Vibrations on a String and Resonance

Umer Hassan and Muhammad Sabieh Anwar
LUMS School of Science and Engineering

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How does our radio tune into different channels? Can a music maestro shatter a crystal glass by beating the *tabla* with a particular frequency and pitch? How does our ear distinguish between tones in the multitude of sounds we hear every day? The answer to all of these questions lies in understanding the concept of resonance. The idea was discovered by Galileo Galilei with his investigations of pendulums, beginning in 1602. The present experiment gives an introduction to the phenomenon of resonance, and the frequencies at which it occurs by visualizing the stationary waves formed on a vibrating string.

KEYWORDS

Transverse wave · longitudinal wave · wave interference · resonance · stationary waves · circular modes · normal modes

APPROXIMATE PERFORMANCE TIME 4 hours.

1 Conceptual Objectives

In this experiment we will learn about,

1. the concept of a wave,
2. the difference between transverse and longitudinal waves,
3. the phenomenon of wave interference,
4. the concept of wave vector and wave number,
5. derivation of the wave speed, and
6. the formation of stationary waves.

2 Experimental Objectives

The experimental objectives for the experiment include,

1. exciting and detecting standing waves on a string,

2. being able to identify where resonance occurs,

3. distinguishing linear from nonlinear behaviors, and

4. correlating experimental plots with mathematical relationships.

3 Waves and their different types

3.1 What is a wave?

A wave is a disturbance or variation that transfers energy progressively from point to point in a medium. It may take, for example, the form of an elastic deformation or a variation of pressure, electric intensity, magnetic intensity, electric potential, or temperature. A medium is a substance or material which carries the wave.The medium through which wave propagates may experience some local oscillations as the wave passes but the particles of the medium don't travel along with the wave. Remember that waves involve the transport of energy without the transport of matter. However, we all know that waves can also travel through vacuum.

3.2 Mechanical Waves

A mechanical wave is a wave which is not capable of transmitting its energy through vacuum. Mechanical waves require a medium in order to transport their energy from one location to another. Sound waves, water waves, and rope waves excited on a length of rope are examples of mechanical waves. Mechanical waves can be categorized into the following two main groups.

1. **Longitudinal Waves**
   A longitudinal wave is a wave in which particles of the medium move in a direction parallel to the direction which the wave moves. For example, when a spring under tension is set oscillating back and forth at one end, a longitudinal wave travels along the spring. It is composed of compressions and rarefactions, and is shown in Figure 1.

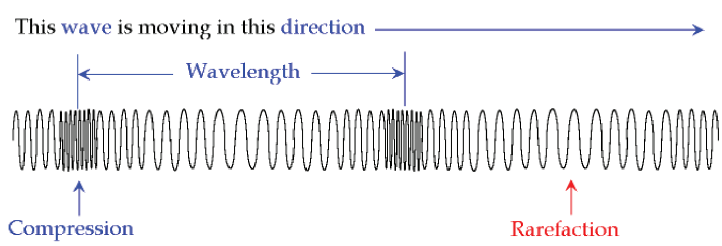

Figure 1: Longitudinal wave in a helical spring.

2. **Transverse Waves**
   A transverse wave is a wave in which particles of the medium move in a direction perpendicular to the direction which the wave moves. For example, when a string under tension is set oscillating at one end, a transverse wave travels along the string; the disturbance moves along the string but the string particles vibrate at right angles to the direction of propagation of the disturbances. The waves on a vibrating string are of this kind. These are the waves we will encounter in the present experiment. We may recall that light or electromagnetic waves are also transverse waves.

3.3 Sinusoidal or harmonic waves

The sine wave is a mathematical function that describes a smooth repetitive oscillation. Mathematically, it can be expressed as,

$$y(t) = Asin(\omega t + \phi), \tag{1}$$

where, A the amplitude, is the peak deviation of the function from its center position, ω, the angular frequency, specifies how many oscillations occur in a unit time interval, in radians per second. It is given by $\omega = 2\pi f$, f being the frequency of the wave. Finally ϕ is the phase, specifying where in its cycle the oscillation begins at $t = 0$. These concepts are illustrated in Figure 2.

Oscillations dominate in real life. All electromagnetic energy, including visible light, microwaves, radio waves, and X-rays, can be represented by a sine wave or a combination of sine waves. At the lowest level, even matter oscillates like a wave. This is the realm of quantum physics. Other examples include ocean waves, sound waves, and tides. Given the ubiquitous nature of waves, the current experiment sets to reveal some interesting properties. However, some more background theory is required before we can start our experiment.

4 Wave Interference and Resonance

4.1 Interference of waves

Interference occurs when two waves meet while traveling along the same medium. Constructive interference is a type of interference which occurs at any location along the medium where the two interfering waves have a displacement in the same direction. In other words, when the crest or trough of one wave passes through, or is super positioned upon, the crest or trough respectively of another wave, the waves constructively interfere. When waves interfere, amplitudes add. Figure 3 shows the two waves of different amplitudes constructively interfering to give a resulting wave of increased amplitude.

Destructive interference is the type of interference which occurs at any location along the medium where the two interfering waves have a displacement in the opposite direction. When the crest of one wave passes through the trough of another wave, we say that the waves

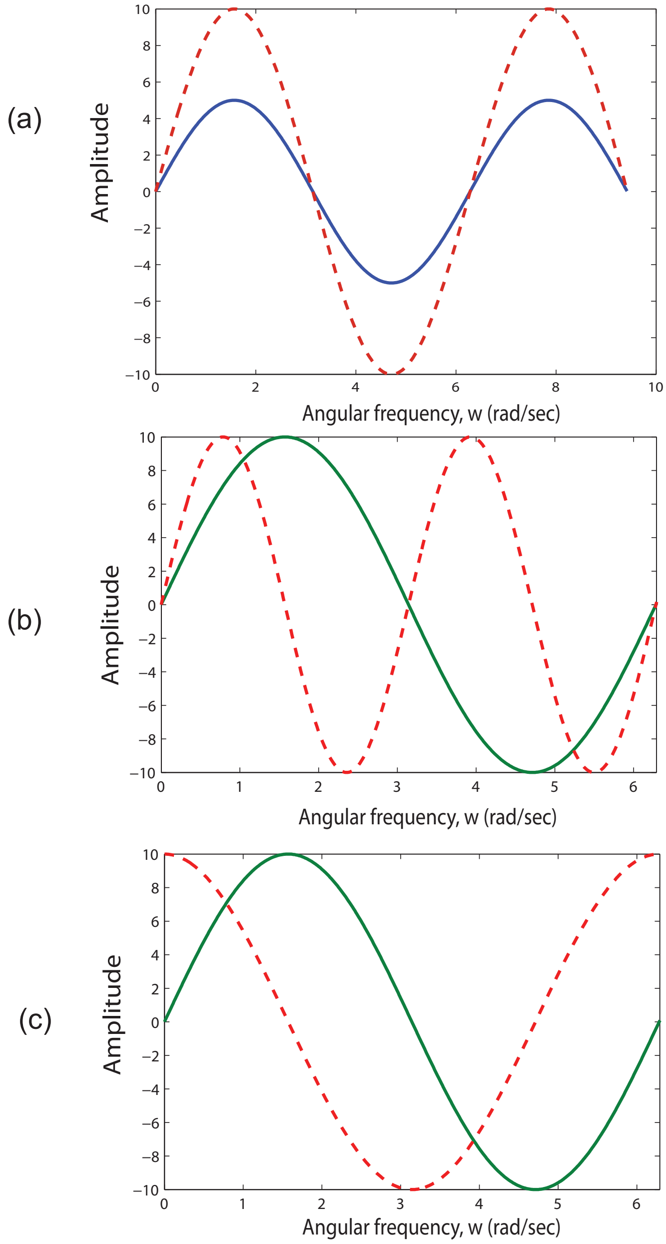

Figure 2: Illustration of variation of amplitude, frequency and phase of harmonic waves, (a) amplitude of one wave is twice the other, (b) frequency of one wave is twice the other, and (c) one wave has a phase difference of $\frac{\pi}{2}$ with the other.

destructively interfere. We often say that when waves interfere, amplitudes add. Refer to Figure 4 for a demonstration of this concept.

If the phase difference is close to 180°, the resultant amplitude is nearly zero. When ϕ is exactly 180°, the crest of one wave falls exactly on the valley of the other. The resultant amplitude is zero, corresponding to total destructive interference.

Q 1. Two waves travel in the same direction and interfere. Both have the same wavelength, wave speed and an amplitude of 10 mm. There is a phase difference of 110° between them. (a) What is the resulting amplitude due to wave interference? (b) How much should the phase difference change so that the resultant wave has an amplitude of 5 mm?

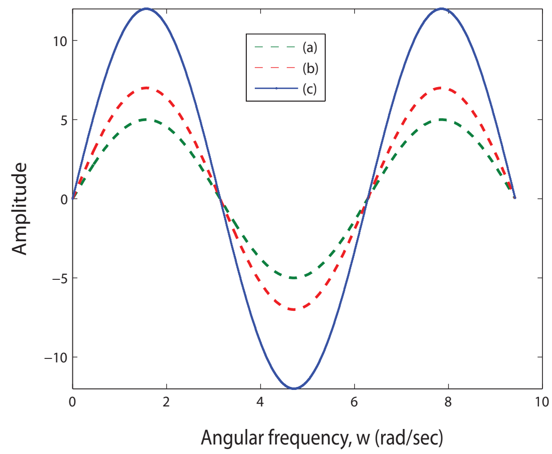

Figure 3: Constructive wave interference phenomenon, (a) a wave of 5 units amplitude, (b) a wave of 7 units amplitude, and (c) resulting wave of 12 units amplitude.

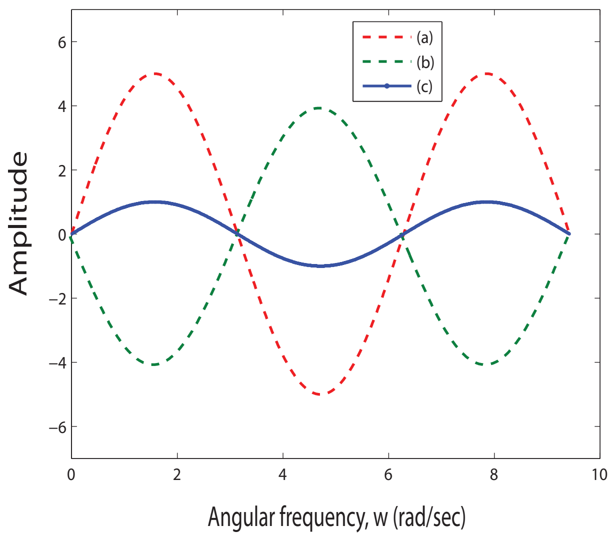

Figure 4: Destructive wave interference phenomenon, (a) a wave of 5 units amplitude, (b) a wave of -4 units amplitude, and (c) resulting wave of 1 unit amplitude.

4.2 Standing or stationary waves

Standing waves are formed by the interference of two harmonic waves of the same amplitude and frequency (and therefore same wavelength), but traveling in opposite directions. Due to the interference of the two waves, there are certain points called nodes at which the total wave is zero at all times. The distance between two consecutive nodes is exactly half the wavelength. The points at the middle between consecutive nodes are called anti-nodes. At the anti-nodes the total wave oscillates with maximum amplitude, equal to twice the amplitude of each wave. Anti-nodes are also half a wavelength apart as shown in Figure 5.

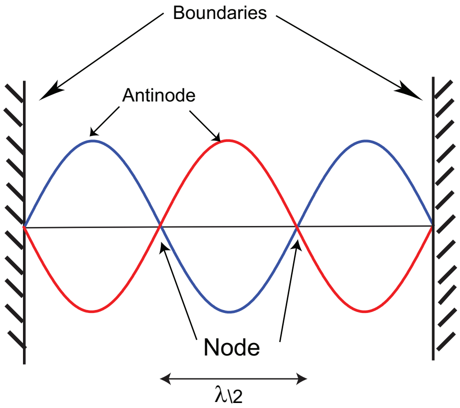

Figure 5: Standing wave.

4.3 Resonance

The frequencies at which we get the stationary waves are the natural frequencies of the oscillating system (in our case vibrating string). If we drive one end of the string and when the frequency of the driving force matches with the natural frequency of the string, standing wave is produced and the string begins to move at large amplitude. This phenomenon is called as resonance.

5 Wave Speed

5.1 Basic definitions

The wavelength λ is the distance between two consecutive crests or troughs in case of transverse wave. The period T of the wave is the time required for any particular point on a wave to undergo one complete cycle of transverse or longitudinal motion. The frequency f is the number of the wave cycles completed in one second. The wave number k is the inverse of the wavelength and is expressed as,

$$k = \frac{2\pi}{\lambda}. \tag{2}$$

Likewise, the angular frequency ω is,

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{3}$$

The length L of the vibrating string in which standing waves are developed can be expressed as integral multiples of half of the wavelength, i.e.,

$$L = \frac{n\lambda}{2}, \tag{4}$$

where $n = 1, 2, 3, \ldots$ is an integer. So, λ can be written as,

$$\lambda = \frac{2L}{n}. \tag{5}$$

Substituting the value of λ into Equation 2, we get,

$$k = \frac{n\pi}{L}. \tag{6}$$

The wave speed v can thus be written as,

$$v = f\lambda = f\frac{2\pi}{k} = \frac{\omega}{k}. \tag{7}$$

5.2 Speed of wave on a string subject to tension

The speed of the waves on a string depends upon the mass of the string element and the tension T under which the string is stretched. The mass of the string element can be expressed in term of the linear mass density μ, which is the mass per unit length.

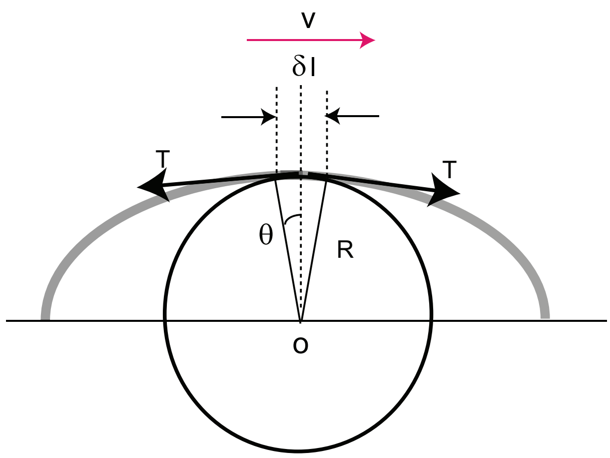

Figure 6: A small portion of string under the action of forces.

Let's consider a small section of string of length δl as shown in Figure 6. Here v represents the wave speed and the direction in which the wave is traveling is identified by the arrow. The element is part of an arc that is part of an an approximate circle of radius R. The mass δm of this element is $\mu\delta l$, where μ is the linear mass density. The tension T in the string is the tangential pull at each end of the segment. The horizontal components cancel since they are equal and opposite to each other. However, each of the vertical components is equal to $Tsin\theta$, so the the total vertical (downward) force is $2Tsin\theta$. Since θ is very small, we can approximate $sin\theta \approx \theta$. Considering the triangle shown in Figure 6, we find that, $2\theta = (\delta l)/R$. So, the net force F acting on the string element can be written as,

$$F = 2Tsin\theta \approx 2T\theta = \frac{T(\delta l)}{R}. \tag{8}$$

This is the force which is supplying the centripetal acceleration of the string particles towards O. The centripetal force F_c acting on the mass $\delta m = \mu\delta l$ moving in a circle of radius R with linear speed v is,

$$F_c = \frac{\mu(\delta l)v^2}{R} \tag{9}$$

Equating the two forces, we get,

$$T\frac{(\delta l)}{R} = \frac{\mu(\delta l)v^2}{R} \tag{10}$$

$$v = \sqrt{\frac{T}{\mu}}. \tag{11}$$

Thus, the wave speed depends upon the tension and the linear mass density of the string.

Q 2. Show that Equation 11 is dimensionally correct.

Q 3. A transverse sinusoidal wave is generated at one end of long horizontal string by a bar that moves the end up and down through a distance of 1.5 cm.The motion is repeated at a rate of 130 times per second. If a string has a linear density of 0.251 kg/m and is kept under a tension of 100 N, find the amplitude, frequency, speed and wavelength of the wave motion.

6 Experimental setup

Consider a stretched string that is fixed from one end through a rigid support, is strung over a pulley and a weight W is hung at the other end. The string can be set under vibrations using a mechanical oscillator, which in our case is a speaker (woofer) fed with a signal generator. Let L be the length between the wedge and the oscillator as shown in Figure 7. **If standing waves are established on the string**, then the wave vectors can be written as,

$$k_n = \frac{n\pi}{L}, n = 1, 2, 3, \ldots \tag{12}$$

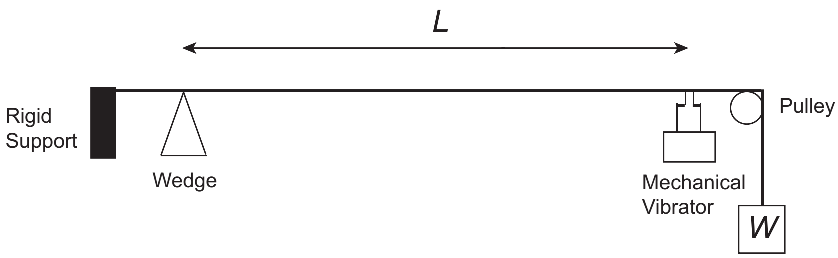

Figure 7: The experimental setup. A string is stretched between a rigid support and a mechanical vibrator, which in our case, is a speaker. Tension in the string is introduced by a hanging mass.

The relation between the angular frequency with the wave vector, is called the dispersion relation and depends upon the effective length and the tension. It can be derived as follows,

Equating Equations 11 and 7, we obtain,

$$\frac{\omega}{k} = \sqrt{\frac{T}{\mu}} \tag{13}$$

Hence, for the bare string, the dispersion relation is given by,

$$\omega(k) = \sqrt{\frac{T}{\mu}} \times k = n\sqrt{\frac{T}{\mu}} \times \frac{\pi}{L}. \tag{14}$$

where is ω is in radians. The equation clearly shows that that the frequency modes $\omega(k)$ are directly proportional to the wave vector. Thus the dispersion relation $\omega(k)$ versus k for the string is linear.

A normal mode of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency and in phase. The frequencies of the normal modes are known as its natural frequencies or resonant frequencies. A normal mode is characterized by a mode number n, and is numbered according to the number of half waves in the vibrational pattern. If the vibrational pattern has one stationary wave, the mode number is 1, for two stationary waves, the mode number is 2 and so on.

Our goal in this experiment is to locate the normal modes of the vibrating string. We will observe the formation of stationary waves on a vibrating string at resonance. The gist of the experiment is that the frequency $f = \frac{\omega}{2\pi}$ will be varied until a pattern of standing waves is observed. When this condition is achieved, there will be an integer number of half-wavelengths formed on the string, which will be vibrating with a large amplitude. This is precisely what a normal mode is! The driving frequency is in resonance with a normal mode frequency. We will note down the frequency f and the number n of half-wavelengths and will verify the relationship given in Equation (14).

6.1 Procedure

Q 4. After receiving all the equipment, set up the apparatus according to the illustration in Figure 7.

Q 5. Carry out the following experimental procedure to find the dispersion relation for the string.

1. Determine the value of μ of the string. (The density of the string is 8000 kg/m^3). What is the uncertainty in the value?
2. Place a driver at any place at the string such that the effective length L is approximately 1.5 m.
3. Attach a 1 kg weight with the other end of the string.
4. Connect the output leads of the signal generated to the woofer. Set the *Amplitude-Offset* at 10 V. Now sweep the frequency slowly of the signal generator and find the frequencies at which the resonance occurs, i.e., where you observe the maximum amplitude standing waves. Start with 1 Hz and increase the frequency with an increment of 0.1 Hz.
5. Note down all the frequencies and plot a graph between frequency and the number of the mode n. This is the desired dispersion relation. Show error bars along the frequency axis. How do you choose this uncertainty?

Q 6. How does your curve fit compare with the theoretical prediction?

Q 7. Change the string tension using different weights (e.g, 0.8, 1.0, 1.2, 1.4 kg and 1.6 kg) and plot the dispersion relation for each case. **Preferably, plot all your results with varying weight on the same graph.** Describe your observations.

Q 8. We would like you to graphically investigate the relationship between the resonant frequency and the tension in the string. Choose your appropriate variables and plot. Show your plots to the demonstrator.

Q 9. You will observe that at resonance, the string is actually exhibiting a circular trajectory instead of oscillating in a purely vertical plane. Can you see this effect? Explain why?

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