears, Am. J. Phys. **37**, 645 (1969).
[7]F. A. McDonald, Am. J. Phys. **48**, 189 (1980).
[8]D. Halliday and R. Resnick, *Physics, Part I*, 2nd ed. (Wiley, New York, 1966), Chap. 15, problem 40.
[9]Richard T. Weidner and Robert L. Sells, *Elementary Physics—Classical and Modern* (Allyn & Bacon, Boston, 1975).
[10]Philip R. Wallace, *Mathematical Analysis of Physical Problems* (Holt, Rinehart & Winston, New York, 1972), Chap. 1.
[11]George H. Shortley, *Principles of College Physics*, 2nd ed. (Prentice-Hall, Englewood Cliffs, NJ, 1967).
[12]It is easily shown that the odd sines given do in fact form a complete set for functions with $f(0) = 0$, $f_x(l_0) = 1$. Allen Schweinsberg, personal communication.
[13]Fritz John, *Partial Differential Equations* (Springer-Verlag, New York, 1971), p. 96.

Nonlinear resonance in vibrating strings

John A. Elliott

Department of Pure and Applied Physics, University of Manchester Institute of Science and Technology, Manchester M60 1QD, England

(Received 20 October 1980; accepted for publication 11 February 1982)

A simple stretched string driven sinusoidally is commonly used to illustrate the concepts of resonance and eigenmodes. In practice, such a system rarely executes planar oscillation, but prefers circular motion. This is shown to be a consequence of the intrinsic nonlinearity of the stretched string oscillator, and the associated parametric coupling between the two transverse polarizations.

INTRODUCTION

Every child who tries to create standing waves on a clothesline soon discovers that it is very difficult to maintain a plane polarized wave. University students commonly do the same experiment in more detail, using controlled tension and sinusoidal excitation. The excitation can be transverse or longitudinal, as in Melde's classic experiment (described by Rayleigh[1]). In all cases it is commonly observed that the string prefers circular motion to simple transverse; sometimes more complicated phenomena are seen, such as the periodic cycling of the system between states with large and small amplitude. The purpose of this paper is to discuss the origins of these phenomena in this simple system. They prove to be interesting examples of nonlinear effects intrinsic to the stretched string system, arising from the simple fact that the tension cannot be constant if the string is varying in length due to finite amplitude of displacement. The effect of this nonlinearity on the free oscillations of a string has been discussed in an earlier paper.[2] Keller[3] also has discussed the large-amplitude oscillations of strings, although his analysis is restricted to a very specific stress–strain law, not appropriate to conventional, real strings. Here we restrict detailed analysis to the steady-state resonance behavior; transient effects are mentioned briefly.

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EQUATIONS OF MOTION

The equations have been derived in the earlier paper[2] for the free oscillations, and need be modified only by the addition of a forcing term. We assume that the string is in the z direction, and can be displaced in the x and y directions. We assume standing waves in each polarization, with wavenumbers $k_x = k_y = k = 2\pi/\lambda$ and oscillating amplitude x and y. It is convenient to define dimensionless parameters $X = kx$ and $Y = ky$. We further assume that the string is being sinusoidally excited in the x direction. Including this forcing term, the equations of motion become

$$\ddot{X} + \omega^2 X[1 + \sigma(X^2 + Y^2)] = A\cos pt, \tag{1}$$

$$\ddot{Y} + \omega^2 Y[1 + \sigma(X^2 + Y^2)] = 0, \tag{2}$$

where $\omega^2 = T_0 k^2/\mu$ and $\sigma = 3Y\mathscr{A}/8T_0$. Here T_0 is the static tension, μ the mass/unit length, Y, Young's modulus (the context will not allow confusion with the displacement), and $\mathscr{A}$ the cross-sectional area of the string. σ clearly is an indication of the nonlinearity and is greatest at low tensions.

Equation (1) is a version of Duffing's equation[4] with the addition of nonlinear coupling to Eq. (2). The system consists of two nonlinear oscillators parametrically coupled. Exciting a nonlinear resonance in one oscillator causes the parametric excitation of nonlinear oscillation in the second, which then couples back into the first.

APPROXIMATE SOLUTION

Duffing's method[4] is applicable here: we rearrange Eqs. (1) and (2) so that only $\ddot{X}$, $\ddot{Y}$ remain on the left-hand side, and substitute sinusoidal forms for X and Y on the right-hand side. We put $X = a\cos pt$ and $Y = b\cos(pt + \varphi)$, since we are looking for motion at the driving frequency. Equation (1) becomes

$$\ddot{X} = A\cos pt - \omega^2 a\cos pt - \omega^2\sigma[a^3\cos^3 pt + ab^2\cos pt\cos^2(pt + \varphi)]. \tag{3}$$

Expanding the trigonometric powers and integrating twice (ignoring constants of integration) yields

$$\begin{aligned} X = &-\cos pt/p^2[A - \omega^2 a - \tfrac{3}{4}\omega^2\sigma a^3 \\ &- \omega^2\sigma ab^2(\tfrac{3}{4}\cos^2\varphi + \tfrac{1}{4}\sin^2\varphi)] \\ &- \sin pt/p^2(\tfrac{1}{2}\omega^2\sigma ab^2\cos\varphi\sin\varphi) \\ &- \cos 3pt/ap^2[-\tfrac{1}{4}\omega^2\sigma a^3 - \omega^2\sigma ab^2(\tfrac{1}{4}\cos^2\varphi - \tfrac{1}{4}\sin^2\varphi)] \\ &- \sin 3pt/ap^2(-\tfrac{1}{2}\omega^2\sigma ab^2\cos\varphi\sin\varphi). \end{aligned} \tag{4}$$

The basis of the method is not to insist that $X = a\cos pt$ is a good approximation, and hence the coefficient of $\cos pt$ in Eq. (4) can be put equal to a:

$$\begin{aligned} -p^2 a = A - \omega^2 a - \tfrac{3}{4}\omega^2\sigma a^3 \\ - \omega^2\sigma ab^2(\tfrac{3}{4}\cos^2\varphi + \tfrac{1}{4}\sin^2\varphi). \end{aligned} \tag{5}$$

The same condition requires that we put the coefficient of $\sin pt$ equal to zero:

$$\cos\varphi\sin\varphi = 0, \quad \text{i.e., } \varphi = n\pi/2,$$

where n is an integer. If $\varphi = 0$ or π we simply have planar motion with $X = \pm Y$. The discussion given previously[2] indicates, and experiment confirms, that this is a metastable condition which cannot persist in practice. We shall assume the solution $\varphi = \pi/2$ for the remaining discussion. Equation (5) now becomes

$$p^2 = -A/a + \omega^2(1 + \sigma b^2/4) + \tfrac{3}{4}\omega^2 a^2\sigma. \tag{6}$$

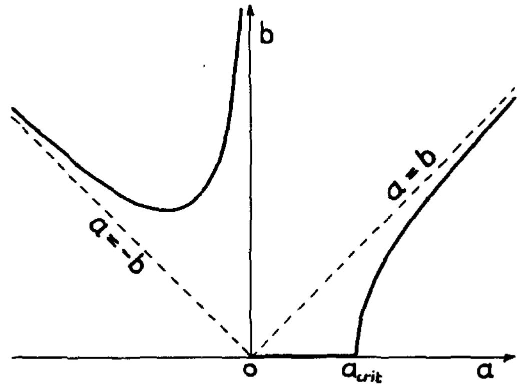

Fig. 1. Graph relating the two orthogonal amplitudes a and b.

Before we examine the physical content of this equation, we should note that the amplitudes a and b are not independent, but are linked via Eq. (2) describing the orthogonal mode of oscillation. If we substitute $X = a\cos pt$ and $Y = b\cos(pt + \varphi)$ in Eq. (2), as before, and put $\varphi = \pi/2$, using a similar procedure to the above, we finally obtain

$$p^2 = \omega^2(1 + \sigma a^2/4) + \tfrac{3}{4}\omega^2 b^2\sigma. \tag{7}$$

The relationship between a and b is now found by eliminating p from Eq. (6) and (7), and yields the trajectory traced by the string in the x–y plane:

$$b^2 = a^2 - 2A/a\sigma\omega^2. \tag{8}$$

Equation (8) is plotted in Fig. 1 as a graph of b against a. There is a forbidden region for b: if $a < (2A/\sigma\omega^2)^{1/3} = a_{crit}$ then b becomes imaginary, and has no real solution. Hence experimentally b is zero, and oscillations transverse to the driving motion cannot be excited. For $a > a_{crit}$ the stable configuration is for b to increase from zero following the curve of Fig. 1, with $\pi/2$ phase difference from the X motion. An elliptical trajectory ensues, which rapidly tends to circularity with increasing a. This is the commonly observed effect we mentioned earlier.

DISCUSSION

We may now return to Eq. (6) and observe its physical significance. For $a < a_{crit}$ the Y motion is not excited, b is zero, and the sytem is a simple nonlinear resonator as is treated in many texts.[5] Figure 2 shows $|a|$ as a function of p. We have an infinite resonance (we have neglected damping) and the cubic nonlinearity in the X motion has caused the resonance to bend over. It is easy to see qualitatively the effect of damping by analogy with the linear resonator.

Above a_{crit}, however, when b becomes finite, the simultaneous existence of the Y motion causes the entire resonance curve to shift towards higher values of p. The locus of the resonance peak meets the p axis at $p = \omega\sqrt{(1 + \sigma b^2/4)} \simeq \omega(1 + \sigma b^2/8)$. The resonance frequency thus increases under the combined but different effects of the finite X and Y amplitudes.

Since the amplitudes a and b are frequency dependent (or, more correctly, the frequency is amplitude dependent), as the driving frequency p is scanned through the resonance the system state migrates along the curve of Fig. 1. The amplitudes reach a maximum when the peak of the distorted resonance curve (Fig. 2) is attained: for further increase of frequency the system may retrace the curve of Fig. 1 in reverse if the nonlinearity is small. More usually, when the

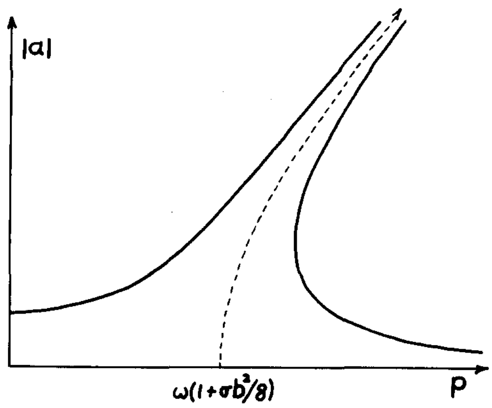

Fig. 2. Nonlinear resonance curve for the amplitude $|a|$ of the X oscillation, for zero damping.

system is sufficiently nonlinear for the resonance peak to be concave, as shown, the system "jumps" down to the lowest amplitude available at that frequency, and the circular motion collapses. At the same time, the entire resonance curve jumps backwards in frequency as the amplitude b collapses.

These results describe only the steady-state behavior. Since the Y motion is excited parametrically, the pure X motion can theoretically continue as a metastable steady state even for large a. However, any slight perturbation will "seed" the growth of the Y oscillation. The motion described by Fig. 1 is the one invariably observed in practice.

From Eq. (4) we might expect an appreciable contribution to the X motion from the third harmonic; the $\cos 3pt$ term is not required to be zero, and is to first order in σ. However, putting $\varphi = \pi/2$ as before, the ratio of the amplitudes of the third to the first harmonic is $\sim \sigma(a^2 - b^2)/36$ (taking a as the amplitude of the $\cos pt$ term). We observe that this tends to zero as the system state migrates along the curve of Fig. 1. Even for $b = 0$, and $a = a_{\text{crit}}$, if we choose a practical example[2] we find the ratio to have a typical value $\sim 10^{-5}$, since a (and b) is a small dimensionless amplitude. Thus harmonic effects are negligible.

We also note that the curve of Fig. 1 has a second branch for negative a. Since the sign of a has little significance, we must ascribe some physical meaning to this branch. It describes a motion in which the Y oscillation has greater amplitude than the X (driving) oscillation. In the undamped case considered here, that configuration is possible in the steady state if energy is stored initially in the Y motion. However, the symmetry of the system indicates that the slightest trace of damping will make this branch unstable. This is best seen by considering the region close to $(-a) \to 0$: b becomes very large, tending to an infinite Y motion for zero driving force in the X direction, which is physically unreasonable. A proper analysis of this branch will require an investigation of the transient behavior not attempted here.

Many readers will, like the author, have a system similar to that described readily to hand in a teaching laboratory. The effects discussed are all readily observable, but may be accompanied in practice by other more complicated phenomena such as an oscillatory precessional motion of the elliptic trajectory, or a periodic rise and collapse of the circular motion. These phenomena are found in those systems where the string tension is maintained by a spring or a weight, as opposed to a fixed pin. In such systems the average tension remains constant as the amplitude varies; this condition has not been incorporated here and is presumably responsible for the additional effects, which will be the subject of future studies.

[1] Lord Rayleigh, Philos. Mag. **15**, 229 (1883).
[2] J. A. Elliott, Am. J. Phys. **48**, 478 (1980).
[3] J. B. Keller, Am. J. Phys. **27**, 584 (1959).
[4] J. J. Stoker, *Nonlinear Vibrations* (Interscience, New York, 1950).
[5] The nonlinear resonance of a stretched string in planar motion has been treated by A. B. Pippard, *The Physics of Vibration* (Cambridge University, Cambridge, 1978).

Constant power equations of motion

Roger Stephenson
Physics Department, University of Wisconsin at Platteville, Platteville, Wisconsin 53818

(Received 13 July 1981; accepted for publication 29 January 1982)

Correct solutions for the common problems associated with bodies accelerating under the application of constant power are generally missing from physics textbooks. The kinematic equations for this motion are not overly difficult to derive and bear strong similarities to their familiar constant force counterparts. The concept of "zip" is here defined as the square root of the power to mass ratio, and is shown to be a useful parameter of this kind of motion. Several specific problems are solved and the results compared with available real data.

I. INTRODUCTION

Most physics texts discuss the motion of automobiles because they are an important part of the student's experience of acceleration, velocity, distance, power, etc. Yet quite often the equations used are completely inappropriate.[1] In particular, the assumption that the force acting on the car remains constant is realistic only during braking

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