Rotational Dynamics, Moment of Inertia, Torque and Rotational Friction

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When you push the throttle to accelerate a moving car, all you are doing is using the engine to apply more force. But why is there no acceleration after a certain speed is reached, even when the engine is still applying the force? Why do you find it difficult to stop a rotating wheel even if there is no translational motion? Why are the brake linings of formula 1 cars made differently from ordinary cars? Why is it easy to accelerate a bicycle having smaller wheels but harder when it has larger wheels? Why is the lower gear capable of imparting more acceleration than the higher ones? Study of the quantities like moment of inertia, torque, angular speed and speed dependent friction may help us find the answers to these questions. This is the intent of the present experiment.

KEYWORDS

Rigid Body · Angular Momentum · Angular Velocity · Angular Acceleration · Moment of Inertia · Torque

1 Conceptual Objectives

In this experiment, we will,

1. learn to appreciate the similarities, differences and relationship between rotational and translational motion,
2. investigate energy loss due to friction,
3. fit experimentally observed curves with mathematically modeled solutions, and
4. see how uncertainties propagate from measured to inferred quantities.

2 Experimental Objectives

The experiment is divided into two sections: determining the frictional losses and measuring the moment of inertia of a circular disc. In the first section, height loss measurements will be done by correlating the rotation of a disc with the loss in the height of a mass attached to a thread wound over a pulley on the disc. After having established the nature of the relationship between angular speed and frictional losses taking place in the system, a value of the moment of inertia of the disc will be determined.

3 Theoretical Introduction

This experiment introduces you to basic concepts of rotational motion. We will touch upon a number of topics and discuss how a large complex object can be considered to be composed of a large assemblage of ideal particles. We will elaborate that a full description of a body's motion must include linear as well as rotational motion. Furthermore, we will discuss torque as it applies to our experiment.

3.1 Angular Momentum

Consider a circular disk made up of small infinitesimal particles of masses $m_1, m_2, m_3, \ldots m_i, \ldots$. Their placement may be defined with the position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, ... $\vec{r}_i$, ... and when rotating, their instantaneous velocities may be defined as $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, ... $\vec{v}_i$, ... The index i shows one of the many particles.

Figure 1 illustrates the ith particle rotating about the $\hat{z}$ axis.

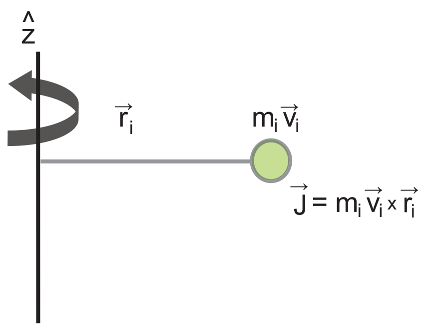

Figure 1: A representative particle rotating about the z-axis; $m_i v_i$ is the linear momentum and J is the angular momentum.

The angular momentum of the particle about the **z axis** is given by,

$$\vec{J}_i = m_i(\vec{v}_i \times \vec{r}_i) \tag{1}$$

where $\times$ denotes the *vector* or *cross product*. The direction of the angular momentum is normal to both $\vec{v}_i$ and $\vec{r}_i$, and hence along the **z axis**. The unit vector $\hat{z}$ points along the z direction. Now, for a particle rotating with an angular velocity ω about **z axis**, we also have

$$v_i = r_i \omega \tag{2}$$

Now consider a circular disk that can be decomposed into many small particles. Using Equations (1) and (2), we can write for the ith particle,

$$\vec{J}_i = m_i r_i^2 \omega \hat{z} \tag{3}$$

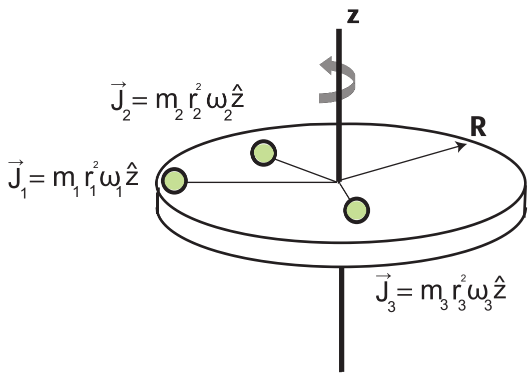

Figure 2: The disk can be considered to comprise of a large number of particles. The individual angular momentums of these particles will all vectorially add up resulting in a total angular momentum.

The total angular momentum of a disk about an axis is simply the vectorial sum of all the angular momentums for the infinitesimal particles,

$$\vec{J} = \sum_{i=1}^{N} m r_i^2 \omega \hat{z} \tag{4}$$

3.2 Moment of Inertia

In our disk which is confined to rotate in the xy plane, the angular momentum $\vec{J}$ has only a **z** component as shown in Equation 4. We can lump the quantity $\sum m_i r_i^2$ replacing it by the moment of inertia **I**. Doing so yields,

$$\vec{J} = \sum_{i=1}^{N} (I_i) \omega \hat{z} \tag{5}$$

where $\vec{\omega} = \omega \hat{z}$ is pointing in the z-direction. The moment of inertia of a solid circular disk is

$$I = \frac{1}{2} M R^2 \tag{6}$$

where M is the mass and R is the radius of the disk.

Q 1. Show that the kinetic energy of a rotating disk is given by,

$$KE = \frac{1}{2} I\omega^2 \tag{7}$$

(HINT: Kinetic energy for a particle is given by, K $= \frac{1}{2}mv^2$. Sum for all the particles and use the fact, $I = \sum m_i r_i^2$.)

Moment of inertia is analogous to *inertia* in linear kinematics. The moment of inertia of a particular body is defined with respect to a particular rotation axis and is different for a body when it is rotating about x, y or z axes. Table 1 provides a brief comparison of linear and rotational motions and their characteristics.

Concepts and quantities	Linear Motion	Rotational Motion
Position	x	θ
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
Motion Equations	$x = vt$	$\theta = \omega t$
Newton's 2^{nd} Law	$F = ma$	$\tau = I\alpha$
Momentum	$p = mv$	$J = I\omega$
Work	Fx	$\tau\theta$
Kinetic Energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$

Table 1: Comparison between Linear and Rotational Motion.

Q 2. A circular disk of mass 0.2 kg and radius 0.1 m is rotating at 10 revolutions per second. Calculate the angular frequency, moment of inertia and kinetic energy of this disk.

3.3 Torque

Q 3. Define torque.

Mathematically, torque (τ) is given by,

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{8}$$

where $\vec{r}$ is the displacement between the line of action of force and the particle and $\vec{F}$ is the force applied. Expanding the cross-product we obtain,

$$\tau = rF\sin(\theta) \tag{9}$$

where θ is the angle between $\vec{F}$ and $\vec{r}$, i.e., between the line of action of the force and the position vector. In a gravity driven system, we may replace $\vec{F}$ using Newton's second law and express Equation 8 as,

$$\tau = mgr\sin(\theta) \tag{10}$$

where m is the mass used to drive the mechanism and g is the acceleration due to gravity.

3.4 Angular Acceleration

Q 4. Define angular acceleration.

Mathematically, *angular acceleration* α is given by

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} \tag{11}$$

You may want to refer to Table 1 to become more comfortable with this seemingly new term which is just an equivalent of *linear acceleration* adapted for rotational motion. Newton's second law for rotational motion states that,

$$\vec{\tau} = I\,\vec{\alpha} \tag{12}$$

where τ is the applied torque, α is the angular acceleration and I is the moment of inertia—the rotational equivalent of mass. Note its similarity to Newton's law for linear motion $F = ma$, establishing the moment of inertia as the analogue of mass and torque as the analogue of force.

If we substitute the magnitudes from Equation (10) and Equation (11) in Equation (12) we obtain,

$$mgr\sin(\theta) = I\frac{d\omega}{dt} \tag{13}$$

Hope you are familiar with the basics given above. You may also like to refer to your favorite physics textbook for consulting the relevant sections on 'rotational motion'.

4 Introduction to the Apparatus

1. **Rotational Motion Apparatus** This apparatus, ME-9341 was procured from *PASCO Scientific*. The contents of the apparatus are in Figure 3.

2. **LabPro and Motion Sensor:** Vernier's LabPro shown in Figure 3(b) is a versatile data-collection interface that can be used to collect data in a variety of ways. It can be used with a computer or as a standalone data logger. More than 60 Vernier sensors are available for use with it. The interface has three buttons, three LEDs, four analog channels (CH 1, CH 2, CH 3, and CH 4), two digital channels (DIG/SONIC 1 and DIG/SONIC 2), a serial computer connection, a USB computer connection, a piezo speaker, and a calculator I/O port.

   Vernier's Ultrasonic Motion Detector (MD-BTD) shown in Figure 3(b) is used to collect position, velocity and acceleration data of moving objects. It emits short bursts of ultrasonic waves from the gold foil of its transducer. These waves fill a cone-shaped area about 15 to 20 off the axis of the centerline of the beam. It then listens for the echo of these ultrasonic waves returning to it. The device measures how long it takes

Figure 3: (a) The components of the rotational motion apparatus. Note the arrows showing the particulars of the components, (b) Vernier's LabPro and Ultrasonic Motion Detector (MD-BTD), (c) PASCO's (ME-8930) Smart Timer.

for the ultrasonic waves to make the trip from the Motion Detector to an object and back. Using this information and the speed of sound in air, the distance to the nearest object is determined.

3. **Smart timer, photogate and super pulley:** PASCO's (ME-8930) Smart Timer shown in Figure 3(c) uses a photogate to send a narrow beam of infrared radiation from one arm which is detected by a detector in the opposite arm. When the beam is blocked, the LED at the back of the photogate lights and a signal is sent to the smart timer (ST).

   We can use this device to measure the speed of a pulley rotation in units of radians/second. To take readings the timer has to be set in the **Pulley (rad/s)**. One measurement will be taken each time the Start/Stop switch is pressed.

5 Experimental Method

5.1 Preparation

1 Place the bubble level on the base. See if the bubble is in the inner ring. If not, adjust the levelling screws to bring the bubble in the inner most ring.

2 Find the outer diameter of the main platter (D_{mp}) and the step pulleys (D_p) on the main platter using vernier callipers. Refer to Figure 3 (a) to identify the step pulley.

You are now ready to perform the experiment. In the first part, we will measure the frictional losses and in the second part, we will determine the platter's moment of inertia.

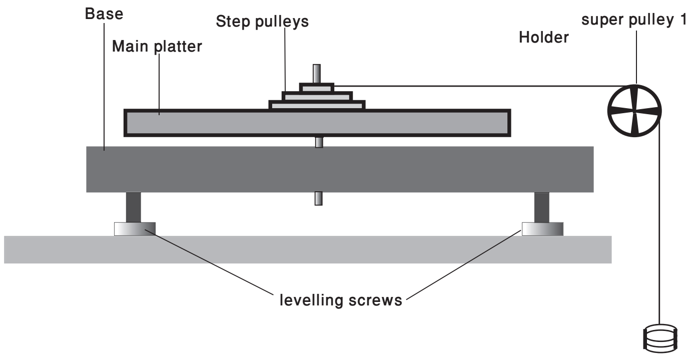

Figure 4: Arrangement of the experimental set up (side view).

5.2 Frictional Losses

We know that a mechanical system involves losses due to friction. In order to measure the moment of inertia of the disc accurately, we have to quantify those losses, so that an energy balance equation can be written and solved. In first section, you will measure the frictional losses by relating friction to the speed of rotation ω of the disc.

In order to proceed, follow the steps given below.

1 Take a thread of suitable length e.g. 85 cm. Tie one end of the thread to the screw protruding from the smallest pulley on the main platter. Then pass the thread from the hole right beside the screw. Pass it through only as many holes as suitable for the pulley that you choose for this part of the experiment.

Fasten the free end of the thread to the hanger and put two 100 g masses in it as shown in Figure 5.

2 Check that the motion detector is connected to the LabPro through the DIG/SONIC2 port which is connected to the computer using the USB cable. Place the motion detector on the floor with its sensor exactly below the mass-hanger. Make sure that when data is being recorded, the line of sight between the mass hanger and the detector is not intercepted, or you will end up with incorrect height measurements.

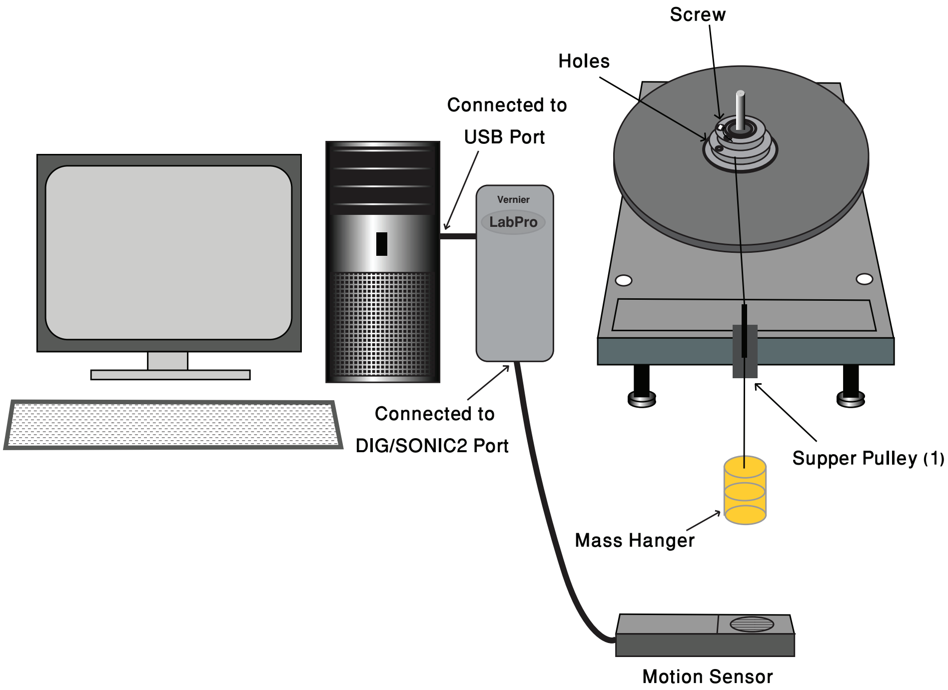

Figure 5: Experimental arrangement for frictional loss measurement.

3 Position the super pulley (1) such that it is exactly in front of the axle of the disc. Pass the thread over the pulley and wind the thread on the chosen step pulley until the mass hanger is at a suitable height, e.g. 65 cm above the floor. The overall arrangement is shows in Figure 5.

4 Before you run the LabView program named **Rotational**, make sure that you have entered the value for the radius of the step pulley and have also given the output path and filename for your data collection. Note the sample time of 0.01 s.

4 Now rotate the main platter to move the mass-hanger to its maximum height. Click on the 'Run' button in LabvIEW and let go of the main platter. It will start rotating about its axis due to the pull force of the mass-hanger. The program will record variations in the height of the mass-hanger with respect to time. After you have recorded the data for 10-15 up-down trips of the mass-hanger, STOP the program and import your data into MATLAB. Once plotted, your data should look like Figure 6(c). Notice that the horizontal axis should have the units of time and its scale will depend on the sampling time used in the LabVIEW program.

Using the "Marker", measure the maximum height achieved by the mass-hanger in every trip. Tabulate these measurements and calculate the differences between subsequent maxima to find out the height loss in every trip.

5 Using the diameter value of the step pulley, you can calculate the number of revolutions of the main platter that took place during every round trip of the mass-hanger.

6 The plotted data shown in Figure 6(b) gives you the values of instantaneous angular velocities. The angular velocity ω is obtained from Equation 2, where v is numerically computed from the derivative of the height data, which itself is shown in Figure 6(a). Use the "Marker" again to measure the values at the maximum angular speeds. These values represent the

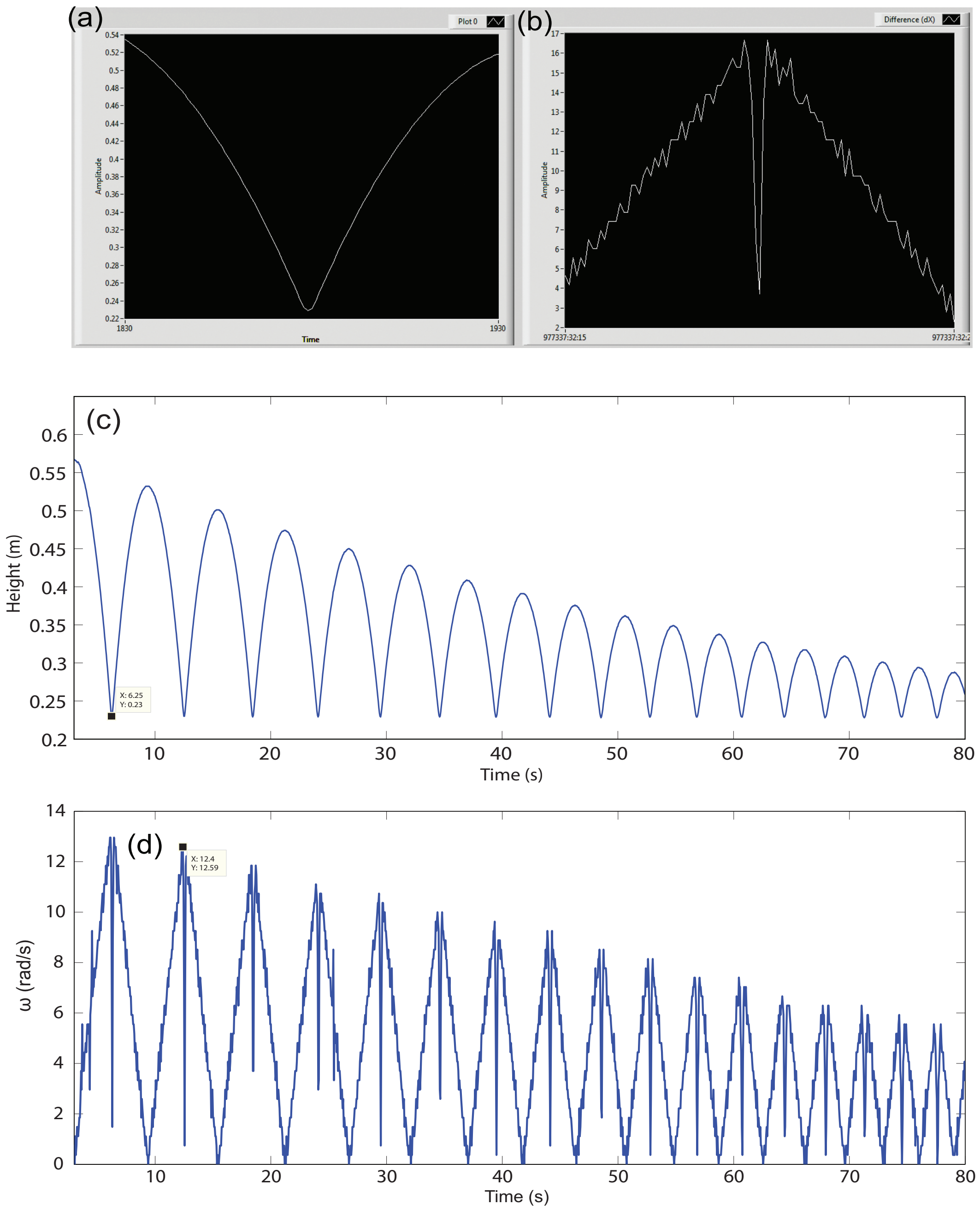

Figure 6: LabView panels showing (a) height and (b) angular velocity, (c) complete graph of the time course of height of the mass-hanger, and (d) graph of the time course of angular velocity of the main platter.

maximum speed achieved by the platter in the respective cycle. To calculate the average speed of the platter during the whole cycle, the following formula is used:

$$\omega_{avg} = \frac{\omega_{max}}{2} \tag{14}$$

7 Once you have measured the height losses for every cycle and the average angular speed, you can construct a table that resembles the one in Table (2). Here, Δh, ΔE, N and ΔE_R represent, respectively, the corresponding height loss, energy loss in a respective cycle (=

Cycle n	h (m)	Δh (m)	N	ΔE (J)	ΔE_R (J)	ω_{avg} (rad/s)
0	0.277					
1	0.257	0.020				
2	0.238	0.019				
3	0.220	0.018				
4	0.203	0.017				
5	0.188	0.015				
6	0.175	0.013				
7	0.163	0.012				
8	0.152	0.011				
9	0.141	0.011				

Table 2: Sample data table for calculating frictional loss. Refer to the text for a meaning of the variables h, N, ω_{avg}, ΔE, ΔE_R

$mg\Delta h$), number of rotations of the main platter that occurred during a round trip of the mass hanger and the energy loss per rotation.

8 Plot the energy loss per rotation against the average angular speed, with ω_{avg} on the x-axis. Ideally, you should get a straight line, showing a linear proportion between the two quantities.

Q What does the relationship between ΔE_R and ω_{avg} signify?

Q What do the spikes in the graph of ω represent?

5.3 Determination of the Moment of Inertia (Design your experiment!)

In this section you will consolidate your knowledge from the previous section and use this to calculate the moment of inertia of the platter. The idea is simple, but in this section, we invite you to come up with an experimental design of your own. So don't expect us to provide a detailed procedure! Of course, we will help. Release the mass-hanger from rest at a maximum height and let it fall onto the ground. The falling mass converts its potential energy into kinetic energy of the rotating platter. However, some energy is also lost in the overcoming rotational friction at the platter, during the descent of the mass. Conceptually, the interconversion is described by,

$$\Big\{\text{P.E. lost by falling mass}\Big\} = \Big\{\text{K.E. gained by platter}\Big\} + \text{ Energy lost} \qquad (15)$$

In this part of the experiment, you will measure the maximum K.E. gained by the platter, which is of course, at the instant the mass hits the ground. From the maximum speed, estimate the average energy lost in friction (see previous section). Here are the preparatory steps, you will need to perform to set up the apparatus.

1 The super pulley (2) is attached to a screw rod and fastened to a holder. Slide it into one of the holes on the sides of the base and use a rubber band to keep the rim in contact with the main platter. The arrangement is shown in Figure 7.

2 Place the photogate such that the spokes of the horizontal super pulley block its beam. Check the connections of the photogate with the smart timer.

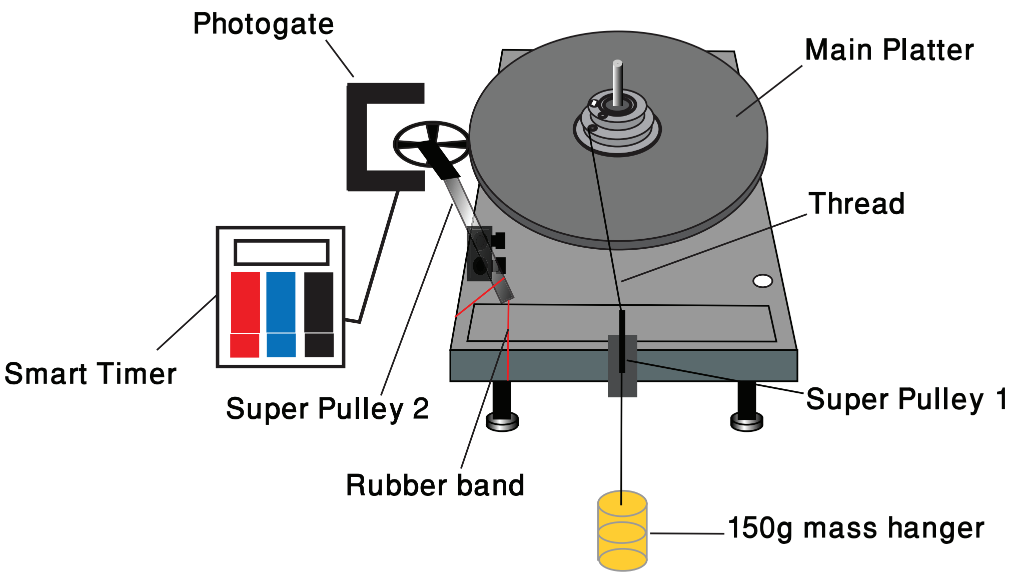

Figure 7: Arrangement of the experimental setup to measure the moment of inertia.

3 Switch the smart timer ON using the power switch on the side. Press 1 to select the quantity to be measured and then 2 to choose the measurement mode. You will take reading using the **Pulley (rad/s)** mode in the speed section.

7 Unfasten the thread from the step pulleys and the mass-hanger. Take a new thread of a suitable length, e.g. 120 cm approximately, and attach it to the mass-hanger. Load a mass of 150 g in the hanger. Wind the free end of the thread around the pulley that you have been using and rotate the main platter to take mass to its maximum height. In this section of the experiment, the mass hanger will hit the ground.

7 From here on, you will have to come up with your own experimental scheme. Best of luck! Show your working and outcomes to your demonstrator.