Mechanics of Precession and Nutation of Rings and Disks

Injila Rasul
Syed Babar Ali School of Science and Engineering, SBASSE
Lahore University of Management Sciences, LUMS

A thesis submitted for the degree of
Bachelor of Science

2017

My work is, and always will be,
dedicated to
the children of APS and Kasur

Acknowledgements

I am thankful to my supervisor, Dr.Sabieh Anwar, for his guidance, patience and commitment to science that is clearly reflected in his work and I hope to learn from him and develop a work ethic as strong as his.
I am also infinitely grateful to my family, for the life they have provided for me.

Abstract

The mechanics and dynamics of a rotating object depend heavily on the physical conditions of it's environment and it's own physical dimensions. In this project I seek to explore and understand how exactly these changes affect it's motion, what are the parameters that influence it, what are the initial conditions for which it behaves the way it does? And as all scientific work aims to do-how can I predict and control the motion?

The motivation for this exploration came form a paper: Terminal Retrograde turn of Rolling Rings (M.A.Jalali et al. [3]) that presented the findings of an unexpected reversal in the spiraling of spinning rings in the final stage of their motion. Using high speed imaging and numerical simulations of models they show that the rotational air drag resistance works in tandem with the frictional force at the contact point with the ground changing its direction at the inflection point and puts the ring on a retrograde spiral trajectory [3].

Our investigation into the behaviour of spinning bodies is built from the ground up, we start with the most simplistic version of the problem and add complexities step by step. At each step we pause and vary the conditions imposed to see how they affect and influence the motion of the object and attempt to understand how these conditions work together and influence each other to produce the results we obtain.

Contents

List of Figures

Chapter 1

Dynamics of Rigid Bodies

1.1 Introduction

To be able to study and model rings, disks and their motion it is imperative to begin with a thorough understanding of rigid body dynamics and it's intricacies. Rigid Bodies are a collection of N particles, given that the distance between the particles is invariant. Due to this invariance property, the position of every particle can be described in relation to the position of one point of the body, this reference point is usually the center of mass of the body(abbreviated: CoM or CM), in addition to the orientation about the specified reference point. This point behaves in response to forces as if the entirety of the force were directed at it. With rigid bodies the possible motions are translation of the CM and rotations about the CM [1]. To be able to progress and discuss the different kinds of motion that rotating rigid bodies exhibit,including but not limited to: precession, nutation, spin, it is important to build up from how the physics of a moving body is set up. This includes understanding how to identify points on the body, locate the body in the lab frame, build coordinate frame transformations to study the vectors involved and their evolution as seen from different perspectives, understand how rotational motion is projected down onto the two-dimensional plane and how to gather and process the related data via video tracking.

1.2 Properties of the Center of Mass

A total of six coordinates are required to specify the motion of any rigid body. The position of the first particle needs three coordinates, the second particle's position can be identified with two angular coordinates with respect to the first particle and the third particle needs one additional coordinate because it's position relative to

the first two is fixed. Consequently six equations of motion are needed to control, describe and predict the motion.

In the case of a two body system a complete analysis is possible given [1]:

$$\vec{F}_i^{ext} + \vec{F}_i^{int} = m_i \ddot{\vec{r}_i} \tag{1.1}$$

where $\vec{F}_i^{ext}$ is the force applied externally and $\vec{F}_i^{int}$ is the force the i^{th} particle experiences due to the other particles surrounding it:

$$\vec{F}_i^{int} = \sum_{i \neq j} \vec{F}_i^{[j]} \tag{1.2}$$

where $\vec{F}_i^{[j]}$ is the force on the i^{th} particle due to the j^{th} particle [1]

$$\vec{F}_i^{ext} + \vec{F}_i^{int} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_i m_i \ddot{\vec{r}_i} \tag{1.3}$$

Let $M = \sum_i m_i$ and $\vec{R} = \frac{\sum_i m_i r_i}{\sum_i m_i} = \frac{\sum_i m_i r_i}{M}$ then:

$$\sum_i \vec{F}_i^{ext} = M \ddot{\vec{R}} \tag{1.4}$$

In continuous media the mass element would be replaced by the mass density and the differential volume element integrated over the boundary of the object [1]. The CoM vector is given by:

$$\vec{R} = \frac{\int \vec{r} \rho(\vec{r}) dV}{\int \rho(\vec{r}) dV} \tag{1.5}$$

If we consider $\vec{r_i} = \vec{R} + \vec{r_i}'$ then

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum m_i \vec{r_i} = \frac{\mathrm{d}}{\mathrm{d}t} M \vec{R} \tag{1.6}$$

1.3 Energy of the system

The time derivative of ((1.6)) leads us to the system's momentum:

$$\vec{P} = \sum m_i \dot{\vec{r_i}} = M \dot{\vec{R}} = M\upsilon \tag{1.7}$$

The kinetic energy is then:

$$K = \frac{1}{2} \sum_i m_i \vec{\upsilon_i}^2 \tag{1.8}$$

$$K = \frac{1}{2}\sum_i m_i \vec{v}^2 + 2.\vec{v}.\vec{v}' + \vec{v}_i'^2 \tag{1.9}$$

The component $2.\vec{v}.\vec{v}$ goes to zero because of ((1.10)) [1]

$$\sum_i \vec{p_i}' = 0 \tag{1.10}$$

Allowing us to treat the center of mass as point carrying the momentum of the body.

The kinetic energy of the center of mass in addition to the kinetic energy relative to the center of mass is given by:

$$K = \frac{1}{2}MV^2 + K_{cm} \tag{1.11}$$

where K_{cm} is defined as $\frac{1}{2}\sum_i m_i v_i'^2$.

The total system's angular momentum will equate to the summation of the angular momentum about the center off ass and the angular momentum of the center of mass about the origin(of the vector $\vec{R}$) [1]:

$$\vec{L} = \sum_i m_i \vec{r_i} \times \vec{v_i} \tag{1.12}$$

$$\vec{L} = \sum_i m_i(\vec{R} + \vec{r_i}') \times (\vec{V} \times \vec{v_i}')$$

whereby

$$\vec{L} = M(\vec{R} \times \vec{V}) + \vec{L}'$$
$$\vec{L}' = \sum_i \vec{r_i}' \times \vec{v_i}'$$

As basic mechanics dictates, the translational motion should depend on the force applied and the rotational motion should depend on angular momentum [1]. Angular momentum is a measure of the rotational motion of an object as it is the product of the moment of inertia and the angular velocity. Angular momentum is constant when acted on by an external force–a torque in this case–and so it's rate of evolution depends solely on the torque applied giving us the following set of equations. For the translational motion we have:

$$\dot{\vec{P}} = \vec{F} \tag{1.13}$$

For rotational motion we have:

$$\dot{\vec{L}} = \vec{\tau} \tag{1.14}$$

where τ is the torque. These equations codify the important conservation law: momentum, be it rotational or translational, is always conserved.

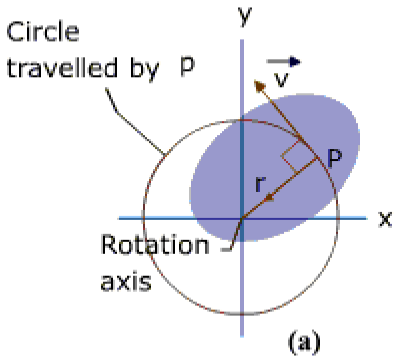

Figure 1.1: Rotation of a point P [6]

This is the complete set of six required equations that describe the motion of the body in all of it's degrees of freedom; $\dot{\vec{P}}$ depends on the components x, y, z while $\dot{\vec{L}}$ depends on the components r, θ, ϕ.

As any non-zero force implies the existence of an acceleration, the net external forces must reduce to zero for the center of mass to move with a constant velocity. The torque about the CoM point must be zero as a non-zero torque implies a change in the angular momentum about the CoM and thus is an indicator of rotational motion.

1.4 Rotation of Rigid Bodies

Rotation is defined as the motion of a point P about a line such that the distance form P to each point on the line is constant [1]. In an infinitesimal rotational displacement $\vec{dr}$ is perpendicular to the vector $\vec{r}$ which goes form the origin to the point P. $\vec{dr}$ is also perpendicular to the unit vector $\widehat{n}$

This gives us the following:

$$\vec{dr} \equiv \widehat{n} \times \vec{r} d\phi \tag{1.15}$$

$$\vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t} = \widehat{n} \times \vec{r}\dot{\phi} \tag{1.16}$$

We define ρ as the perpendicular radius of the rotation given by

$$\rho = \mid \widehat{n} \times \vec{r} \mid = r \sin(\theta)$$

This leads us to define the angular velocity vector, ω, as

$$\vec{\omega} = \omega\widehat{n} \equiv \widehat{n}(t)\phi \tag{1.17}$$

And then in relation to that, the rotational velocity vector about the $\widehat{n}$ is defined as:

$$\vec{v}_{rot} = \vec{\omega} \times \vec{r} \tag{1.18}$$

1.5 Moment of Inertia

We now move on to discussing the angular momentum and forming the inertia tensor. The angular momentum vector is not parallel to ω and can be written as follows:

$$\vec{L} = \sum_i m_i(\vec{r}_i \times \vec{v}_i) = \sum_i m_i \vec{r}_i \times (\vec{\omega}_i \times \vec{r}_i) \tag{1.19}$$

Then employing the triple product identity, and expanding it we obtain:

$$\vec{L}_x = \omega_x \sum_i m_i(y_i^2 + z_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i \tag{1.20}$$

$$\vec{L}_y = -\omega_x \sum_i m_i y_i x_i + \omega_y \sum_i m_i(x_i^2 + z_i^2) - \omega_z \sum_i m_i y_i z_i \tag{1.21}$$

$$\vec{L}_z = -\omega_x \sum_i m_i x_i z_i - \omega_y \sum_i m_i y_i z_i + \omega_z \sum_i m_i(x_i^2 + y_i^2) \tag{1.22}$$

We know that $\vec{L} = I\vec{\omega}$ so we note that the equations $(1.20, 1.21)$ and (1.22) are the result of a multiplication of a matrix containing information about Inertia with a vector representing $\vec{\omega}$. So we separate these terms and write the coefficients that form the inertia tensor as follows:

$$I_{xx} = \sum_i m_i(y_i^2 + z_i^2) \qquad I_{xy} = -\sum_i m_i x_i y_i \qquad I_{xz} = -\sum_i m_i x_i z_i \tag{1.23}$$

$$I_{yx} = -\sum_i m_i y_i x_i \qquad I_{yy} = \sum_i m_i(x_i^2 + z_i^2) \qquad I_{yz} = -\sum_i m_i y_i z_i$$

$$I_{zx} = -\sum_i m_i x_i z_i \qquad I_{zy} = -\sum_i m_i y_i z_i \qquad I_{zz} = \sum_i m_i(x_i^2 + y_i^2)$$

The inertia tensor written compactly is:

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

The angular momentum can then we written compactly as well:

$$\vec{L}_j = \sum_i I_{jk}\omega_k \tag{1.24}$$

It is important to note what the different indices of the inertia components represent:

$$I = \begin{cases} I_{ii}, & \text{moments of inertia} \\ I_{ij}, & \text{products of inertia} \end{cases}$$

Returning to the previous equations of torque and angular momentum we now account for the inertia and angular velocity in the formulae:

$$\begin{aligned} \vec{\tau} &= \dot{\vec{L}} && (1.25) \\ \tau_j &= \frac{\mathrm{d}}{\mathrm{d}t} L_j \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t} I_{jk}\omega_k \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{I}.\vec{\omega} && (1.26) \end{aligned}$$

This large matrix reduces to a simple 3×3 dimensional diagonal matrix, with only the moments of inertia values remaining and the products of inertia values reducing to zero. When this happens when we know that we have chosen a special orientation, called the Principal Axis of rotation, of the rotating body and taken advantage of the body's inherent symmetry.

The inertia tensor takes the form:

$$\begin{bmatrix} I_{xx} & 0 & I0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

And subsequently the kinetic energy becomes:

$$K = \frac{1}{2}(I_{xx}\omega_x + I_{yy}\omega_y + I_{zz}\omega_z) \tag{1.27}$$

1.6 Euler Equations

We once again refer back to the relationship between torque and angular momentum, giving us the Conservation Law:

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t}$$

The conservation of angular momentum is based on the assumption that all quantities were measured in an inertial frame, therefore Newton's second law could be invoked

[5]. If the exterior torques about the CoM are equal to zero then $\vec{L}$ is conserved [5]. To transform it to a frame that is rotating we need the equation transforming any vector in-between inertial and non-inertial frames of reference given by:

$$\left.\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right|_{fixed} = \left.\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right|_{rotational} + \vec{\omega}_{rel} \times \vec{A} \tag{1.28}$$

Where ω is the relative rotational velocity of the frames. And so for our case it takes the form:

$$\left.\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right|_{fixed} = \left.\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right|_{rotational} + \vec{\omega}_{rel} \times \vec{L} \tag{1.29}$$

The equation for τ transforms to:

$$\tau_x = I_x \frac{\mathrm{d}\omega_x}{\mathrm{d}t} - \omega_y \omega_z (I_y - I_z) \tag{1.30}$$

$$\tau_y = I_y \frac{\mathrm{d}\omega_y}{\mathrm{d}t} - \omega_x \omega_z (I_z - I_x) \tag{1.31}$$

$$\tau_z = I_z \frac{\mathrm{d}\omega_z}{\mathrm{d}t} - \omega_x \omega_y (I_x - I_y) \tag{1.32}$$

These equations are known as **Euler's Equations.**

1.7 Euler Angles

To be able to describe the orientation of the rotating body we need to construct Euler Angles, that in combination completely describe the new orientation of a rotating body with respect to any frame of reference. This can be the fixed lab frame, the initial body-fixed axes or any other orientation.

The first rotation is denoted by $\mathbf{R}(\phi)$ and is about the body-fixed z-axis. This rotates the x and y axes, and we denote the rotated axes by a primed notation.

$$x \rightarrow x'$$
$$y \rightarrow y'$$
$$z = z'$$

The second rotation is denoted by $\mathbf{R}(\theta)$ and it is about the body-fixed x' axis. The transformation rotates the y' and z' axes by an angle θ giving us a new set of body-fixed axes expressed in the double-primed notation.

$$x' = x''$$
$$y' \rightarrow y''$$
$$z' \rightarrow z''$$

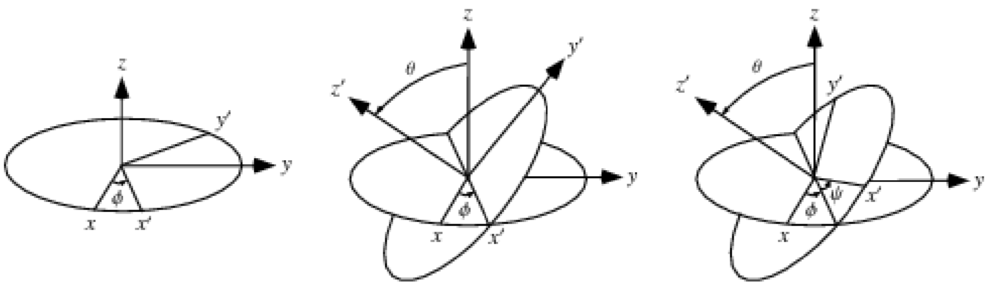

Figure 1.2: In order of appearance:$\mathbf{R}(\phi)$: a rotation about the z axis by an angle ϕ, $\mathbf{R}(\theta)$: a rotation about the $x^{'}$ axis by an angle θ, $\mathbf{R}(\psi)$: a rotation about the $z^{'}$ axis by an angle ψ in addition to the previous two rotations [4]

The third and final rotation is about the new $z^{''}$ axis, allowing for the $x^{''}$ and $y^{''}$ to rotate about the $z^{''}$ axis by an angle ψ. This is represented in the triple-primed coordinate system as:

$$x^{''} \rightarrow x^{'''}$$
$$y^{''} \rightarrow y^{'''}$$
$$z^{''} = z^{'''}$$

A complete rotation of the body would require the supposition of all three rotations and is represented by the matrix multiple off each individual rotation.

Representing these rotations in matrix form in terms of their projections back onto the lab-fixed X-Y-Z axes we have the important results:

$\mathbf{R}(\phi)$:

$$\begin{bmatrix} \cos(\phi)\sin(\phi) & 0 & \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R}(\theta)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$\mathbf{R}(\psi)$:

$$\begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The total rotation $\mathbf{R}(\phi)\mathbf{R}(\theta)\mathbf{R}(\psi)$ is given by:

$$\begin{bmatrix} \cos(\psi)\cos(\phi) - \cos(\theta)\sin(\phi)\sin(\psi) & \cos(\psi)\sin(\phi) + \cos(\theta)\cos(\phi)\sin(\psi) & \sin(\theta)\sin(\psi) \\ -\sin(\psi)\cos(\phi) - \cos(\phi)\sin(\phi)\cos(\psi) & -\sin(\phi)\sin(\psi) + \cos(\theta)\cos(\psi)\cos(\phi) & \sin(\theta)\cos(\psi) \\ \sin(\phi)\sin(\theta) & -\cos(\phi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Chapter 2

Numerical Solution to Top Dynamics

2.1 Analytic Formulation

Previously we worked out the Euler Angles and Euler Equations. It is not important to see how they knit the physics of the problem together. We start with referring back to angular velocity. The angular velocity vector points along the direction of the rotation of the object. Now that we have a system of angles that represent the orientation completely of a rotating object, let us find the angular velocity with respect to this efficient representation.

$$\begin{aligned} \vec{\Omega} &= \frac{\mathrm{d}\vec{\phi}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\theta}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\psi}}{\mathrm{d}t} \\ &= \dot{\vec{\phi}}\vec{e_z} + \dot{\vec{\theta}}\vec{e_{x''}} + \dot{\vec{\psi}}\vec{e_{z'''}} \end{aligned} \tag{2.1}$$

Here $\vec{e_i}$, where i is any index, denotes the unit vector in the direction of that specific axis, i.e, $\vec{e_{z'''}}$ is the unit vector pointing along the triple-primed z-axis.

We can construct a frame transformation, take the projections of each rotations on the frame whose representative ω we want to consider and take our work forward with that. Here we state the two important sets of transformations without proof. For the Body-Fixed reference frame:

$$\begin{aligned} \vec{\Omega} = & \Big(\sin(\theta)\sin(\psi)\dot{\phi} + \cos(\psi)\dot{\theta}\Big)\vec{e}_x \\ & + \Big(\sin(\theta)\cos(\psi)\dot{\phi} - \sin(\psi)\dot{\theta}\Big)\vec{e}_y \\ & + \Big(\cos(\theta)\dot{\phi} + \dot{\psi}\Big)\vec{e}_z \end{aligned} \tag{2.2}$$

For the Lab-Fixed reference frame:

$$\begin{aligned} \vec{\Omega} = & \Big(\cos(\phi)\dot{\theta} + \sin(\theta)\sin(\phi)\dot{\psi}\Big)\vec{e}_X \\ & + \Big(\sin(\phi)\dot{\theta} - \sin(\theta)\cos(\phi)\dot{\psi}\Big)\vec{e}_Y \\ & + \Big(\dot{\phi} + \cos(\theta)\dot{\psi}\Big)\vec{e}_Z \end{aligned} \tag{2.3}$$

Here the $\dot{\phi}, \dot{\theta}, \dot{\psi}$ are the rates of change of angular velocities in inertial co-ordinates.

2.2 Torque-free Motion

We start our work on building a rotating body by imagining the simplest case possible: a body that is rotating somewhere far away from us, so far in-fact that it is free from gravitation and the influence of any and all forces. The absence of external forces renders the torque vector as completely zero. And so our equations take the form:

$$I_x \frac{\mathrm{d}\omega_x}{\mathrm{d}t} = \omega_y \omega_z (I_y - I_z) \tag{2.4}$$

$$I_y \frac{\mathrm{d}\omega_y}{\mathrm{d}t} = \omega_x \omega_z (I_z - I_x) \tag{2.5}$$

$$I_z \frac{\mathrm{d}\omega_z}{\mathrm{d}t} = \omega_x \omega_y (I_x - I_y) \tag{2.6}$$

Giving us the first order differential equations:

$$\frac{\mathrm{d}\omega_x}{\mathrm{d}t} = \omega_y \omega_z \frac{(I_y - I_z)}{I_x} \tag{2.7}$$

$$\frac{\mathrm{d}\omega_y}{\mathrm{d}t} = \omega_x \omega_z \frac{(I_z - I_x)}{I_y} \tag{2.8}$$

$$\frac{\mathrm{d}\omega_z}{\mathrm{d}t} = \omega_x \omega_y \frac{(I_x - I_y)}{I_z} \tag{2.9}$$

We abbreviate the relative moment of inertia terms as γ, giving us the concise form for torque-free Euler equations:

$$\begin{aligned} \frac{\mathrm{d}\omega_x}{\mathrm{d}t} &= \omega_y \omega_z \gamma_x \\ \frac{\mathrm{d}\omega_y}{\mathrm{d}t} &= \omega_x \omega_z \gamma_y \\ \frac{\mathrm{d}\omega_z}{\mathrm{d}t} &= \omega_x \omega_y \gamma_z \end{aligned} \tag{2.10}$$

Where the asymmetry factors are defined as

$$\begin{aligned} \gamma_x &= \frac{(I_y - I_z)}{I_x} \\ \gamma_y &= \frac{(I_z - I_x)}{I_y} \\ \gamma_z &= \frac{(I_x - I_y)}{I_z} \end{aligned} \tag{2.11}$$

2.3 Adding Torques

To add toques we will have to refer back to the transformation equation for vectors in inertial or non-inertial reference frames:

$$\left.\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right|_{fixed} = \left.\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right|_{rotational} + \vec{\omega}_{rel} \times \vec{L} \tag{2.12}$$

If we consider a disk inclined and rotating on a horizontal surface, the torque is about the $\widehat{x}''$ axis and so it's components are:

$$\vec{\tau} = mgl\sin(\theta)\vec{e}_{x''} \tag{2.13}$$

For the simulation that we present in Chapter 3, the frame we worked in was the double-primed frame, so writing the coordinate transformations for the vectors in that frame as well, we have:

$$\vec{\omega} = \dot{\theta}\hat{x}'' + \dot{\phi}\sin(\theta)\hat{y}'' + \big(\dot{\phi}\cos(\theta) + \dot{\psi}\big)\hat{z}'' \tag{2.14}$$

We refer to the third component of $\vec{\omega}$ as $\vec{\omega_s}$. Then for the angular momentum we have:

$$\vec{L} = I\dot{\theta}\hat{x}'' + I\dot{\phi}\sin(\theta)\hat{y}'' + I(\omega_s)\hat{z}'' \tag{2.15}$$

We refer to the third component of I as I_s. Using these quantities and inputting them in (2.12) we construct the set of equations:

$$\tau_x = I\ddot{\theta} + \frac{1}{2}(I_s - I)\dot{\phi}^2\sin(2\theta) \tag{2.16}$$

$$\tau_y = \frac{\mathrm{d}}{\mathrm{d}t} I\dot{\phi}\sin(\theta) + (I - I_s\dot{\theta}\dot{\phi}\cos\theta) \tag{2.17}$$

$$\tau_z = I_s\ddot{\phi}\cos(\theta) - I_s\dot{\phi}\sin(\theta)\dot{\theta} \tag{2.18}$$

We know that all other components of τ are zero and only the x-component contributes and has a value equal to $mgl\sin(\theta)$.

$$mgl\sin(\theta) = I\ddot{\theta} + \frac{1}{2}\big(I_s - I\big)\dot{\phi}^2\sin(2\theta) \tag{2.19}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} I\dot{\phi}\sin(\theta) + \big(I - I_s\dot{\theta}\dot{\phi}\cos\theta\big) \tag{2.20}$$

$$0 = I_s\ddot{\phi}\cos(\theta) - I_s\dot{\phi}\sin(\theta)\dot{\theta} \tag{2.21}$$

These second order differential equations are coupled, and so we rearrange them to be able to solve for the Euler angles.

$$\ddot{\phi} = \frac{1}{I\sin(\theta)}\Big[(I_s - 2I)\dot{\theta}\dot{\phi}\cos(\theta) + I_s\dot{\theta}\dot{\psi}\sin(\theta)\Big] \tag{2.22}$$

$$\ddot{\theta} = \frac{1}{2}\Big[mgl\sin(\theta) + (I - I_s)\dot{\phi}^2\frac{1}{2}\sin(2\theta) - I_s\dot{\phi}\dot{\psi}\sin(\theta)\Big] \tag{2.23}$$

We express the second order derivatives of the Euler angles in terms of w_s which is a constant:

$$\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}t} = \frac{1}{I\sin(\theta)}\Big[I_s\dot{\theta}\omega_s - 2I\dot{\theta}\dot{\phi}\cos(\theta)\Big] \tag{2.24}$$

$$\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = \frac{\sin(\theta)}{I}\Big[mgl + I\dot{\phi}^2\cos(\theta) - I_s\omega_s\dot{\phi}\Big] \tag{2.25}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \omega_s - \dot{\phi}\cos(\theta) \tag{2.26}$$

2.4 Projections and Trajectories

The entirety of the work done until this point was in a body-fixed coordinate system. We now branch out and construct the transformations that represent what we see in the lab. For this we need to write the projections of the disk rotating on the horizontal plane onto the x, y and z axis and be able to trace out the trajectory of the center of mass of the body. We obtain:

$$X = l\sin(\theta)\cos(\phi) \tag{2.27}$$

$$Y = l\sin(\theta)\sin(\phi) \tag{2.28}$$

$$Z = l\sin(\theta) \tag{2.29}$$

Chapter 3

Modeling and Simulation

3.1 Revisiting the Euler Equations

To observe, assess and simulate the motion of a rotating body we refer back to the equations that govern motion, we begin again with the Euler Equations:

$$\tau_x = I_x \frac{\mathrm{d}}{\mathrm{d}t}\omega_x - \omega_y\omega_z(I_y - I_z) \tag{3.1}$$

$$\tau_y = I_y \frac{\mathrm{d}}{\mathrm{d}t}\omega_y - \omega_x\omega_z(I_z - I_x) \tag{3.2}$$

$$\tau_z = I_z \frac{\mathrm{d}}{\mathrm{d}t}\omega_z - \omega_x\omega_y(I_x - I_y) \tag{3.3}$$

These equations are applicable in situations where the net torque is zero, the center of mass is moving with a uniform speed or is completely at rest. We begin our analysis by looking at one such case–torque-free motion.

3.2 Torque-free motion

We begin our simulations with a body that is free of torques, and ergo free of gravity, represented by the Euler Equations in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_x = \omega_y\omega_z\gamma_x \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_y = \omega_x\omega_z\gamma_y \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_z = \omega_x\omega_y\gamma_z \tag{3.6}$$

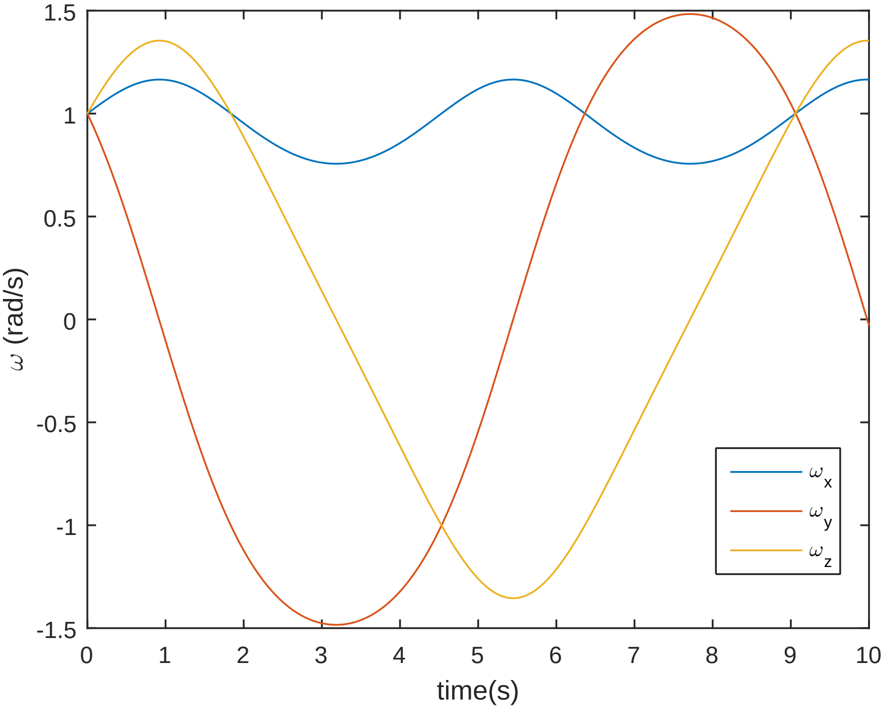

Figure 3.1: Plot of angular velocities for arbitrary values of $\omega_{initial}$ and moment of inertia.

Where the asymmetry factors are given by:

$$\gamma_x = \frac{(I_y - I_z)}{I_x}$$

$$\gamma_y = \frac{(I_z - I_x)}{I_y}$$

$$\gamma_z = \frac{(I_x - I_y)}{I_z}$$

To solve our set of coupled first order differential equations we used ode45 in MATLAB which is Runge Kutta solver. So for the most general case, let's set the Inertia vector as some non-zero value and initiate the $\vec{\omega}$ as a vector of all ones, and set the other parameters to some nonzero value (For the basics of this code please refer to the Appendix). When we solve for the most general case with all our values inputted as arbitrary and non-zero this is what we get Figure3.1

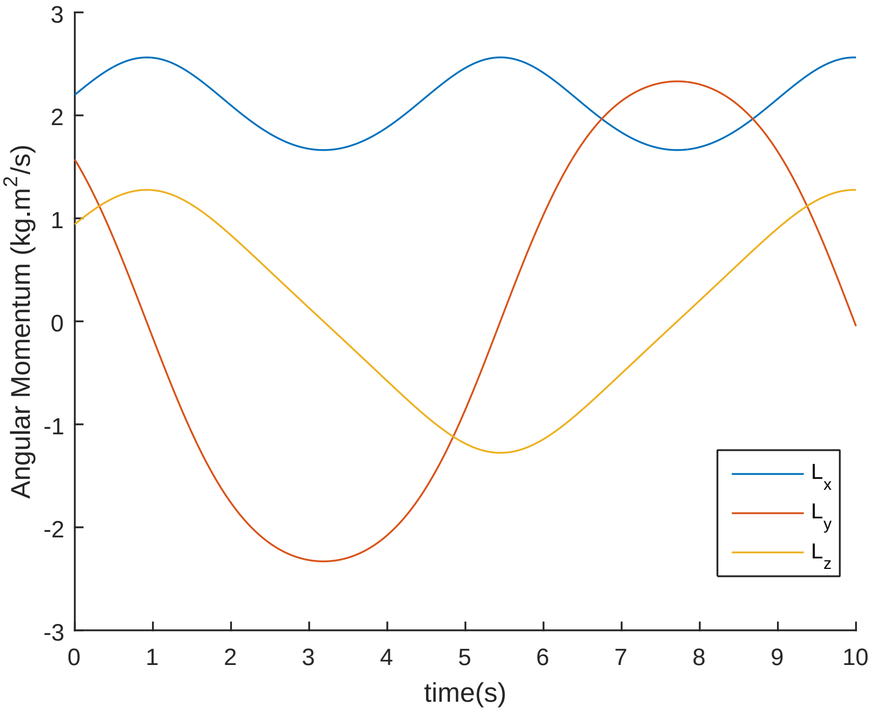

Figure 3.2: Plot of angular momentum versus time for the angular velocities for arbitrary values of $\omega_{initial}$ and moment of inertia

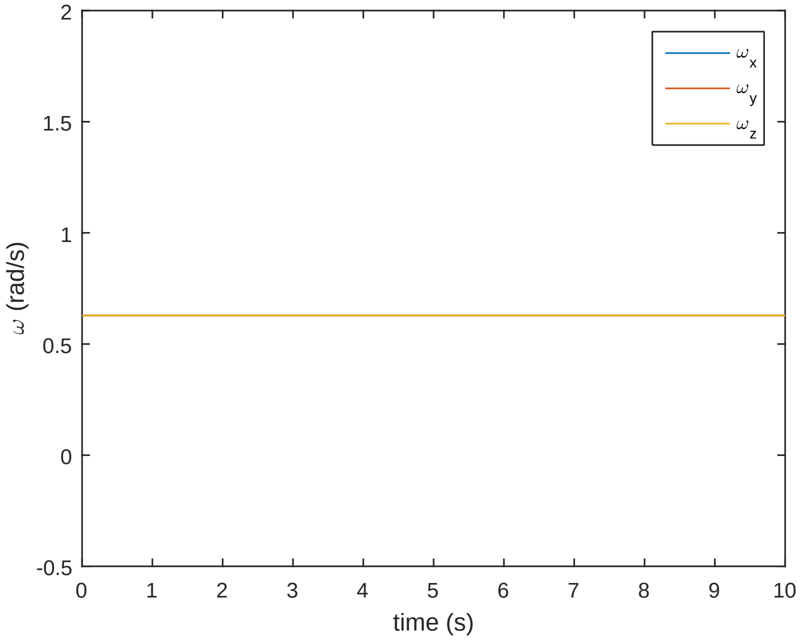

Figure 3.3: **Case 1**:Plot of angular velocity components for values of moment of inertia such that $I_x = I_y = I_z$. All of the angular momentum components reduce to a constant value as expected.

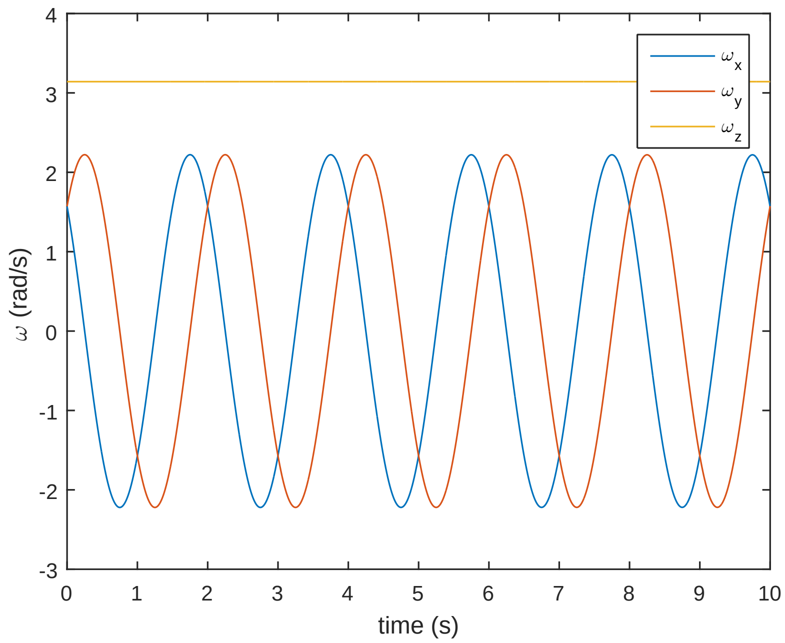

Figure 3.4: **Case 2**: Plot of angular velocity components for values of moment of inertia such that $I_x = I_y \neq I_z$

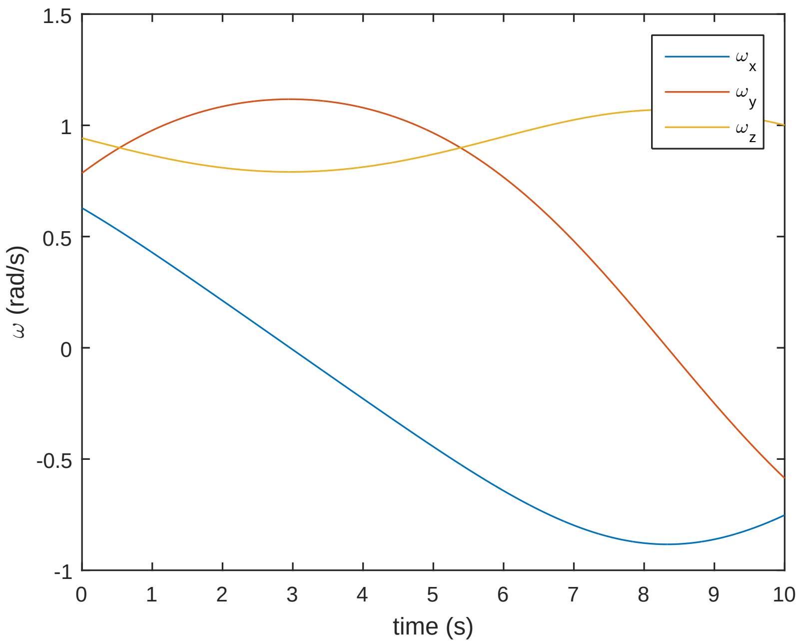

Figure 3.5: **Case 3** : Plot of angular velocity components for values of the moment of inertia that are very close to each other.

3.3 Symmetric Torque-Free Top

Now we add another condition to the previously modified Euler equations, we state that it is symmetric in two dimensions, i.e, $I_x = I_y \neq I_z$. This disrupts the asymmetry factors, reducing γ_z to zero. This sets the right-hand side of (3.6) to zero as well, which tells us that the third angular velocity component is constant as its derivative is zero.

We generate the plots for time versus the three components of omega as we previously did, using ode45 to solve the differential equations obtained from the Euler equations.

We generated a matrix of values evaluated at specific time intervals, and every succeeding value is estimated using the previous point, we further use this iteratively generated set of values to calculate essentially, the area under the graph. For this purpose we use the trapezoidal method of numerical integration and calculate and plot the values obtained, please note that a cumulative summation method was used. A simple summation would yield a final value of the integrand rather than a vector of values.

$$\Omega(\theta_i) = \int \theta_i \tag{3.7}$$

For i = 1, 2, 3. Each of these plotted angular velocities is integrated numerically to yield a corresponding matrix of values.

The third integration in monotonously increasing and this makes intuitive sense because the graph being integrated never dips below the y-axis, to account for this we simply readjust and scale that line such that the original ω_z line lies coincident with the x axis, this yields the more concise graph in Figure (3.8).

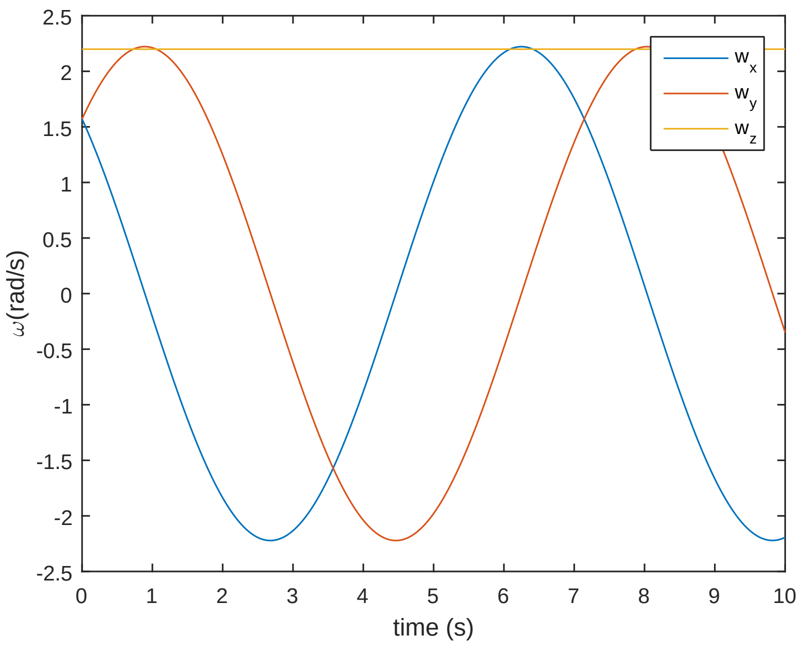

Figure 3.6: Plot of angular velocities versus time for $I_x = 0.5\pi = I_y \neq I_z$, where $I_z = 0.7\pi$

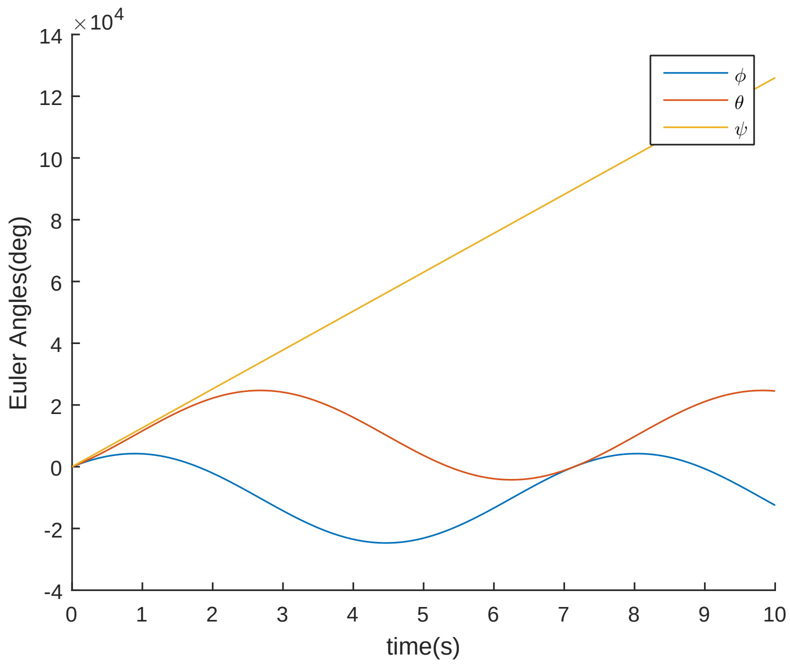

Figure 3.7: Plot of cumulative numerical integration of individual angular velocity components

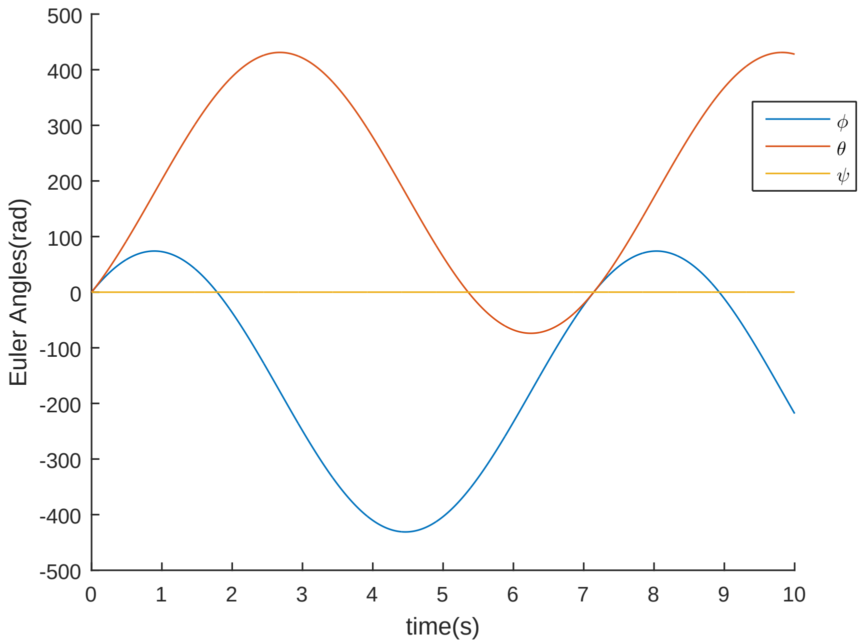

Figure 3.8: Numerical integration of the individual angular velocity components

3.4 Accounting for Gravity and Subsequent Variations

We now return to the analytic analysis of what we have obtained thus far to take one step further: adding gravity to the equation. We started with an arbitrary vector of the initial angular velocities of the body, and a symmetric and toque free system, we calculated the angular velocities as time rolled on, and also formulated calculation of the angular momentum should we need it, and then proceeded to integrate the graphs of each individual angular velocity component to obtain numerically the angle whose change over time it represented. We adjusted for compactness of our results and reset a datum line.

Here we pause to refer back to the (2.2) equation that expressed the relationship between the Euler angles (ϕ, θ, ψ) and the angular velocity components(w_x, w_y, w_z):

$$\omega_x = \dot{\phi}\sin(\theta)\sin(\phi) + \dot{\theta}\cos(\psi)$$
$$\omega_y = \dot{\phi}\cos(\theta)\cos(\psi) - \dot{\theta}\phi$$
$$\omega_z = \dot{\phi}\cos(\theta) + \dot{\psi}$$

The first-order linear differential equations are coupled and so we solve them simultaneously using one again, ode45, to solve for the angles. The next step is to derivate the terms obtained for the Euler angles, shift from radians to degrees and plot our results for the conditions $I = 1, I_s = 1.5, l = 0.1, ws = 0.5, g = 9.8, m = 0.1$:

We see in Figure 3.9 that the function for ψ and ϕ is increasing without bound so we must fix that to allow it to better align with our physical understanding of the seamless way angles and angular rotations are stitched together. We use a wrapping function to generate a saw-tooth shape for the angles ϕ and ψ as expected, as shown in Figure 3.10.

It is important at this point to pause and vary the parameters of the problem and seek to understand the effect on our data and it's consequent physical connotations.

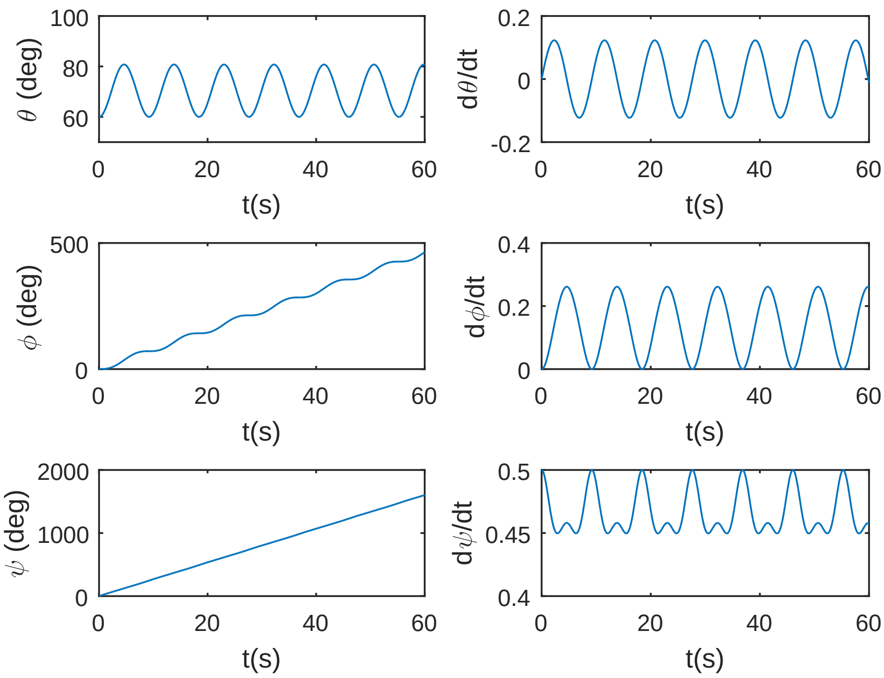

Figure 3.9: Euler angles and their derivatives, for the conditions: $(I = 1, I_s = 1.5, l = 0.1, \omega_s = 0.5, g = 9.8, m = 0.1)$

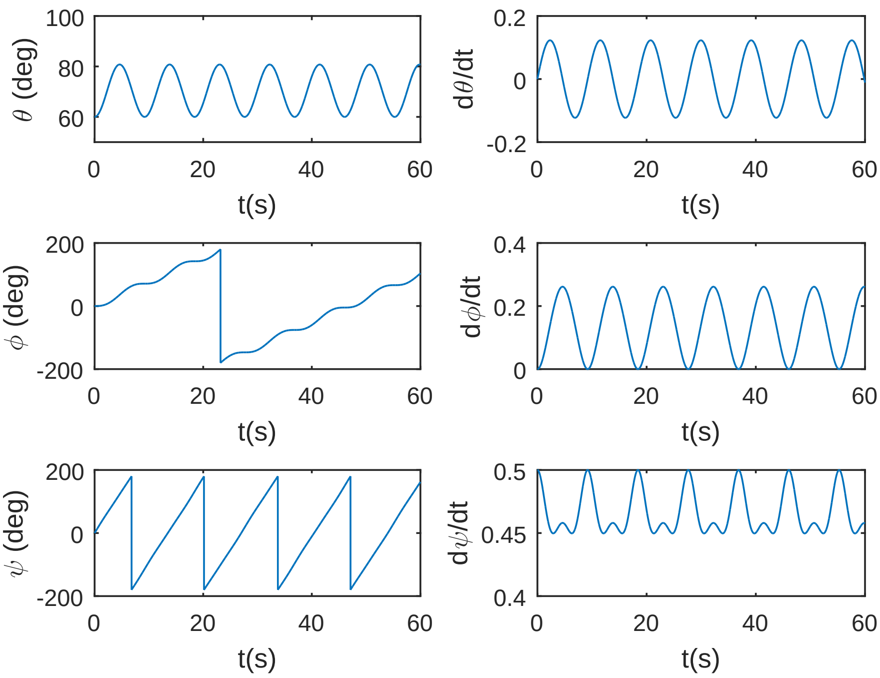

Figure 3.10: Euler angles and their derivatives, readjusted, for the conditions: $(I = 1, I_s = 1.5, l = 0.1, \omega_s = 0.5, g = 9.8, m = 0.1)$

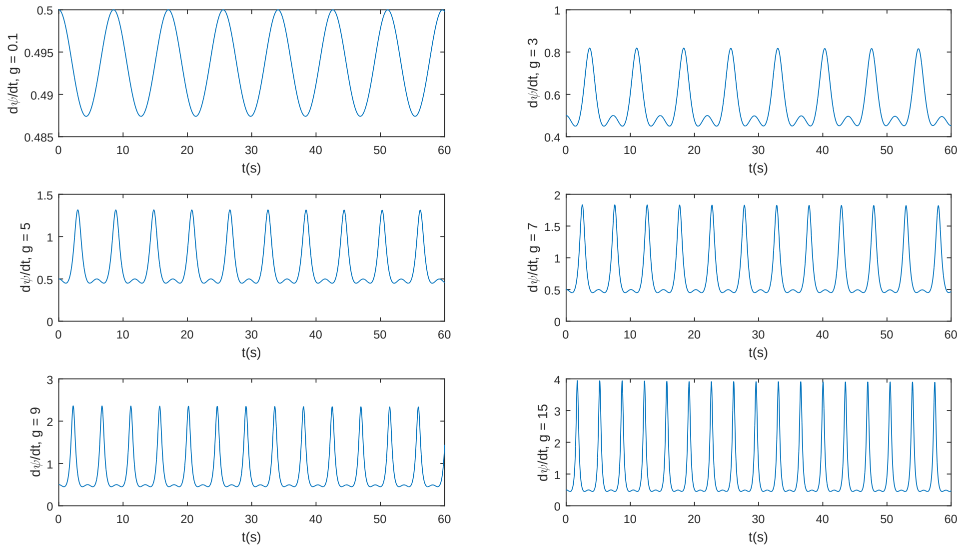

Figure 3.11: Euler angles and their derivatives for the values of $g = 0.1, 3, 5, 7, 9, 15$, all else remains the same ($I = 1, I_s = 1.5, l = 0.1, \omega_s = 0.5, m = 0.1$)

3.5 Variations and Analysis

Let us begin with varying the gravity acting on the system; starting from a value of 0.1 and in successive steps crank it up till 15. The most obvious alteration observed due to this is a change in the shape of the $\frac{d\psi}{dt}$ graph as the value of the gravitational field strength increases; initially there is a single frequency and as the value climbs up the the subsequent graphs have a small bump in the middle, as seen in Fig(3.11). The bump was not present for the value of $g = 0.1$, we can further try and investigate exactly where and at what point the bump starts to occur, we do so in a comparative plot of just the angle for values of g that vary over a small interval with smaller increments in Fig(3.12): The bump originates around the 0.8 mark.

In terms of general changes that we noted were that the range of values of θ increased an in increase in the value of the gravitational constant. The rate of change of θ also responded similarly, increasing in both frequency and amplitude, ϕ reached greater values in smaller time periods.

For a variation of the initial condition of ω_s we see the following evolution in the Euler angles in Fig(3.13) and then we change the value and see the difference it makes in Fig(3.14) and Fig(3.15).

Generating images for the angles that varied from 0.1 to 0.5 by an increment of 0.1 it was observed that the plots got denser and the frequency increased by each incremental step. The amplitude, though, remained unchanged. Fig(**??**) and Fig(**??**) show the corresponding graphs for the minimum and maximum values of mass taken, respectively. What is interesting is the shift in the form of ψ and $theta$ that is clearer in the graph for a higher value of m, and was not evident in the graph for the lower value of m.

We now want to see how the angles change individually over all these various conditions. To have such a holistic view, we can plot together the different graphs we get from varying one condition each time. For the following sets of images, we will vary one condition at a time (labeled on the y-axis) and lump together the evolution of one of the angles over all of these variations to give us a thorough idea of how these changes come about. First, we look at θ and $\frac{d\theta}{dt}$ and how they change.

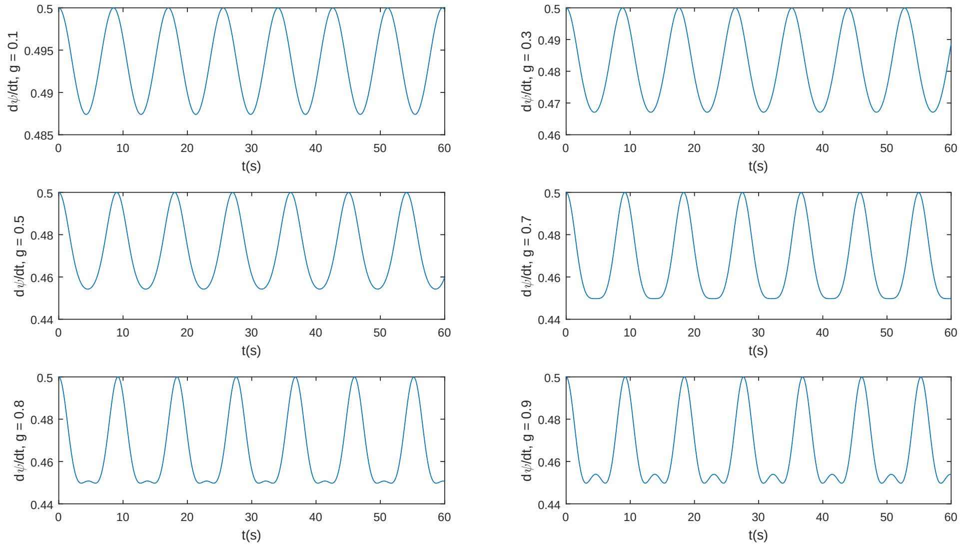

Figure 3.12: Euler angles and their derivatives for the values of $g = 0.1, 0.3, 0.5, 0.7, 0.8, 0.9$, all else remains the same $(I = 1, Is = 1.5, l = 0.1, ws = 0.5, m = 0.1)$

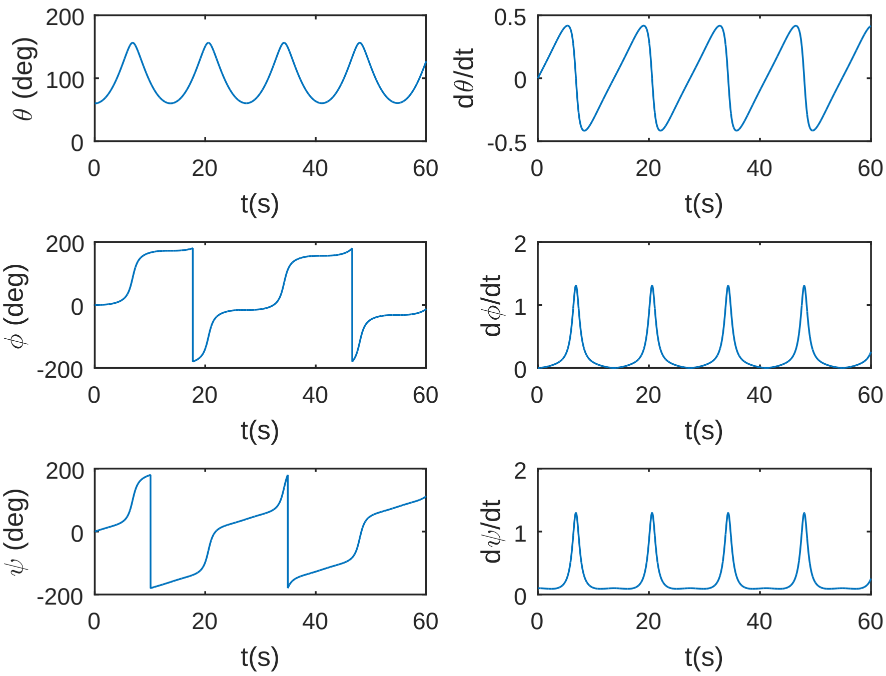

Figure 3.13: Euler angles and their derivatives for the values of $\omega_s = 0.1$, all else remains the same $(I = 1, I_s = 1.5, l = 0.1, g = 9.8, m = 0.1)$

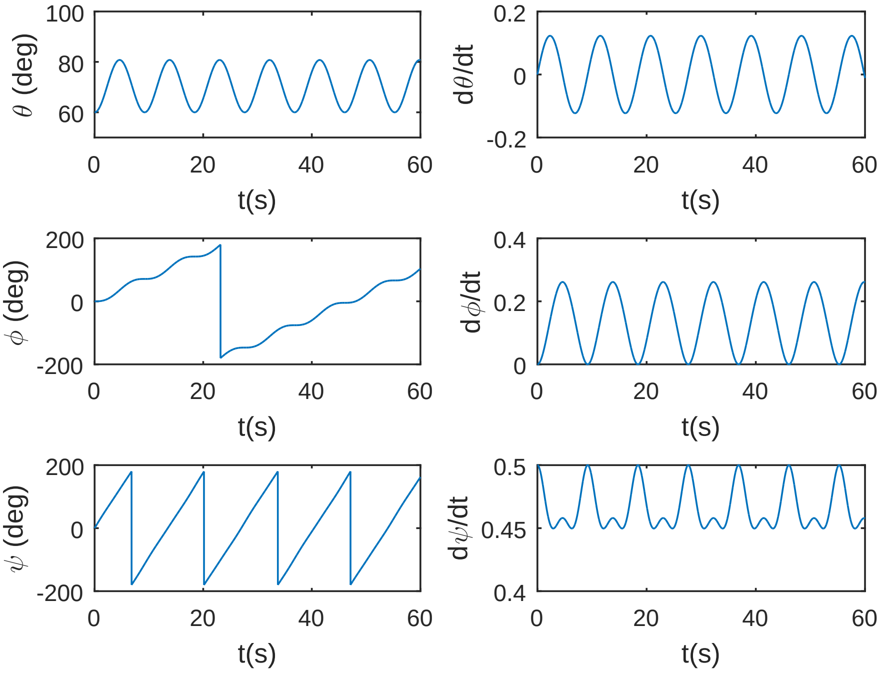

Figure 3.14: Euler angles and their derivatives for the values of $\omega_s = 0.5$, all else remains the same $(I = 1, I_s = 1.5, l = 0.1, g = 9.8, m = 0.1)$

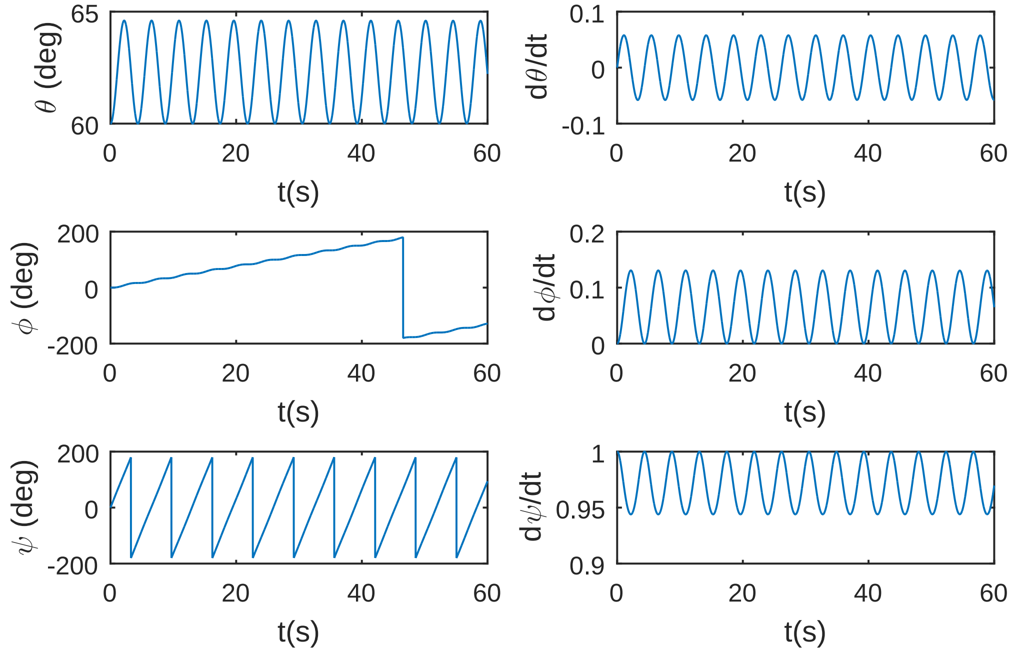

Figure 3.15: Euler angles and their derivatives for the values of $\omega_s = 1$, all else remains the same $(I = 1, I_s = 1.5, l = 0.1, g = 9.8, m = 0.1)$

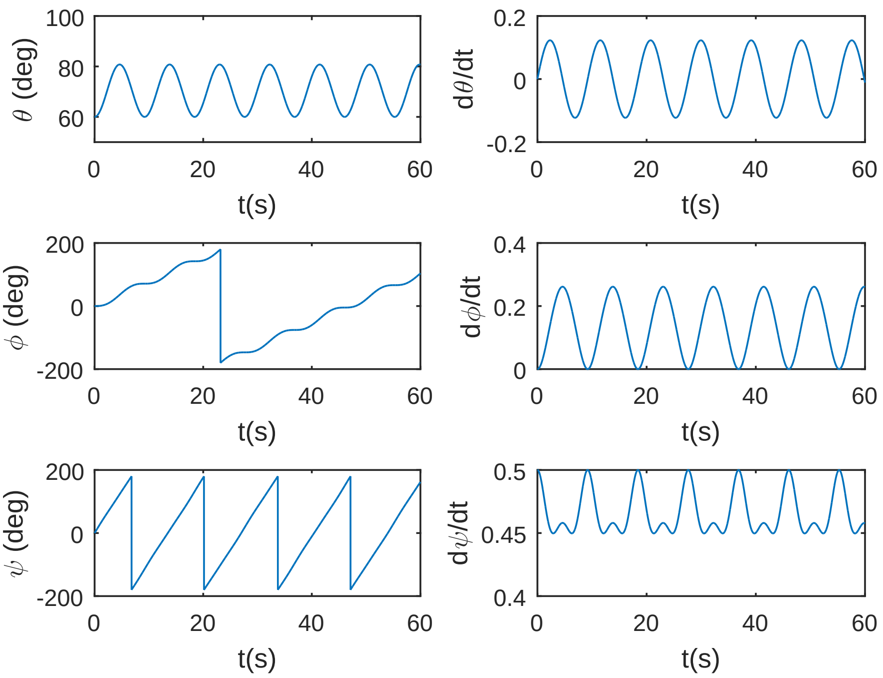

Figure 3.16: Euler angles and their derivatives for the values of $m = 0.1$, all else remains the same $(I = 1, I_s = 1.5, l = 0.1, g = 9.8, \omega_s = 0.5)$

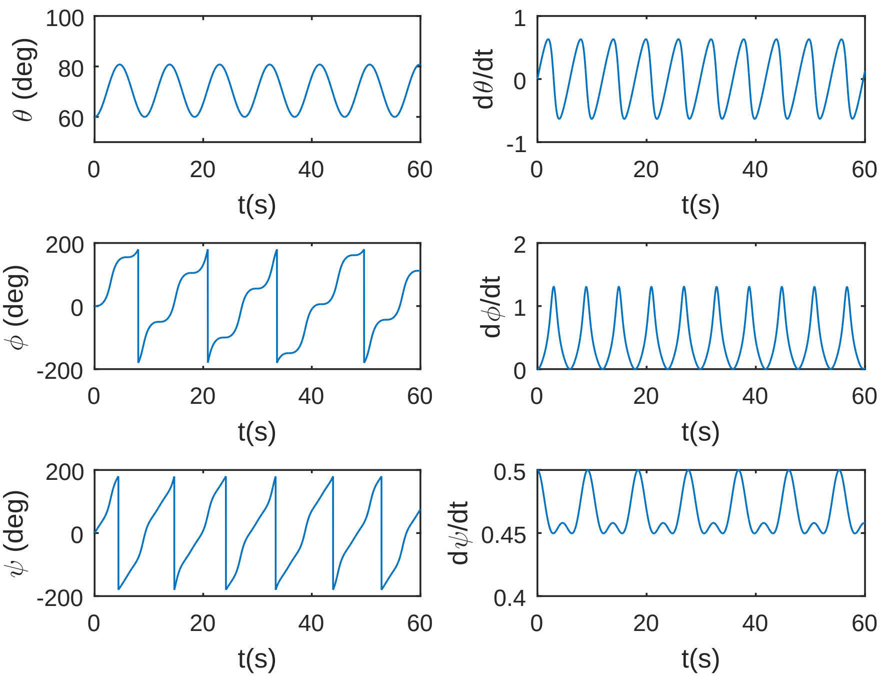

Figure 3.17: Euler angles and their derivatives for the values of $m = 0.5$, all else remains the same $(I = 1, I_s = 1.5, l = 0.1, g = 9.8, \omega_s = 0.5)$

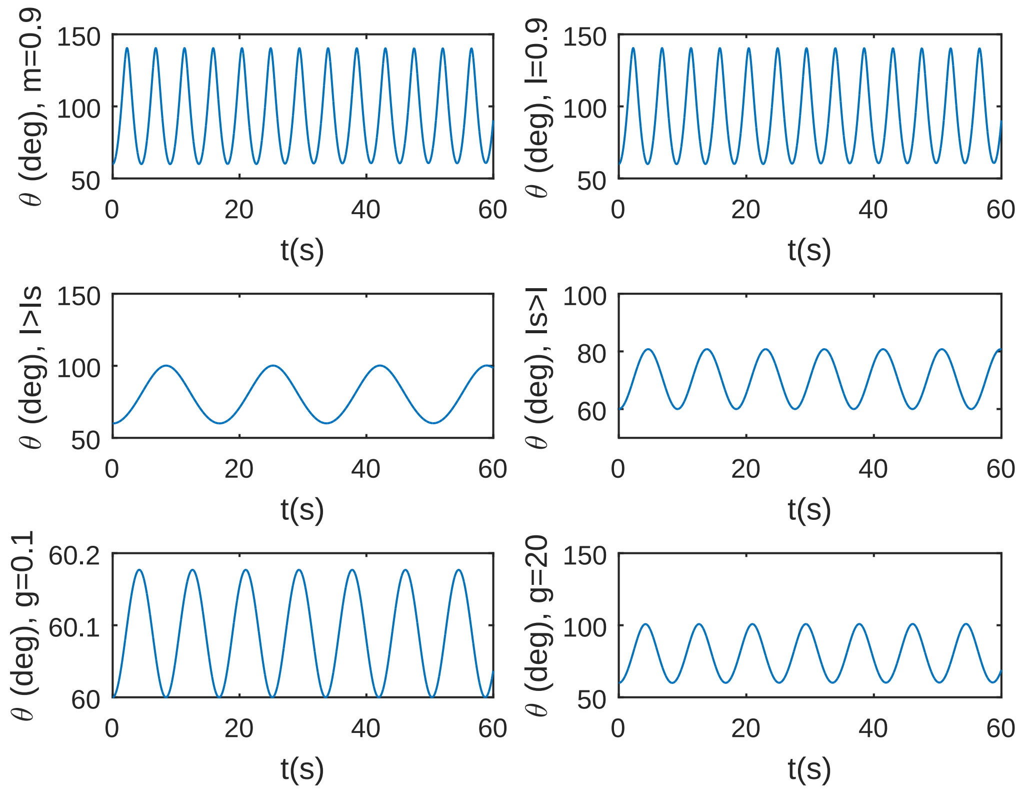

Figure 3.18: The individual evolution of the angle θ over the conditions $m = 0.9, l = 0.9, I > I_s, I < I_s, g = 0.1, g = 20$

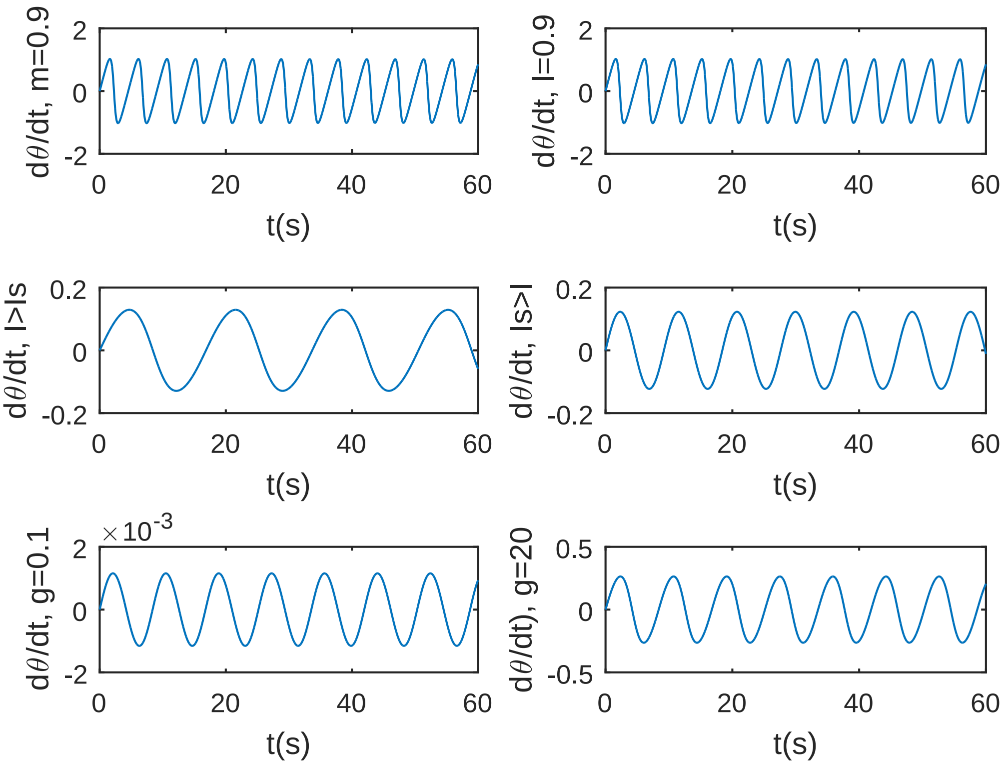

Figure 3.19: The individual evolution of $\frac{d\theta}{dt}$ over the conditions $m = 0.9, l = 0.9, I > I_s, I < I_s, g = 0.1, g = 20$

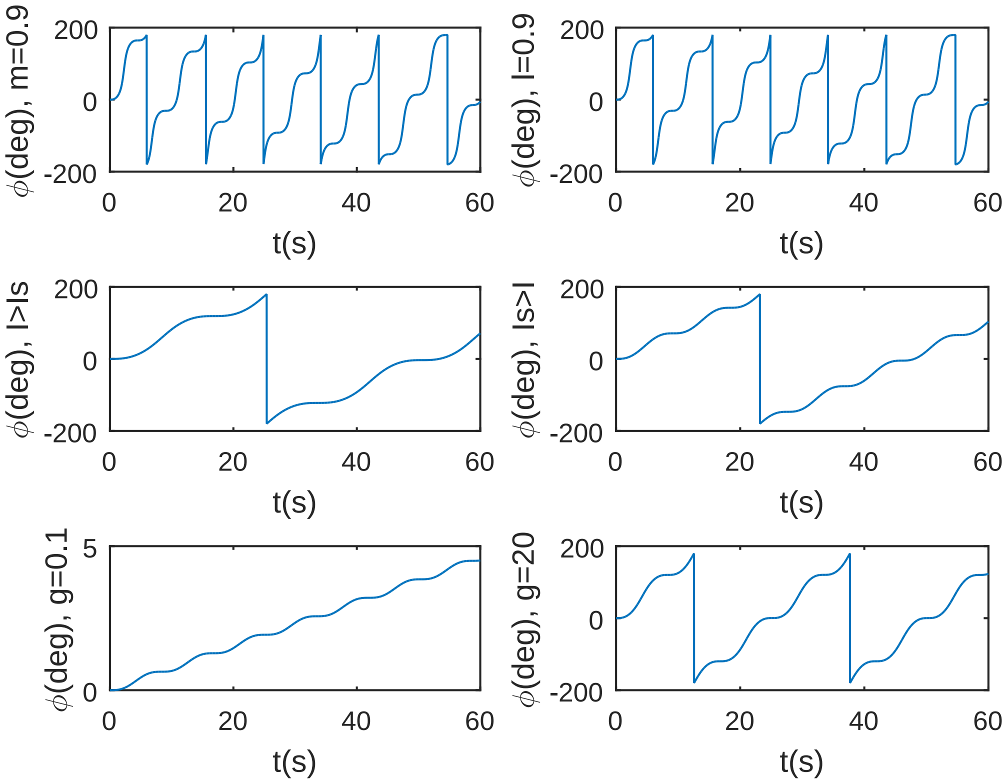

Figure 3.20: The individual evolution of the angle ϕ over the conditions $m = 0.9, l = 0.9, I > I_s, I < I_s, g = 0.1, g = 20$

Similarly, we look at the evolution of ϕ and $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ over the same set of conditions in Fig(3.20) an Fig(3.21) respectively.

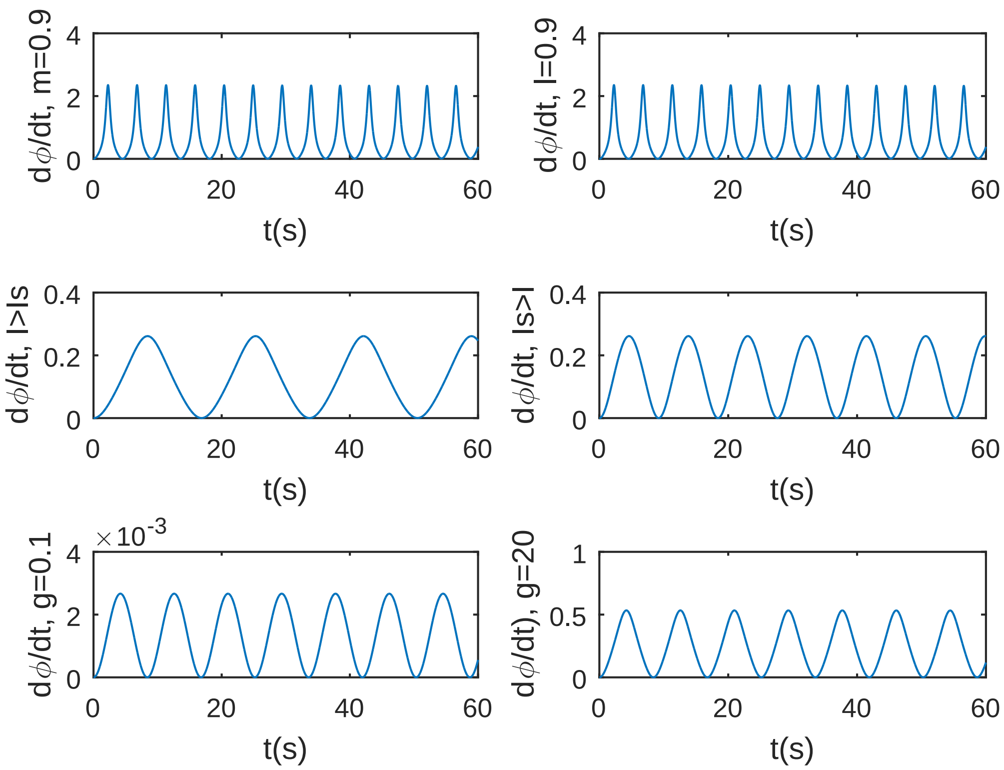

Figure 3.21: The individual evolution of the angle $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ over the conditions $m = 0.9, l = 0.9, I > I_s, I < I_s, g = 0.1, g = 20$

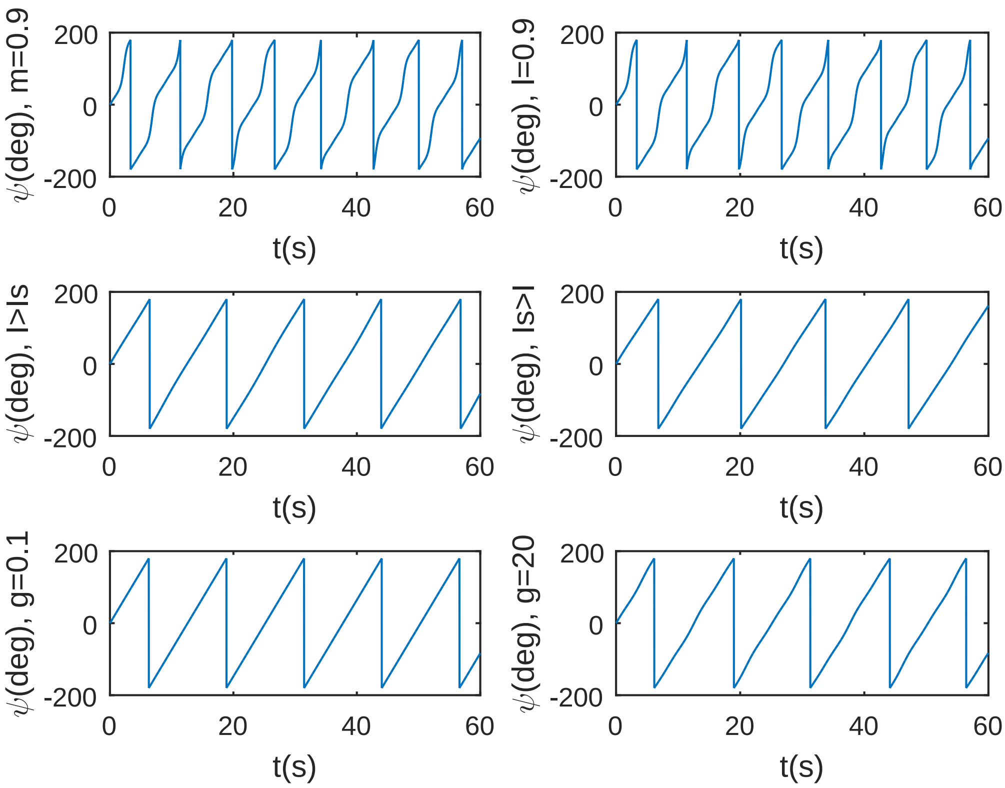

Figure 3.22: The individual evolution of the angle ψ over the conditions $m = 0.9, l = 0.9, I > I_s, I < I_s, g = 0.1, g = 20$

And lastly we look at the changes in ψ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ as shown in Fig(3.22) and Fig(3.23).

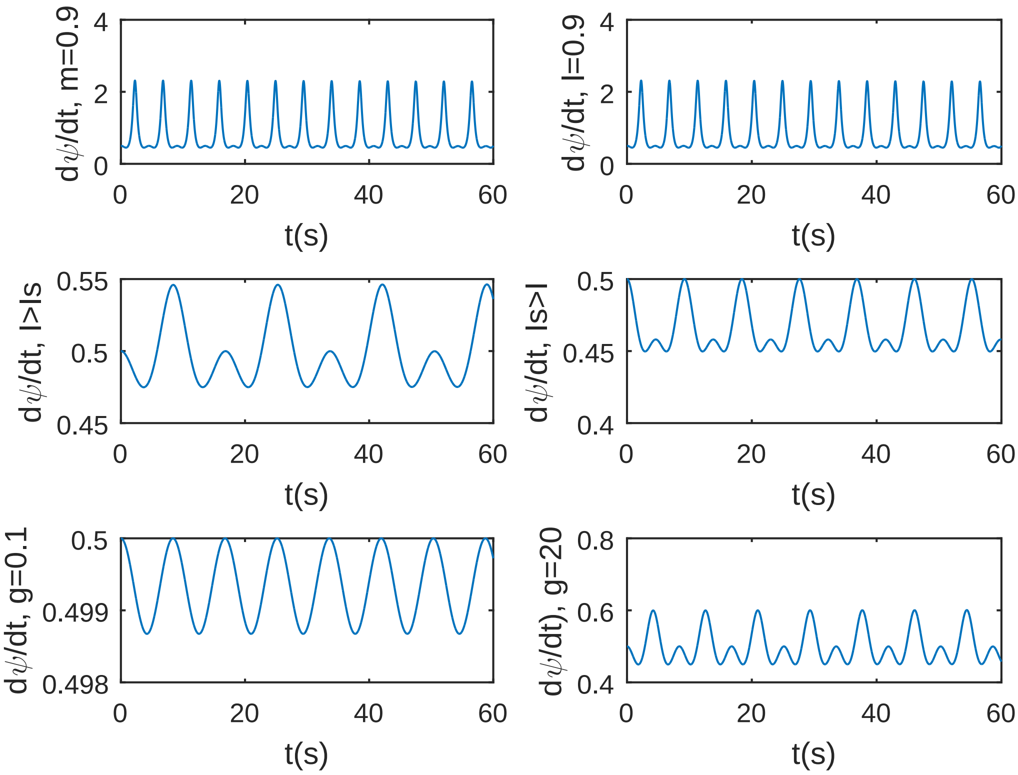

Figure 3.23: The individual evolution of the angle $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ over the conditions $m = 0.9, l = 0.9, I > I_s, I < I_s, g = 0.1, g = 20$

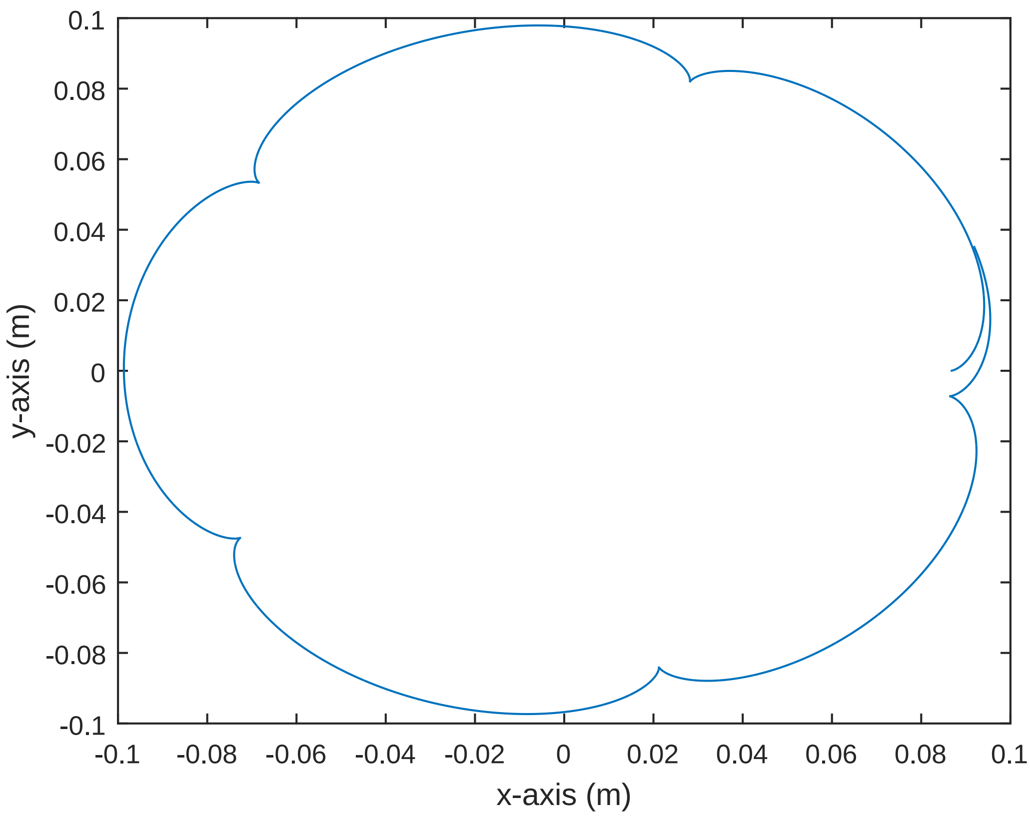

Figure 3.24: Projection of the center of mass of a rotating disc onto the x-y plane, as viewed from the top, for the conditions I=1, Is=1.5, l=0.1, ws=0.5, g=9.8, m=0.1

3.6 Frame Transformations and Projections

For this system let us revisit the starting point of our formulation. We started with our conventions and derivations set in terms of physical quantities as observed in the body fixed frame of reference. But what do we actually observe in the lab? For this we refer back to the the analysis in the previous chapter where we constructed the projections of the Center of Mass onto a plane; it is essentially a representation of what the camera detects and records. So to be able to understand how the data that we collect relates to the quantities we calculated and worked with previously, we need to be able to transform them interchangeably within the two frames as required.These projections for the system we started with and solved for initially are restated here as:

$$x = l\sin(\theta)\cos(\phi) \tag{3.8}$$
$$y = l\sin(\theta)\sin(\phi) \tag{3.9}$$
$$z = l\sin(\theta) \tag{3.10}$$

So if I calculate and plot for what is actually observed in the lab there are basically two things that can be done: (i) a plot from a bird's-eye perspective and (ii) a plot from the side of the rotating body.
The second plot clearly shows nutation of the CoM and this makes intuitive sense because nutation is the change of θ and the xz-plane projection depends solely on a change in θ.

It is these projections, particularly the projection onto the xy-plane that we record in our experimental analysis of the problem. Following is a compilation of the trajectories recorded of

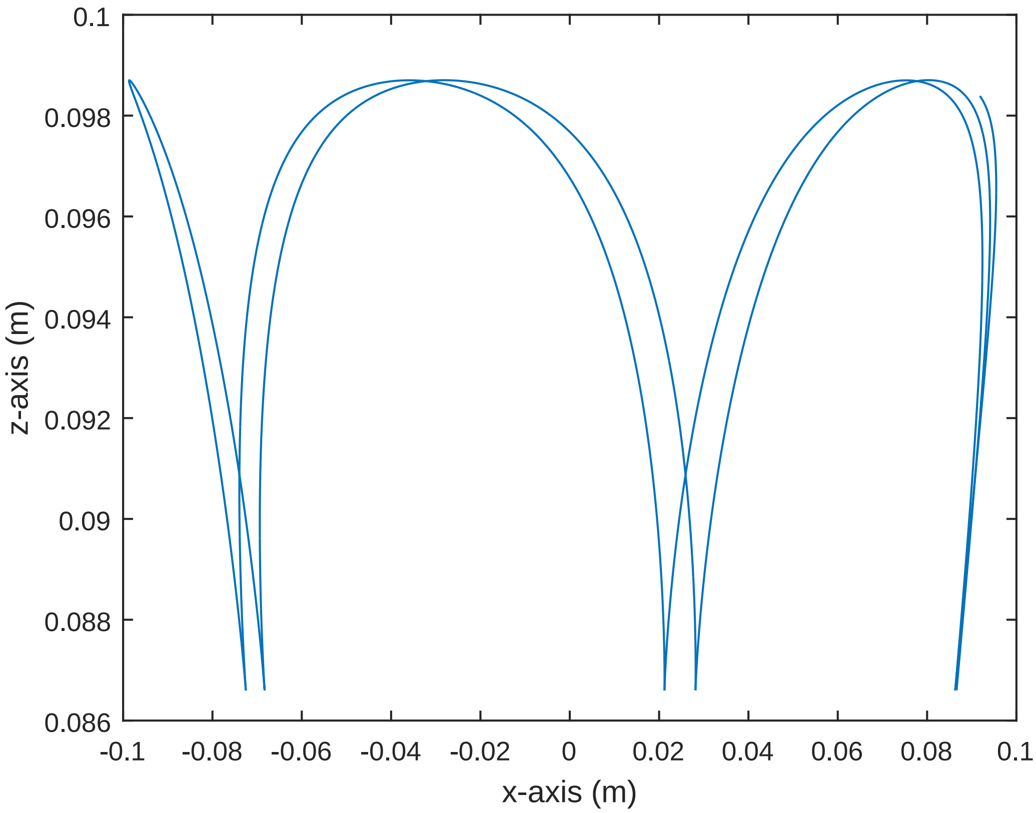

Figure 3.25: Projection of the center of mass of a rotating disc onto the x-z plane, as viewed from the side, for the conditions I=1, Is=1.5, l=0.1, ws=0.5, g=9.8, m=0.1

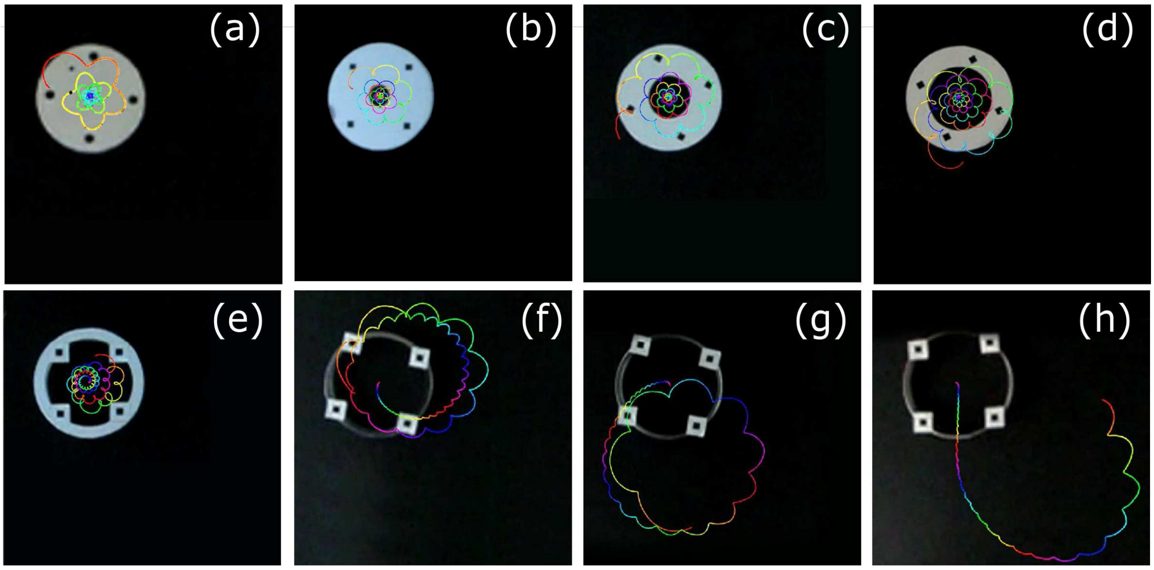

Figure 3.26: Recordings of the trajectories of the center of mass of different discs and rings onto the x-z plane, as viewed from the top

different discs and rings.

To be able to better emulate this problem in our simulations and align it with our experimental recordings and results, we need to account for friction between the disc and the contact point with the surface and also to change the moment of inertia vector and consider bodies of different symmetries and configurations.

A careful case-wise study of the different angles, configurations and situations for the simple case of a rolling disc is done by M. Batista in his paper 'Steady Motion of a Rigid Disk of Finite Thickness on a Horizontal Plane' where he investigates pure rolling and pure spinning about a diameter [2]. In addition to this he discusses and develops the formalism for straight-line and circular motion as well as motion on rough and smooth ground [2]. This is an important study in friction and it's effects on the motion and dynamics of rotational bodies and it's understanding and application is the next step forward in order to be headed to the original goal of discussing and understanding Jalali's findings.

Appendix A

MATLAB Code

A.1 Evolution of arbitrary $\vec{\omega}$ and $\vec{L}$

The code required for plotting the evolution of ω for a given set of initial conditions is as follows; the comments are denoted by a % sign.
%need three things for ode45 %initial values of w, range of t, the function generating w
clc
close all
I = [pi*0.7, pi*0.5, pi*0.3];
wi = [1 ,1, 1]; %initial value of w
% asymmetry gamma factors
g1 = (I(2)-I(3))/I(1);
g2 = (I(3)-I(1))/I(2);
g3 = (I(1)-I(2))/I(3);
%now input into the ode45 solver
$[t, x] = ode45(@omegaDot, [0 : 0.01 : 10], wi, I, g1, g2, g3)$;
%plot the angular velocities figure; plot(t,x); xlabel('time(s)'); ylabel('omega (rad/s)');
legend($'wx','wy','wz'$);
%calculating w, ode45 gave x = omegadot, we need w right now
w1=x(:,1);
w2=x(:,2);
w3=x(:,3);
%angular momentum
L1=I(1)*w1;
L2=I(2)*w2;
L3=I(3)*w3;
figure;
%plot of the omegadot generated by the ode45 code as follows
hold on
plot(t,L1);
plot(t,L2);
plot(t,L3);
legend('Lx','Ly','Lz');
xlabel('time(s)'); ylabel('Angular Momentum ($kg.m^2/s$)');

function $[y]$ = omegaDot(I,w, g1, g2, g3)
y = [0,0,0];%initialize y vector

% y is the omegaDot vector

```
y(1) = w(2)*w(3)*g1;
y(2) = w(1)*w(3)*g2;
y(3) = w(1)*w(2)*g3;
%ode45 needs a column vector so take a transpose, MATLAB is row-major
y = y';
end
```

A.2 Numerical Integration of Angular Velocity

```
clc
close all
I = [pi*0.7, pi*0.5, pi*0.3];
wi = [1 ,1, 1]; %initial value of w
% asymmetry gamma factors
g1 = (I(2)-I(3))/I(1);
g2 = (I(3)-I(1))/I(2);
g3 = (I(1)-I(2))/I(3);
%now input into the ode45 solver
[t,x] = ode45(@omegaDot,[0:0.01:10],wi,I,g1,g2,g3);
w1=x(:,1);
w2=x(:,2);
w3=x(:,3);

% cumtapz is a function that uses the trapezoidal rule to cumulatively integrate over a function in a given interval ang1 = cumtrapz(w1);
ang2 = cumtrapz(w2);
ang3 = cumtrapz(w3);

time = 0:0.01:10;
figure
hold on

plot(time,ang1)
plot(time,ang2)
plot(time,ang3)
title('Numerical integration of the angular velocities');
xlabel('time(s)'); ylabel('Euler Angles(rad)');
legend('phi','theta','psi');
hold off
```

A.3 Euler Angles and their Derivatives

For the inclusion of torques the equations change and so we need to re-write the function that is being fed into the ode45 solver.

```
clc
clear all
I=1;
Is=1.5;
l=0.1;
ws=0.5;
time=0:.01:60;
```

```
g=9.8;
m=0.1;
wi=[0;0;pi/3;0; 0];%phi, dphi, theta, dtheta, psi
[t,w]=ode45(@eulersolv,time,wi,[],I,Is,l,g,ws,m);

%the angles are wrapped at pi using the wrapToPi function, if we do not use this we get monotonously increasing functions
phi= wrapToPi(w(:,1));
%phi= (w(:,1));
dphi = w(:,2);

theta=wrapToPi(w(:,3));
%theta=(w(:,3));
dtheta = w(:,4);

psidot=ws-w(:,2).*cos(w(:,3));
psi = wrapToPi(w(:,5));
%psi = (w(:,5));

figure;

subplot(3,2,1);
plot(t,theta*180/pi);
xlabel('t(s)');
ylabel('theta (deg)');

subplot(3,2,2);
plot(t,dtheta);
xlabel('t(s)');
ylabel('d theta/dt');

subplot(3,2,3);
plot(t,phi*180/pi);
xlabel('t(s)');
ylabel('phi (deg)');

subplot(3,2,4);
plot(t,dphi);
xlabel('t(s)');
ylabel('d phi/dt');

subplot(3,2,5);
plot(t,psi*180/pi);
xlabel('t(s)');
ylabel('psi (deg)');

subplot(3,2,6);
plot(t,psidot);
xlabel('t(s)');
ylabel('d psi/dt');
%the function used in ode45 eulersolv is as follows:
function out = eulersolv(t,wi,I,Is,l,g,ws,m)
out = rand(5, 1);
```

```
phiang = wi(1);%phi angle
phid = wi(2);% phi derivative
thetaang = wi(3);% theta angle
thetad = wi(4); %theta derivative
psid = wi(5);%psi angle

out(1)= phid;
out(2)=(Is*thetad.*ws-2*I*thetad.*phid.*cos(thetaang))./(I*sin(thetaang));
out(3)=thetad;
out(4)=(m*g*l+I*phid.^2.*cos(thetaang)-Is*ws*phid).*sin(thetaang)/I;
out(5)=ws-phid.*cos(thetaang);
end
```

A.4 Projections and Trajectories

```
clc
clear all
I=1;
Is=1.5;
l=0.1;
ws=0.5;
time=0:.01:50;
g=9.8;
m=0.1;
wi=[0;0;pi/3;0; 0];%phi, dphi, theta, dtheta, psi

[t,w]=ode45(@eulersolv,time,wi,[],I,Is,l,g,ws,m);

phi=w(:,1);
dphi = w(:,2);

theta=w(:,3);
dtheta = w(:,4);

psidot=ws-w(:,2).*cos(w(:,3));
psi = w(:,5);

x = l*sin(theta).*cos(phi);
y = l*sin(theta).*sin(phi);
z = l*sin(theta);

figure;
comet(x, y)%comet(a, b) is a function that simulates and traces out the plot of a versus b
title(’Projection of CoM on the X-Y plane, view from the top’);
xlabel(’x-axis (m)’);
ylabel(’y-axis (m)’);
pause(1)
figure;
comet(x, z)
xlabel(’x-axis (m)’);
ylabel(’z-axis (m)’);
title(’View from the side, Nutation of the CoM is clearly visible’);
```

Bibliography

[1] Vernon D. Barger and Martin G. Olsson. *Classical Mechanics: a modern perspective.* McGRAW-HILL INTERNATIONAL EDITIONS, 1995.

[2] Milan Batista. Steady motion of a rigid disk of finite thickness on a horizontal plane. *International Journal of Non-Linear Mechanics*, 41(4):605 – 621, 2006.

[3] Mir Abbas Jalali, Milad S. Sarebangholi, and Mohammad-Reza Alam. Terminal retrograde turn of rolling rings. *Phys. Rev. E*, 92:032913, Sep 2015.

[4] Wolfram Mathworld. Euler angles, 2017.

[5] John R. Taylor. *Classical Mechanics.* University Science Books, 2005.

[6] TutorVista.com. Comparison of linear and rotational motion, 2017.