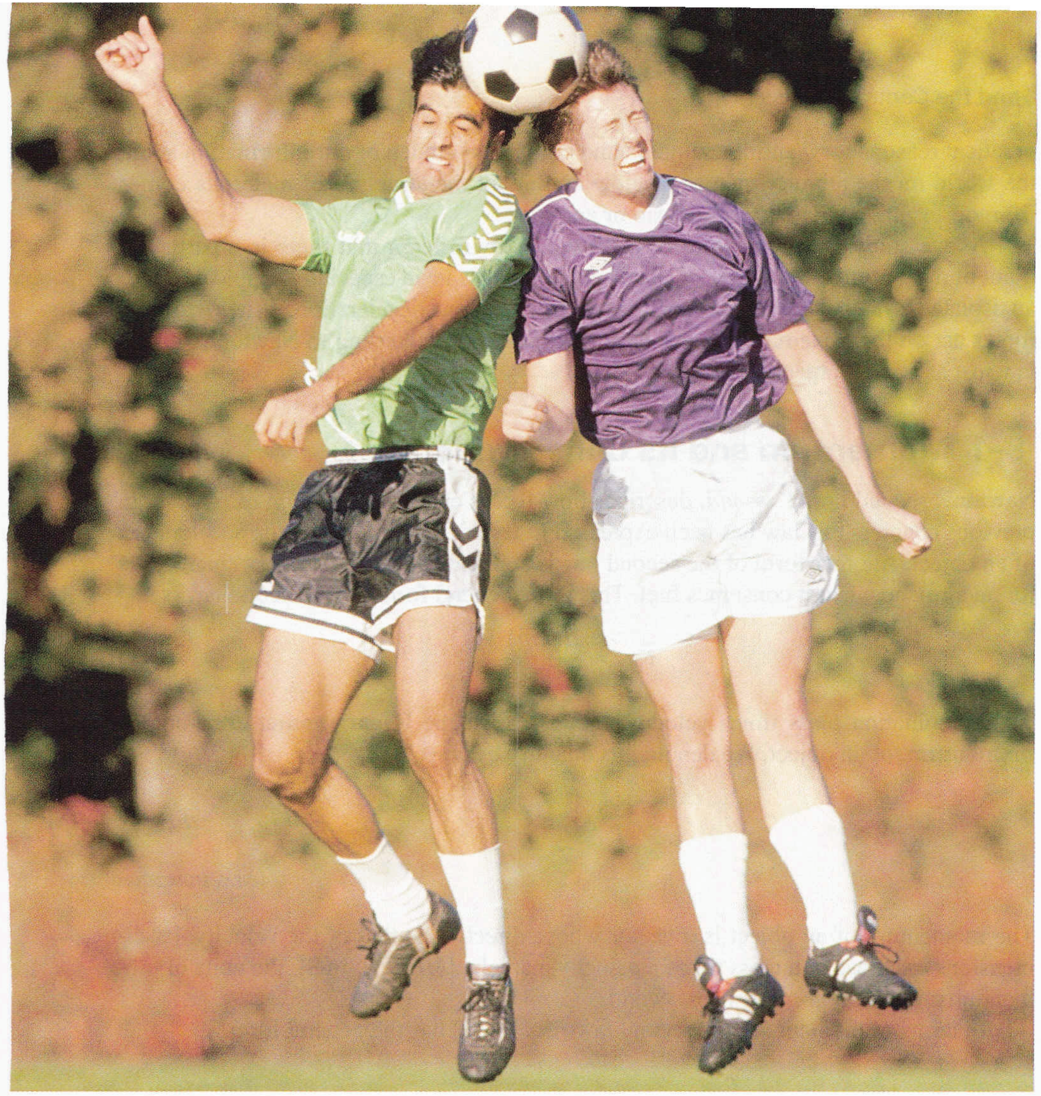


8

Chapter

The two soccer (football) players collide inelastically with each other, but collide almost elastically with the ball. We will study collisions in this chapter and learn that linear momentum is always conserved in such collisions.

Linear Momentum, Collisions, and the Center of Mass

In the world around us collisions are commonplace—think of raindrops colliding with the ground or ocean waves with the shore, the collision of a golf club with the ball, or the gentle collision of a mother taking a baby in her arms. Behind these everyday scenes are the incessant motion of air molecules and their collisions. Even the distant interactions between stars in the galaxy can be thought of as collisions—slow ones, for the most part. Behind the complexity of these collisions we can find a great simplification, the conservation of a new quantity called the momentum. The momentum is simply formed as the product of the mass and velocity of a particle, and its importance is associated with the fact that Newton's laws are very simply restated in terms of it. In particular, Newton's third law then shows us that momentum is conserved within an isolated system of particles—a system free from net external forces, such as when two hockey pucks sliding on ice collide—no matter how complex the internal interactions of the particles making up the system. In other words, the total momentum of such a system is constant. The conservation of momentum is enormously useful for understanding the behavior of colliding objects, and there are good

practical reasons for wanting to understand collisions—much of our information about the world on the atomic scale and below and about the structure of materials comes from observing collisions in one form or another.

As we study the momentum of a system, we will learn that there is a particular point of the system—the *center of mass*—which moves in an especially simple way. For an isolated system, the center of mass moves without acceleration. When external forces act on the system, the center of mass accelerates according to Newton's second law just as a point object does.

8–1 Momentum and Its Conservation

Newton's second law, $\vec{F} = m\vec{a}$, describes how forces change the motion of objects. In previous chapters this law has been expressed in terms of the mass and the acceleration of an object. Another form of the second law is applicable even if the mass changes, as for an airplane when it consumes fuel. This more general form of the second law is

$$\vec{F}_{\text{net}} = \frac{d(m\vec{v})}{dt}. \tag{8–1}$$

The combination mass times velocity, $m\vec{v}$, is called the **linear momentum**, or just **momentum**, of an object. We denote this quantity by $\vec{p}$:

$$\vec{p} \equiv m\vec{v}. \tag{8–2}$$

LINEAR MOMENTUM

The momentum of an object is a vector whose direction is that of the velocity. Its dimensions are those of a mass times a velocity, namely, $[MLT^{-1}]$; in SI, the units of momentum are kilogram-meters per second.

In terms of momentum, the second law [Eq. (8–1)] has the general form

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}. \tag{8–3}$$

NEWTON'S SECOND LAW

The kinetic energy of an object can also be expressed in terms of the momentum:

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}, \tag{8–4}$$

where p is the magnitude of the momentum. From Eq. (8–3) it can be seen that when a large net force acts on an object, the object's momentum will change rapidly and a small net force will result in a slow momentum change.

CONCEPTUAL EXAMPLE 8–1 The same net force acts on a table tennis ball and a bowling ball. Compare the rates at which their momenta change.

Answer This is a bit like the trick question "Which weighs more, a pound of feathers or a pound of nails?" We did not ask for the rate at which the velocity changes—the acceleration—which will be significantly less for the bowling ball than for the table tennis ball. We asked for the rate of change of momentum, and that is precisely given by the force acting [Eq. (8–3)]. The rate of momentum change is the same for each ball since the net force acting on each is the same.

Conservation of Momentum

When objects exert a force on one another, we say they interact. Let's consider the interaction between objects 1 and 2 in both parts of Fig. 8–1. The two objects may be in contact, as in a collision of two billiard balls (Fig. 8–1a), or they may exert a force on each other at a distance, as in the gravitational attraction between Earth and the Moon, or they may be connected by a spring (Fig. 8–1b). Let $\vec{F}_{12}$ denote the force exerted on object 1

by object 2 and let $\vec{F}_{21}$ denote the force exerted on object 2 by object 1 (the first subscript always labels the object that is acted *upon*). Then, Newton's third law states that

$$\vec{F}_{12} = -\vec{F}_{21}. \tag{8–5}$$

When it is expressed in terms of momentum, Newton's second law [Eq. (8–3)] tells us that the rate of change of each object's momentum is the force acting on it:

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{12}, \tag{8–6}$$

$$\frac{d\vec{p}_2}{dt} = \vec{F}_{21} = -\vec{F}_{12}. \tag{8–7}$$

Addition of these two equations leads to

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \vec{F}_{12} - \vec{F}_{12} = 0;$$

$$\frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = 0. \tag{8–8}$$

As a consequence,

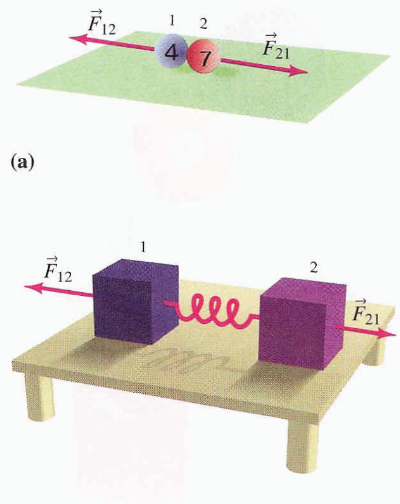
for zero net external force: $\quad \vec{p}_1 + \vec{p}_2 = \text{a constant}. \quad$ (8–9)

CONSERVATION OF MOMENTUM

We will see below that this result is not confined to two objects. In words, Newton's third law implies that

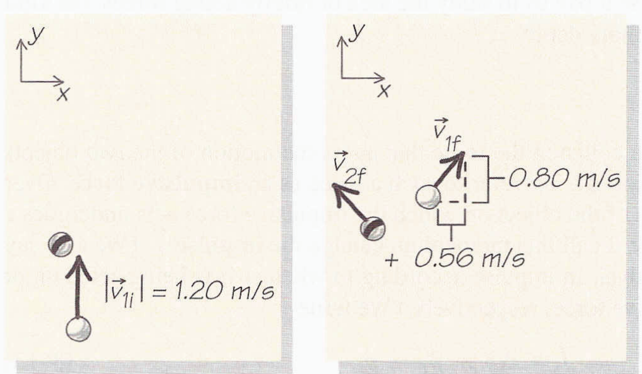
> **the sum of the momenta of an isolated system of objects is a constant, no matter what forces act between the objects making up the system.**

This is called the **principle of conservation of momentum**. Like the principle of conservation of energy, the conservation of momentum is important both as a general principle and as a powerful tool for solving problems.



▲ **FIGURE 8–1** (a) Two billiard balls at the moment of collision and (b) two masses connected by a spring. In each case, the objects exert forces on one another. Here $\vec{F}_{12}$ is the force exerted on object 1 by object 2, and $\vec{F}_{21}$ is the force exerted on object 2 by object 1. In these collisions, the forces are contact forces although, in general, physical contact is not necessary for two objects to exert forces on one another.

EXAMPLE 8–2 A cue ball moves with a velocity of 1.20 m/s in the $+y$-direction on a billiard table and strikes an equally massive ball initially at rest (Fig. 8–2a). The cue ball is deflected so that its velocity has a component of 0.80 m/s in the $+y$-direction and a component of 0.56 m/s in the $+x$-direction (Fig. 8–2b). What is the velocity of the struck ball immediately after the collision?



Before

After

(a) (b)

▲ **FIGURE 8–2** Two colliding billiard balls. (a) Before the collision. (b) After the collision.

Setting It Up The figures label the initial and final velocities of the cue ball as $\vec{v}_{1i}$ and $\vec{v}_{1f}$, respectively, and the final velocity of the struck ball as $\vec{v}_{2f}$. The given velocities are $\vec{v}_{1i} = (1.20\ \text{m/s})\hat{j}$ and $\vec{v}_{1f} = (0.56\ \text{m/s})\hat{i} + (0.80\ \text{m/s})\hat{j}$.

Strategy We can find the initial and final momenta of the sum of the two balls and use the conservation of momentum, which will relate the velocities in question. The only unknown is $\vec{v}_{2f}$.

Working It Out If m is the mass of each ball, the initial momentum is $\vec{p}_i = m\vec{v}_{1i}$ because the struck ball has an initial velocity of zero. Then the conservation of momentum reads

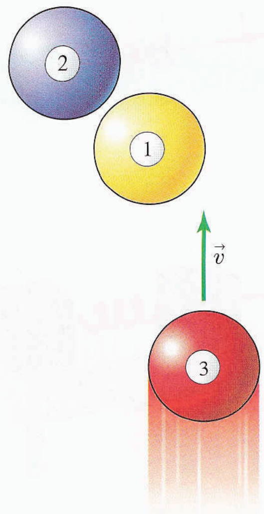
$$m\vec{v}_{1i} = m\vec{v}_{1f} + m\vec{v}_{2f}.$$

This simplifies to $\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f}$ or $\vec{v}_{2f} = \vec{v}_{1i} - \vec{v}_{1f}$. Numerically,

$$\begin{aligned}\vec{v}_{2f} &= \vec{v}_{1i} - \vec{v}_{1f} = (1.20\ \text{m/s})\hat{j} - [(0.56\ \text{m/s})\hat{i} + (0.80\ \text{m/s})\hat{j}] \\ &= (-0.56\ \text{m/s})\hat{i} + (0.40\ \text{m/s})\hat{j}.\end{aligned}$$

This corresponds to a final speed of $v_{2f} = \sqrt{(v_{2f,x}^2 + v_{2f,y}^2)} =$ 0.69 m/s.

What Do You Think? In this example we specified some information about the final state (after the collision) as well as all the information about the initial state. If no information had been supplied about the final state, could the statement that kinetic energy is conserved, if it were true here, have allowed a full solution? *Answers to What Do You Think? questions are given in the back of the book.*



▲ **FIGURE 8–3** System of three interacting objects. Any time one ball touches another, there is a force between those two balls.

Conservation of Momentum for a System of Many Objects: The conservation of momentum is not confined to a system of two interacting objects. Suppose that there are three objects in a system on which no external forces act. [For example, you could imagine three billiard balls on a table, the 1 ball and the 2 ball quite close to one another and the 3 ball coming in to interact with both of them (Fig. 8–3).] Then the total force $\vec{F}_1$ on object 1 is given by the sum of the forces on object 1 due to objects 2 and 3:

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}. \tag{8–10a}$$

Similarly, the total force $\vec{F}_2$ on object 2 is given by the sum of the forces on object 2 due to objects 1 and 3; so

$$\vec{F}_2 = \vec{F}_{21} + \vec{F}_{23}, \tag{8–10b}$$

and similarly,

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32}. \tag{8–10c}$$

(Note that in Fig. 8–3 the 3 ball may not ever directly hit the 2 ball, so that in this case $\vec{F}_{23}$ and $\vec{F}_{32}$ would be zero.) Adding these three equations and using $\vec{F}_{12} = -\vec{F}_{21}$, $\vec{F}_{13} = -\vec{F}_{31}$, and $\vec{F}_{23} = -\vec{F}_{32}$ yield

$$\begin{aligned}\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{21} + \vec{F}_{23} + \vec{F}_{31} + \vec{F}_{32} = 0. \end{aligned} \tag{8–11}$$

Consequently, the sum of the momenta of the three objects is constant throughout the motion:

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \text{a constant}. \tag{8–12}$$

We can easily extend this demonstration to N interacting objects and prove that the sum of the objects' momenta is constant throughout the motion.

8–2 Collisions and Impulse

What happens when objects *collide*? The word "collision" evokes the image of an action with a short, sharp contact, such as the collision between two billiard balls or two automobiles. We can more formally think of a collision between objects as an interaction between them—a set of forces—that is limited in time. (Of course, the word "limited" allows us a lot of leeway.) We will want to think of the colliding objects as otherwise isolated in order to be able to apply the conservation of momentum to the situation. Before we do so, it will pay us to study the idea of briefly acting forces, the kind that occur in collisions, in more detail.

Impulsive Forces

We'll suppose that during a collision the force that alters the motion of the two objects is active for only a short time Δt. We refer to such a force as an **impulsive force**. Over the time Δt the momentum of the object on which the impulsive force acts undergoes a momentum change $\Delta\vec{p}$. We'll call this momentum change the **impulse** $\vec{J}$. (We also say that an object receives or gives an impulse according to whether it is being acted on or is the source of the impulsive force, respectively.) We write

$$\vec{J} \equiv \Delta\vec{p} = \vec{p}_f - \vec{p}_i. \tag{8–13}$$

A little calculus allows us to express the impulse for an impulsive force in terms of that force. The change in momentum is the integral of the rate of change of momentum between the initial and final times that the force acts, and from Newton's second law, the rate of change of momentum is the (impulsive) force $\vec{F}$ that acts on the object:

$$\vec{J} = \int_{t_i}^{t_f} \left(\frac{d\vec{p}}{dt} \right) dt = \int_{t_i}^{t_f} \vec{F}\, dt. \qquad (8\text{–}14)$$

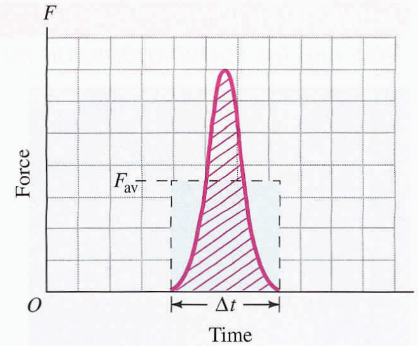
IMPULSE AND FORCE

We have used the fact that the force is zero outside the time interval $\Delta t = t_f - t_i$.

Because the integral of force over time is the area under a curve of force versus time, we can also represent the impulse as the product of the time interval Δt and a quantity that we may call the *average force* $\vec{F}_{av}$ (Fig. 8–4). The average force refers to the average value of the force over the time interval Δt. In this case, the impulse or change in momentum can be written in the form

$$\Delta\vec{p} = \vec{J} = \int_{t_i}^{t_f} \vec{F}\, dt = \vec{F}_{av}\, \Delta t. \qquad (8\text{–}15)$$

Remember that this equation is a *vector* equation. In Example 8–3, however, the vector aspect does not play a crucial role.



▲ **FIGURE 8–4** If an impulsive force acts on an object, its momentum will be changed. That force acts over a time period Δt, varying with time as it acts. The same change in momentum is produced by another force that is constant over Δt and takes the actual force's average value over that time period.

EXAMPLE 8–3 The magnitude of the average force exerted by a bat on a baseball during the time of contact (a period of 2.00×10^{-3} s) is 6660 N. The mass of the baseball is 0.145 kg and its speed is 33.5 m/s just before the bat collides with it. What is the velocity of the ball when it leaves the bat? Assume that the ball leaves the bat along the same line of direction from which it is pitched.

Setting It Up Figure 8–5 illustrates this problem. The x-axis is horizontal and to the right. We know the average force magnitude F_{av} and the time Δt over which it is applied. We are also given the ball's mass m and its initial speed v_0. The motion is one dimensional, in a direction that we label as the x-axis, with the original direction of the ball—the direction of the pitch—in the $+x$-direction. We want to find the final velocity of the ball.

Strategy Because we know both F_{av} and Δt, we can find the ball's impulse—the change in its momentum. Then we can use the value of the impulse to find the ball's final momentum knowing its initial momentum. We can then use its final momentum to find its final speed.

Working It Out The momentum change $\vec{p}_f - \vec{p}_i$ of the ball is given by Eq. (8–15), with $\vec{p}_i = mv_0\hat{i}$. The direction of the force (and hence of the impulse) is in the $-x$-direction, so from Eq. (8–15),

$$\vec{p}_f = \vec{p}_i + \vec{F}_{av}\,\Delta t = (mv_0 - F_{av}\,\Delta t)\hat{i}.$$

The final velocity is the final momentum divided by m:

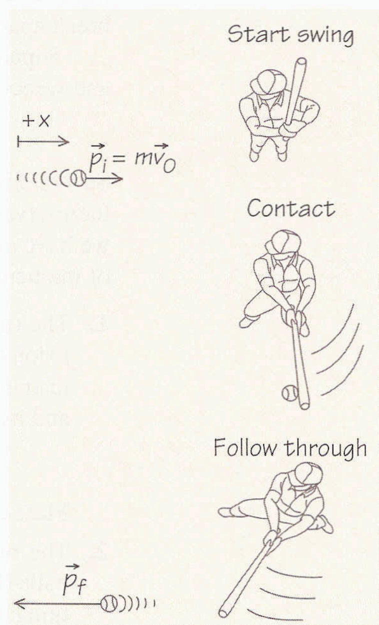
$$\vec{v}_f \equiv v_f\hat{i} = \frac{\vec{p}_f}{m} = \frac{mv_0 - F_{av}\,\Delta t}{m}\hat{i} = \left(v_0 - \frac{F_{av}\,\Delta t}{m} \right)\hat{i}.$$

Thus the final velocity is oriented along the x-axis, with x-component

$$v_f = +33.5\ \text{m/s} - \frac{(6660\ \text{N})(2.00 \times 10^{-3}\ \text{s})}{0.145\ \text{kg}} = -58.4\ \text{m/s}.$$

The minus sign indicates that the ball moves in the negative direction, back toward the pitcher.

What Do You Think? Which sort of racket will allow you to return a tennis ball with more velocity: an ordinary strung tennis racket or a solid wooden paddle of the same shape and weight?



▲ **FIGURE 8–5** The direction of the impulse imparted to a ball hit by a bat is to the left, in the $-x$-direction.

THINK ABOUT THIS . . .

HOW ARE THE FOUNDATIONS FOR TALL BUILDINGS MADE?

▲ **FIGURE 8–6** The pile driver is one of the devices that enables us to build large structures.

One way of constructing foundations (Fig. 8–6) is to drive vertical "piles" into the soil down to bedrock and then anchor the building to them. These piles are often made of steel. A pile driver is used to force them into the ground, and momentum and impulse are key to the operation of this device. In Fig. 8–6, there is a weight sitting within a cylinder that encloses the top of the pile. The weight is lifted by exploding fuel and then drops back onto the pile. During the fall, the weight gains momentum. When it reaches the top of the pile, it is brought to rest in an exceedingly short time (determined by the compressibility of the weight and the pile, and the distance the pile moves). Thus the impulse to the dropped cylinder is large, and by Newton's third law there is an equal and opposite impulse to the pile. In turn, the pile delivers an impulse to the rock and soil beneath it, breaking the rock and soil and allowing the pile to move down until friction and normal forces bring it back to a stop. The action is repeated, driving the pile further down with each repetition. ■

Classification of Collisions

Attempts to understand collisions were carried out by Galileo and his contemporaries. The description of collisions in one dimension were formulated by John Wallis, Christopher Wren (best remembered today as an architect), and Christian Huygens in 1668, and the principle of the conservation of momentum plays a central role in understanding their results.

We can recap our previous discussion on the collision of two objects as follows: The two objects move freely before the collision—no net forces act on either of them—and each has its own constant momentum. During the brief interaction, their individual momenta change because each object experiences an impulsive force due to the other object. After the collision, the two objects are again free but have momenta that differ from those they had before the collision. However, the impulses of the two objects are equal and opposite because the forces each exerts on the other are equal and opposite, so that the change in the momentum of one object is equal and opposite to the change in momentum of the other. In other words, *the sum of their momenta is unchanged*. This feature, the conservation of total momentum of the isolated system, provides a governing constraint. It will hold even if the objects stick together or, at the other extreme, break apart into many pieces.

Suppose that initially object 1 has mass m_1 and velocity $\vec{v}_1$ and object 2 has mass m_2 and velocity $\vec{v}_2$. The total initial momentum is given by

$$\vec{p}_{\text{init}} = m_1\vec{v}_1 + m_2\vec{v}_2. \qquad (8\text{–}16)$$

Several distinct and interesting possibilities for what the final state can look like present themselves. Figure 8–7 shows these cases and we enumerate them below. In the figure, we have *drawn* the collision in one dimension, although we'll express the conservation of momentum in more general form.

1. The two masses hit each other and stick together, coalescing into one, as in the collision of two blobs of putty or a comet colliding with a planet. Figures 8–7a and b illustrate the before and after for this case. If the mass of the single final object is M and its velocity is $\vec{v}$, momentum conservation reads

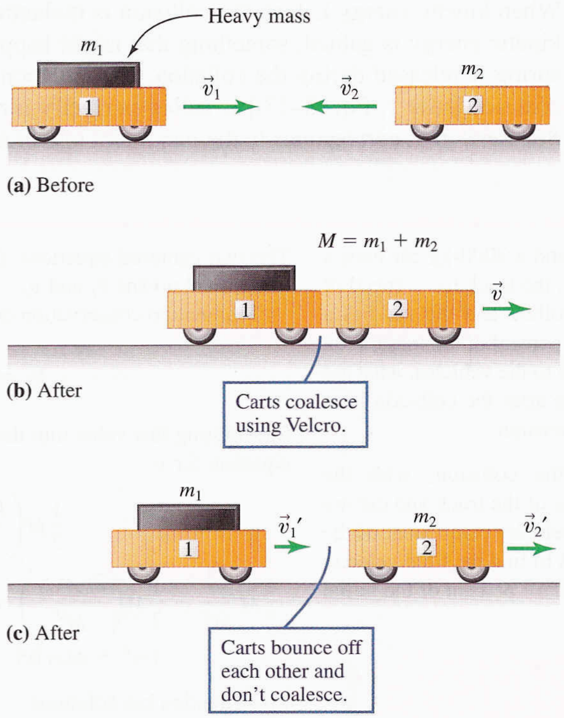
$$M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2. \qquad (8\text{–}17)$$

Mass conservation gives us the additional information† that $M = m_1 + m_2$.

2. The two masses can remain distinct and unchanged, as in the collision of billiard balls (Fig. 8–7c). We label the final velocities $\vec{v}_1'$ and $\vec{v}_2'$, respectively, and momentum conservation takes the form

$$m_1\vec{v}_1' + m_2\vec{v}_2' = m_1\vec{v}_1 + m_2\vec{v}_2. \qquad (8\text{–}18)$$

†Because special relativity plays an important role in nuclear or subnuclear collisions, the conservation of mass per se is only an approximation, and sometimes a very bad one, in those cases.



◀ **FIGURE 8–7** A collision between two objects, as shown before the collision in part (a), can have several outcomes. The two objects shatter into several pieces (not shown). (b) The two masses combine as the objects coalesce into one object. (c) Two objects leave the collision, each with the mass of the original objects respectively, drawn for the case of object 1 continuing to move in the original direction.

3. Mass can be transferred from one object to the other such that after the collision one would have two objects with masses m_3 and m_4. As an example, a carbon atom can collide with a molecule of carbon dioxide to make two carbon monoxide molecules. We label the final velocities of masses m_3 and m_4 as $\vec{v}_3$ and $\vec{v}_4$, so that momentum conservation reads

$$m_3\vec{v}_3 + m_4\vec{v}_4 = m_1\vec{v}_1 + m_2\vec{v}_2. \tag{8–19}$$

   In this case mass conservation would add the information that $m_3 + m_4 = m_1 + m_2$.
4. One or both of the objects can shatter into several pieces (a more complex possibility that we won't deal with in detail).

The vector equations above are indispensable tools for understanding collisions. However, even given all the information about the initial state, they are generally not enough to determine everything about the final state. Only in the case of coalescence [Eq. (8–17)] are the three vector equations sufficient to determine the three components of the single final velocity. In all the other cases, more information is required to understand the details of the motion, and this is usually information about the energy of the objects.

Energy Considerations in Collisions

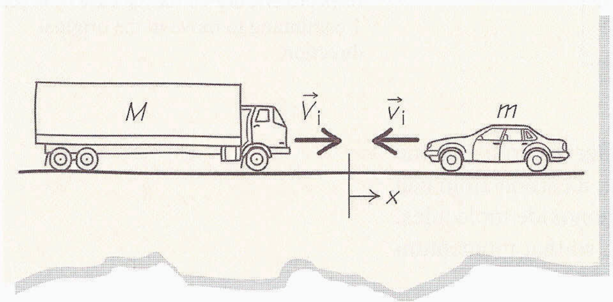
The degree to which the kinetic energy is conserved in a collision provides us with another piece of information that we can use to help us understand the process. Although for the reasons described in Chapter 7 the *total* energy is always conserved in a collision, some of the energy may be *dissipated* in ways that make it lost to us. For example, when friction is present, some of the energy goes into heating. To take another example, in a collision between two cars, some energy goes into the crumpling of metal, and as a result there is less energy available for motion—the kinetic energy after the collision. If the kinetic energy decrease is known (as in Example 8–4 below), we can use this information to help us calculate the details of the motion after a collision.

The degree to which kinetic energy is conserved provides another way to classify collisions. Kinetic energy is conserved if all the initial kinetic energy of the two colliding objects goes into the kinetic energy of the objects present after the collision, and we call the collision **elastic**. (We are assuming the objects retain their identities as well, as in case 2 above, although that case also includes the possibility that energy is lost.)

When kinetic energy is lost, the collision is **inelastic**—we would also use this term if kinetic energy is gained, something that might happen if the equivalent of an internal spring is released during the collision. The situation in which two objects collide and coalesce [case 1, Eq. (8–17)] is called **perfectly inelastic** because, as we shall see in Section 8–6, it corresponds to the maximum loss of kinetic energy.

EXAMPLE 8–4 A 14,000-kg truck and a 2000-kg car have a head-on collision. Despite attempts to stop, the truck has a speed of 6.6 m/s in the $+x$-direction when they collide and the car has a speed of 8.8 m/s in the $-x$-direction. If 10 percent of the initial total kinetic energy is dissipated through damage to the vehicles, what are the final velocities of the truck and the car after the collision? Assume that all motion takes place in one dimension.

Setting It Up Figure 8–8 shows the collision, with the $+x$-direction to the right. The given masses of the truck and car are M and m, respectively. The known initial velocity component of the truck is V_i and that of the car is v_i. We want to find the final velocity components of the truck V_f and the car v_f if 10 percent of the initial kinetic energy is lost.



▲ FIGURE 8–8 Head-on collision between large truck and small car.

Strategy We can use the conservation of momentum, which, because the motion is one dimensional, is a single equation for the x-component of velocity. The energy information we have is that the final mechanical energy (all kinetic) is 90 percent of the initial energy. Both the initial momentum and initial energy are easily calculated from the given information. Thus our two conditions should be enough to solve for the two unknowns.

Working It Out Momentum conservation reads

$$MV_f + mv_f = MV_i + mv_i \equiv p_i.$$

The initial total energy is the sum of the kinetic energies of the two vehicles, $K_i = (MV_i^2/2) + (mv_i^2/2)$. The sum of the final kinetic energies is 90 percent of this quantity, so

$$K_f = (MV_f^2/2) + (mv_f^2/2)$$
$$= 0.9[(MV_i^2/2) + (mv_i^2/2)] \equiv 0.9K_i.$$

The two centered equations above are two algebraic equations for the two unknowns V_f and v_f.

Momentum conservation directly yields

$$V_f = \frac{p_i - mv_f}{M}.$$

Substituting this value into the energy relation, we have a quadratic equation for v_f:

$$\frac{1}{2}M\left(\frac{p_i - mv_f}{M}\right)^2 + \frac{1}{2}mv_f^2 - 0.9K_i = 0;$$

$$\frac{1}{2}M\frac{p_i^2}{M^2} - \frac{1}{2}M\left(\frac{2p_imv_f}{M^2}\right) + \frac{1}{2}M\frac{m^2v_f^2}{M^2} + \frac{1}{2}mv_f^2 - 0.9K_i = 0;$$

$$(m^2 + Mm)v_f^2 - 2mv_fp_i + p_i^2 - 2(0.9)MK_i = 0.$$

This equation has solutions

$$v_f = \frac{2mp_i \pm \sqrt{(2mp_i)^2 - 4(m^2 + Mm)[p_i^2 - 2(0.9)MK_i]}}{2(m^2 + Mm)}.$$

When numbers are inserted for the two possible solutions represented by the $\pm$ sign, the solution with a minus sign gives a negative velocity for the car and a positive velocity for the truck. This means that the car continues its motion to the left, going "through" the truck, while the truck similarly goes "through" the car. Since this is not possible, the correct solution has the plus sign. Inserting numbers, we find

$$v_f = 17 \text{ m/s} \quad \text{and} \quad V_f = 2.9 \text{ m/s}.$$

The truck continues in the $+x$-direction with a speed less than its initial speed; the car has completely reversed its direction and is moving even faster than its initial speed. The car has "bounced" from the much more massive truck, much like a tennis ball against a tennis racket. Analysis of this type is used by crash-scene investigators. Adding information such as stopping distance under friction, they can learn, for example, the speeds and directions of the vehicles just before the crash.

What Do You Think? Assuming that they are protected from injury directly associated with the automobile collapsing on them, which would be better for the occupants of the car, an elastic or an inelastic collision?

8–3 Perfectly Inelastic Collisions; Explosions

Perfectly Inelastic Collisions

There is a range of possible collisions from elastic to inelastic to perfectly inelastic, according to how much kinetic energy is lost. The simplest is the case of *perfectly inelastic* collisions in one dimension, which are those in which the objects coalesce as a result of the collision. For example, an asteroid hitting Earth would be a perfectly inelastic collision. Momentum conservation for these collisions are described by Eq. (8–17).

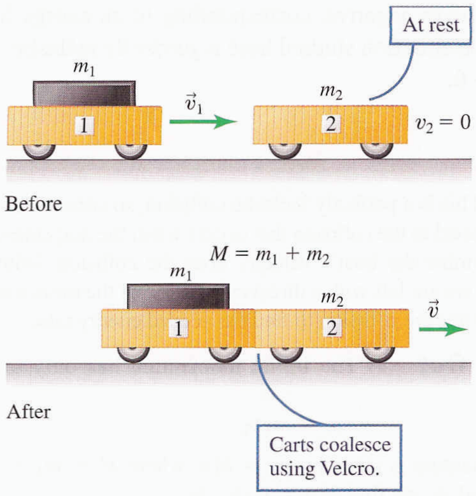


FIGURE 8–9 A cart with mass m_1 and velocity $\vec{v}_1$ collides with a cart of mass m_2 at rest. The two carts stick together and move on together with velocity $\vec{v}$.

Mass conservation implies that $M = m_1 + m_2$. We can divide Eq. (8–17) by this factor. The velocity of the coalesced object then becomes (in one-dimensional motion)

$$v = \frac{m_1 v_1 + m_2 v_2}{M}. \tag{8–20}$$

Let's analyze this result for some special cases.

If one of the objects (m_2) is at rest $(v_2 = 0)$ and the other (m_1) runs into it (Fig. 8–9), then

$$v = \frac{m_1}{M} v_1. \tag{8–21}$$

If $m_1 \gg m_2$, the "composite" object will move with a velocity nearly equal to that of the initially moving object; a car colliding with a bug does not slow down very much. In contrast, when $m_1 \ll m_2$, as when a stationary athlete catches a ball, we get the opposite effect; that is, the athlete will recoil with only a low velocity, just the fraction $m_1/(m_1 + m_2) \cong m_1/m_2$ of the velocity of the ball.

Next, consider the case of a head-on collision in which the two objects have equal and opposite velocities $(v_2 = -v_1)$. In this case, Eq. (8–20) becomes

$$v = \frac{m_1 - m_2}{m_1 + m_2} v_1. \tag{8–22}$$

In the special case that $m_1 = m_2$, the two objects have equal and opposite *momenta* because

$$m_1 v_1 + m_2 v_2 = m_1 v_1 + m_1 v_2 = m_1(v_1 + v_2) = 0. \tag{8–23}$$

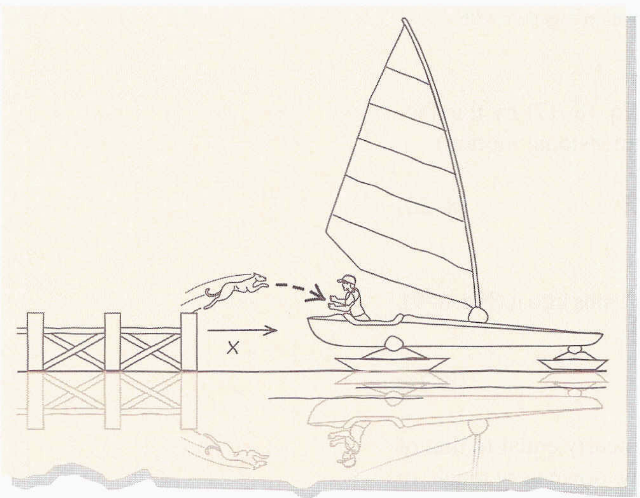
In that case, the final momentum must be zero and thus $v = 0$, as Eq. (8–22) verifies. The objects collide and come to rest.

Energy Loss in Perfectly Inelastic Collisions: Let's find the change in energy for the collision described above. Before the collision, the total energy E_i is the sum of the kinetic energies of the two objects. The final energy E_f is the kinetic energy of the composite object of mass $M = m_1 + m_2$. The change in energy $\Delta E = E_f - E_i$ is, using Eq. (8–20),

$$\begin{aligned}
\Delta E &= \frac{1}{2}Mv^2 - \left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right) \\
&= \frac{1}{2}\frac{M(m_1 v_1 + m_2 v_2)^2}{M^2} - \left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right) \\
&= \frac{1}{2}\frac{m_1^2 v_1^2 + 2m_1 m_2 v_1 v_2 + m_2^2 v_2^2 - M(m_1 v_1^2 + m_2 v_2^2)}{M} \\
&= \frac{1}{2}\frac{m_1 m_2(-v_1^2 - v_2^2 + 2v_1 v_2)}{M} = -\frac{1}{2}\frac{m_1 m_2}{M}(v_1 - v_2)^2.
\end{aligned} \tag{8–24}$$

The right-hand side of Eq. (8–24) is always negative, corresponding to an energy loss, or to an inelastic collision. As to why the situation studied here is *perfectly* inelastic, see the Think About This box in Section 8–6.

EXAMPLE 8–5 A dog who jumps into the interior of a stationary ice boat is moving at $v_1 = 32$ km/h when he enters the boat (Fig. 8–10), and his landing on the boat can be regarded as a collision. The dog's mass is 14 kg and that of the boat plus boater is 160 kg. You can assume all the motion is horizontal. (a) Assuming that the ice surface is frictionless, what is the velocity of the boat after the collision? (b) What is the ratio of the energy loss to the initial energy?



▲ **FIGURE 8–10** A dog jumping into the boat changes the boat's momentum.

Setting It Up We specify that the motion is in the x-direction by adding an axis to Fig. 8–10. We know the dog's mass m_1 and that of the boat and person, m_2. We also know the dog's velocity before the collision, or equivalently its x-component v_1, as well as the boat's ($v_2 = 0$). We want the velocity x-component v of the boat after the collision and the ratio of the energy loss ΔE to the initial kinetic energy K_i.

Strategy This is a perfectly inelastic collision, so conservation of momentum applied to the collision that occurs when the dog enters the boat will determine the boat's velocity after the collision. With all speeds known, we are left with a direct calculation of the kinetic energy before and after the collision to find the needed energy ratio.

Working It Out (a) The initial momentum has only the x-component,

$$p_i = m_1 v_1.$$

The final momentum is given by $p_f = Mv$, where $M = m_1 + m_2$. Equating p_f and p_i, we find $Mv = m_1 v_1$, or

$$v = \frac{m_1 v_1}{M} = \frac{(14\ \text{kg})(32\ \text{km/h})}{174\ \text{kg}} = 2.6\ \text{km/h} = 0.72\ \text{m/s}.$$

(b) The initial energy is the kinetic energy of the dog, $K_i = \frac{1}{2}m_1 v_1^2$. The final energy is again all in the form of kinetic energy:

$$K_f = \frac{1}{2}Mv^2 = \frac{1}{2}M\left(\frac{m_1 v_1}{M}\right)^2 = \frac{1}{2}\left(\frac{m_1}{M}\right)m_1 v_1^2 = \frac{m_1}{M}K_i.$$

Thus the energy loss is given by

$$\Delta E = K_i - K_f = K_i - \frac{m_1}{M}K_i = K_i\left(1 - \frac{m_1}{M}\right),$$

and the ratio of the energy loss to the initial energy is

$$\frac{\Delta E}{K_i} = 1 - \frac{m_1}{M} = \frac{M - m_1}{M} = \frac{m_2}{M},$$

a number less than 1. The energy has decreased. Numerically,

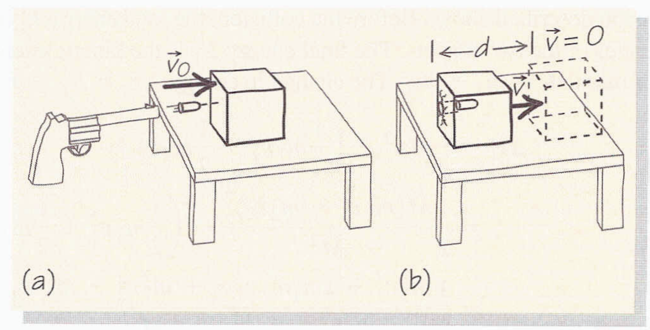
$$\frac{\Delta E}{K_i} = \frac{160\ \text{kg}}{174\ \text{kg}} = 0.92.$$

What Do You Think? Kinetic energy is lost. Which of the following is correct? (a) It went into gravitational energy. (b) It went into the dog and boater "giving and stretching" as the dog lands. (c) It went into melting the ice.

EXAMPLE 8–6 A 10-g bullet is fired in the $+x$-direction into a stationary block of wood that has a mass of 5.0 kg. The speed of the bullet before entry into the wood block is 500 m/s. What is the speed of the block just after the bullet has become embedded? What distance will the block slide on a surface with a coefficient of friction equal to 0.30?

Setting It Up We have drawn the situation in Fig. 8–11. Positive x is to the right. We label the known mass of the bullet as m_1, its known initial velocity as $\vec{v}_1 = v_1\hat{i}$, and the known total mass of block and bullet as M. We want to find the distance d the block will slide on a surface for which there is a known coefficient of kinetic (sliding) friction μ_k.

Strategy We can use conservation of momentum to find the velocity of the block and bullet immediately after the collision, then Newton's second law to find the friction-induced negative acceleration of the block. Knowing the acceleration and the initial condition, kinematic relations will give us the distance d the block moves.



▲ **FIGURE 8–11** A bullet fired into a stationary block of wood moves the block.

Working It Out The x-component of the initial momentum is $p_i = m_1 v_1$, while the momentum immediately after the collision is Mv. Momentum conservation gives

$$m_1 v_1 = Mv;$$
$$v = \frac{m_1}{M} v_1 = \frac{10\ \text{g}}{5010\ \text{g}}(500\ \text{m/s}) = 1.0\ \text{m/s}.$$

We now turn to the problem of finding the acceleration given the value v of the initial speed of the bullet–block composite. The normal force N on the block from the table has magnitude Mg, so the force of friction between block and table is $-\mu_k N = -\mu_k Mg$. (The minus sign indicates that friction points to the left, along the $-x$-direction.) The friction force has constant magnitude and leads to a constant acceleration a of the block, according to Newton's second law, namely $F_{\text{net}} = Ma$ reads $-\mu_k Mg = Ma$, or

$$a = -\mu_k g.$$

The negative sign means the block slows down, traveling a distance d before it stops (Fig. 8–11b). Because the acceleration is uniform, we can use the relation $v_f^2 - v_i^2 = 2ad$ [from Eq. (2–24)]. With $v_f = 0$ and the initial speed $v_i = v$ above, we have

$$d = -\frac{v_i^2}{2a} = \frac{1}{2}\frac{v^2}{\mu_k g} = \frac{1}{2}\frac{(1.0\ \text{m/s})^2}{(0.30)(9.8\ \text{m/s}^2)} = 0.17\ \text{m}.$$

What Do You Think? In what way could we have used the fact that the mass of the bullet is much less than the mass of the block?

Explosions

Imagine that we were to film a perfectly inelastic collision in a frame of reference in which the total momentum is zero. In this reference system, the two objects approach each other and merge, leaving a composite object at rest. If we ran the film in reverse, it would look like a film of an explosion. The "initial" object of mass $M = m_1 + m_2$, at rest, breaks up into two objects, m_1 and m_2, with their total momentum equal to zero,

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = 0. \qquad (8\text{–}25)$$

Energy conservation tells us that an explosion is possible if there is an initial potential energy U within the "unexploded" system that can be converted into kinetic energy. For the case we are referring to here, this could be as simple as a compressed spring between two masses that is then released. We'll have

$$U = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2. \qquad (8\text{–}26)$$

An explosion can involve many more than two objects in the final state. In the explosion of, say, dynamite, the potential energy is stored in its molecules, in the form we call chemical energy. To take another example, Fig. 8–12 shows the remnants of a stellar explosion far from Earth. In these more complicated cases, we will always have the overriding simplicity that the initial momentum of the system before it explodes is the same as the sum of the momentum of all the fragments after the explosion. Let's next take a look at an explosion that occurs when an unstable atomic nucleus disintegrates—a nuclear decay.

▲ **FIGURE 8–12** A small portion of the Cygnus Loop supernova blast wave passes through clouds of interstellar gas. The collision heats and compresses the gas, which causes the glow. Such images taken by the Hubble Space Telescope reveals the structure of the interstellar medium.

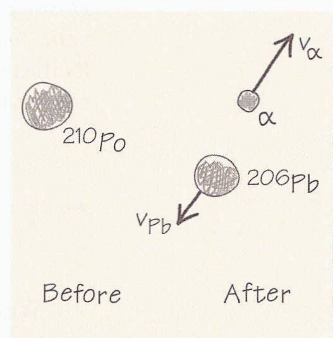
EXAMPLE 8–7 One type of polonium nucleus (symbol ^{210}Po), with mass 3.49×10^{-25} kg, can decay into an α particle (actually a helium nucleus), mass 6.64×10^{-27} kg, and a certain type of lead nucleus (symbol ^{206}Pb), mass 3.42×10^{-25} kg:

$$^{210}\text{Po} \rightarrow \alpha + {}^{206}\text{Pb}.$$

In this process, the final decay products have a kinetic energy of 8.65×10^{-13} J if the polonium nucleus decays at rest, the situation we consider here. What are the speeds of the α particle and the lead nucleus?

Setting It Up We show the decay in Fig. 8–13. With a decay from rest into two bodies, the momenta of the two decay products must go off back to back because that is the only way they can add to zero. We call this direction the x-axis. We label the known masses of the alpha particle, the polonium nucleus, and the lead nucleus as M_α, M_{Po}, and M_{Pb}, respectively, and we denote the known kinetic energy in the final state as Q. We want to find the speeds v_α and v_{Pb} of the alpha particle and lead nucleus, respectively.

Strategy Conservation of (one-dimensional) momentum is a single equation involving both speeds, as is the known final energy value. Therefore we have two equations for the two unknowns.



▲ **FIGURE 8–13** Decay of a nucleus. Before: Original nucleus. After: Two outgoing fragments that result from the decay (explosion).

Working It Out Conservation of momentum and the expression for the kinetic energy read, respectively,

$$M_\alpha v_\alpha = M_{\text{Pb}} v_{\text{Pb}};$$
$$Q = \tfrac{1}{2} M_\alpha v_\alpha^2 + \tfrac{1}{2} M_{\text{Pb}} v_{\text{Pb}}^2.$$

These two equations can be solved for the two variables v_α and v_{Pb}.

(continues on next page)

We find that the speeds are

$$v_\alpha = \sqrt{\frac{2Q}{M_\alpha(1 + M_\alpha/M_{\text{Pb}})}},$$

$$v_{\text{Pb}} = \sqrt{\frac{2Q}{M_{\text{Pb}}(1 + M_{\text{Pb}}/M_\alpha)}}.$$

Substitution of the known numerical values gives $v_\alpha = 1.60 \times 10^7$ m/s and $v_{\text{Pb}} = 3.10 \times 10^5$ m/s. The speed of the α is about 5 percent of the speed of light, and this is where special relativity begins to play a role—we have ignored that here of course.

This decay is a form of radioactivity; the fragments could conceivably be dangerous. A relevant element is the energy carried by the two fragments, and you should be able to calculate the kinetic energy of each one.

What Do You Think? An object at rest explodes into two fragments of unequal mass. Which of the following is true? (a) The lighter fragment could have the same speed as the heavier fragment but most often moves more quickly. (b) The lighter fragment always moves off more quickly. (c) The lighter fragment can move off more slowly, depending on the details of the explosion.

THINK ABOUT THIS . . .

HOW DOES A JET ENGINE WORK?

▲ **FIGURE 8–14** A jet engine propels an airplane through the operation of the conservation of momentum.

Conservation of momentum is one way to understand the operation of a jet engine (Fig. 8–14). Air is brought into the front of the engine by intake fans. The air, containing oxygen molecules (O_2), is then mixed with fuel. Among chemical reactions that occur during the ensuing combustion, there is the production of two water molecules (H_2O) for each O_2. This doubles the volume of that part of the oxygen from the air that combines with the hydrogen. Other combustion reactions leave the number of molecules and hence the volume of the air unchanged, and some part of the air, principally the nitrogen, undergoes no chemical reaction. Because of the water-producing reaction, the net effect is that a bigger volume of gas must leave the engine than entered it. To be able to keep up the continuous action of the engine, the outgoing gas must therefore leave with a velocity greater than the incoming velocity; the combustion provides the necessary kinetic energy to enable this to happen. Furthermore, the mass of the departing gas is larger, since the mass of the fuel used in combustion has been added to it. Thus the outgoing gas has substantially larger momentum in the backward direction than the incoming gas. To conserve the momentum of the entire system, the airplane gains momentum in the forward direction. The force that makes the escaping gas accelerate out the back has its third law partner in a forward force on the airplane. ■

8–4 Elastic Two-Body Collisions in One Dimension

Let's continue to work in one dimension. As usual, the word "velocity" will mean the velocity *component* in the direction of motion; this can be positive or negative. In an *elastic collision*, there is no mass transfer from one object to another. Further, *all the kinetic energy in the initial state goes into kinetic energy in the final state.* If the final velocities of objects 1 and 2 are denoted by v'_1 and v'_2, then, in addition to the momentum conservation equation for one dimension,

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2, \tag{8–27}$$

we have the energy conservation equation

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2. \tag{8–28}$$

With this information we can find the final velocities of the colliding objects if their initial velocities are known. We rewrite the momentum conservation equation (8–27) as

$$m_1(v_1 - v'_1) = -m_2(v_2 - v'_2). \tag{8–29}$$

We use the fact that $v_1^2 - v_1'^2 = (v_1 - v'_1)(v_1 + v'_1)$ and $v_2^2 - v_2'^2 = (v_2 - v'_2)$ $(v_2 + v'_2)$ to rewrite the energy conservation equation (8–28) in the form

$$\tfrac{1}{2}m_1(v_1 - v'_1)(v_1 + v'_1) = -\tfrac{1}{2}m_2(v_2 - v'_2)(v_2 + v'_2). \tag{8–30}$$

Dividing both sides of Eq. (8–30) by the two sides of Eq. (8–29) leads to the equation

$$v_1 + v'_1 = v_2 + v'_2. \tag{8–31}$$

If we use the letter u to denote the *relative velocity* of the two colliding objects, then

$$u_i = v_1 - v_2 \quad \text{and} \quad u_f = v_1' - v_2'.$$

Using these quantities, Eq. (8–31) can be written in the form

$$u_i = -u_f. \tag{8–32}$$

Equation (8–32) states that *when the collision is elastic, the relative velocity of the colliding objects changes sign but does not change magnitude.* A simple way to remember this result is that the relative velocity behaves like the velocity of a perfectly elastic rubber ball hitting a brick wall.

We may solve Eq. (8–31) for one of the unknown variables, v_2', for example,

$$v_2' = v_1 - v_2 + v_1',$$

and substitute this value into the momentum conservation equation (8–27). We then have

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2(v_1 - v_2 + v_1'),$$

which may be rewritten in the form

$$(m_1 + m_2)v_1' = (m_1 - m_2)v_1 + 2m_2 v_2;$$

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2. \tag{8–33}$$

A similar calculation leads to the formula

$$v_2' = \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_2 - m_1}{m_1 + m_2}v_2. \tag{8–34}$$

These equations are complicated and it is useful to consider two special cases that simplify them.

1. *Object 2 is initially at rest.* We set $v_2 = 0$, so Eqs. (8–33) and (8–34) become

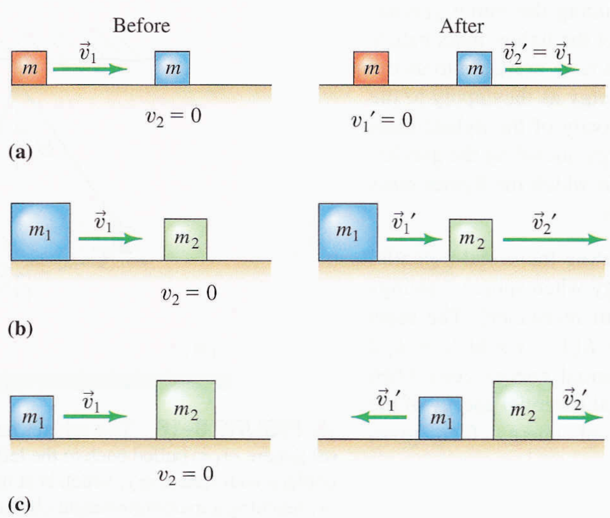
$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 \tag{8–35a}$$

and

$$v_2' = \frac{2m_1}{m_1 + m_2}v_1. \tag{8–35b}$$

Let's consider the following situations (in all of which object 2 is initially at rest):

a. The objects have equal masses (Fig. 8–15a). In this case, $v_1' = 0$ and $v_2' = v_1$. The two objects in effect change roles: The moving object comes to rest and the object that was initially at rest moves with the initial velocity of the first object. This effect can be seen vividly in hard billiard shots along a line.



◀ FIGURE 8–15 Two objects collide elastically in one dimension, with the second object initially at rest. (a) If the two masses are equal, the objects simply exchange velocities. (b) If $m_1 \gg m_2$, both objects move off to the right. (c) If $m_2 \gg m_1$, mass m_1 reverses its direction and m_2 moves slowly off to the right.

b. Mass $m_1 \gg$ mass m_2 (Fig. 8–15b). In this case, Eqs. (8–35) yield $v_1' \cong v_1$ and $v_2' \cong 2v_1$. The velocity of the moving object decreases a little, while the object that was at rest picks up almost twice the velocity of the incoming object.

c. Mass $m_2 \gg$ mass m_1 (Fig. 8–15c). In this case, Eqs. (8–35) yield $v_1' \cong -v_1$ and $v_2' \cong (2m_1/m_2)v_1$. The moving object very nearly reverses its velocity, while the object initially at rest recoils with a very small velocity. In the limit that m_2 approaches infinity, the recoil velocity can be neglected and the final velocity of the first object is equal and opposite to its incident velocity. This is just what happens when a tennis ball is bounced off a wall.

2. *The initial total momentum is zero.* The two objects approach each other with velocities such that the initial total momentum is zero, $m_1v_1 + m_2v_2 = 0$. Thus

$$v_2 = -\frac{m_1}{m_2}v_1. \tag{8–36}$$

When this value is substituted into Eq. (8–33), we find

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 + \left(\frac{2m_2}{m_1 + m_2}\right)\left(-\frac{m_1}{m_2}\right)v_1 = \left(\frac{m_1 - m_2 - 2m_1}{m_1 + m_2}\right)v_1 = -v_1. \tag{8–37}$$

The initial total momentum was zero and so, by momentum conservation, the final total momentum $(m_1v_1' + m_2v_2')$ is also zero and

$$v_2' = -\frac{m_1}{m_2}v_1' = \frac{m_1}{m_2}v_1 = -v_2. \tag{8–38}$$

Therefore, in the case where the total momentum is zero, the velocities of the objects are unchanged in magnitude but they change sign. In effect, under these circumstances, each of the objects acts as if it hit an infinitely massive brick wall.

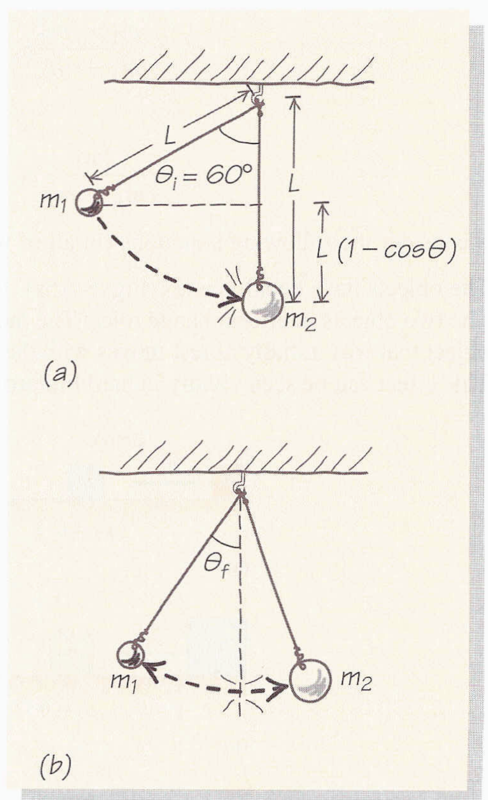
EXAMPLE 8–8 Two spheres with masses of 1.0 and 1.5 kg hang at rest at the ends of strings that are both 1.5 m long. These two strings are attached to the same point on the ceiling. The lighter sphere is pulled aside so that its string makes an angle $\theta_i = 60°$ with the vertical. The lighter sphere is then released and the two spheres collide elastically. When they rebound, what is the largest angle with respect to the vertical that the string holding the lighter sphere makes?

Setting It Up We specify in Fig. 8–16a an initial angle θ_i and in Fig. 8–16b a final angle θ_f. We know the values of the light mass m_1 and the heavy mass m_2. The string length L and the initial angle θ_i are also known. We want the rebound angle, θ_f, of the lighter mass after the collision.

Strategy Conservation of energy, including the initial gravitational potential energy, gives us the speed of the lighter mass before the collision. The collision is elastic, so we have available to us the conservation both of momentum and of energy as they apply to the collision. These will give us the recoil velocity of the lighter mass and we can then apply conservation of energy, including the gravitational potential energy, to find the height to which the lighter mass rises. Geometry then gives us the final angle.

Working It Out We begin by calculating the initial potential energy, which is converted to kinetic energy when sphere 1 swings down to the minimum point (neglecting air resistance). The mass m_1 is raised a distance $L(1 - \cos\theta_i) = L(1 - \cos 60°) = L/2$ above the minimum point. With the potential energy zero when the spheres are hanging vertically, the initial potential energy of the system is $U_i = m_1gL/2$. Conservation of energy then gives $\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1gL$, or

$$v_1 = \sqrt{gL}.$$



▲ **FIGURE 8–16** Two spheres hang from strings of equal length. (a) Sphere m_1 is pulled back to the left at an angle θ_i and released. It collides with sphere m_2, which is at rest. (b) The balls recoil, with sphere m_1 reaching a maximum height characterized by the angle θ_f.

We now treat the collision, which takes place at the bottom of the swing, where $\vec{v}_1$ is horizontal, $\vec{v}_1 = v_1 \hat{i}$. (In fact, the collision itself is a one-dimensional one, with all motion aligned with the x-axis, so we'll drop the explicit vector notation and write all our equations for the collision in terms of the x-components of the vectorial quantities momentum and velocity.) The initial momentum is

$$p_i = m_1 v_1 + m_2 v_2 = m_1 v_1 .$$

To find the velocities v_1' and v_2', we use conservation of momentum, which states $p_f = p_i$, or

$$m_1 v_1' + m_2 v_2' = m_1 v_1 . \tag{8–39}$$

The collision is elastic, so we can also use Eq. (8–32), which is a consequence of energy conservation and which states that the initial relative velocity and the final relative velocity are equal in magnitude but opposite in sign. With

$$v_{\text{rel, initial}} = v_1 - v_2 = v_1 ,$$

we have a final relative velocity of

$$v_{\text{rel, final}} = v_1' - v_2' = -v_{\text{rel, initial}} = -v_1 . \tag{8–40}$$

Equations (8–39) and (8–40) are two simultaneous equations that can be solved for v_1' and v_2'. We are interested only in the final velocity of the lighter sphere, and the solution for this quantity is

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}\sqrt{gL}.$$

With m_2 larger than m_1, this quantity is negative, indicating that the lighter sphere recoils back to the left.

We next find the height to which m_1 recoils. The kinetic energy right after the collision, $m_1 v_1'^2/2$, is converted into gravitational potential energy $U = m_1 g h$ as the sphere rises by h. At the top of the recoil motion, all the kinetic energy is converted to potential energy. As Fig. 8–16b shows, the height risen is $L(1 - \cos\theta_f)$, so we have

$$m_1 g L(1 - \cos\theta_f) = \tfrac{1}{2} m_1 v_1'^2 .$$

Numerically,

$$1 - \cos\theta_f = \frac{v_1'^2}{2gL} = \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2} gL \frac{1}{2gL} = \frac{(m_1 - m_2)^2}{2(m_1 + m_2)^2}$$
$$= \frac{(1.0\ \text{kg} - 1.5\ \text{kg})^2}{2(1.0\ \text{kg} + 1.5\ \text{kg})^2} = 0.020,$$

or $\theta_f = 11°$.

What Do You Think? Describe the same process in the case that the two masses are equal.

8-5 Elastic Collisions in Two and Three Dimensions

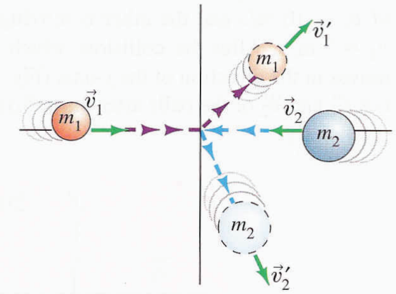
In one dimension, the possible motion of colliding objects is limited. When collisions are no longer restricted to lie along a line, as when billiard balls collide on a billiard table, the vector nature of the mathematical equations becomes important. We work here with collisions in which the identities of the two objects are preserved and in which kinetic energy is conserved.

The law of conservation of momentum [Eq. (8–9)] for the collision of two objects of masses m_1 and m_2, with initial velocities $\vec{v}_1$ and $\vec{v}_2$ and final velocities $\vec{v}_1'$ and $\vec{v}_2'$, reads

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1' + m_2 \vec{v}_2' . \tag{8–41}$$

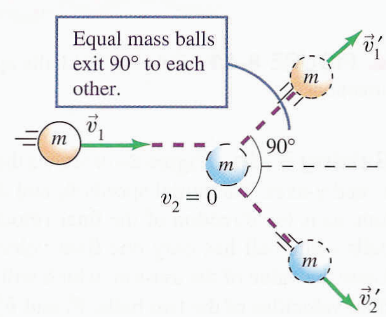
The energy conservation law is

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 . \tag{8–42}$$



▲ **FIGURE 8–17** An object of mass m_1 and velocity $\vec{v}_1$ collides with another object of mass m_2 and velocity $\vec{v}_2$ in two dimensions.

Let's first consider collisions in two dimensions, where everything happens in a plane, such as the xy-plane shown in Fig. 8–17. The collision of billiard balls on a billiard table represents this case, but with equal masses. Given information about the initial motions, we want to find the magnitudes of the final velocities and the angles they make with the x-axis. Equivalently, we want the x- and y-components of the final velocities of objects 1 and 2. There are four unknowns but only three equations. [Equation (8–41) is a vector equation and actually comprises two equations: one for the x-components and one for the y-components.] Therefore, for a given set of initial velocities, *there is no unique solution for the final velocities* and the final objects can move in a variety of directions with a variety of speeds. Nevertheless, the three equations do impose substantial constraints.



▲ **FIGURE 8–18** An object of mass m_1 and velocity $\vec{v}_1$ collides elastically with an object of the same mass at rest. The angle between the final velocities is 90°.

An interesting example of these constraints occurs when the two masses have identical values m and one of the objects is initially at rest. This is very literally the billiards case, in which the projectile is the cue ball and the target is initially stationary. Figure 8–18 shows the geometry—we have chosen object 2 to be initially at rest, $v_2 = 0$. After canceling a common factor of m, we take the square of Eq. (8–41) (the square of a vector equation $\vec{A} = \vec{B}$ implies $\vec{A} \cdot \vec{A} = \vec{B} \cdot \vec{B}$). With $v_2 = 0$, we find

$$v_1^2 = v_1'^2 + 2\vec{v}_1' \cdot \vec{v}_2' + v_2'^2 . \tag{8–43}$$

We can also cancel a common factor of $\frac{1}{2}m$ from Eq. (8–42), leaving, with $v_2 = 0$,

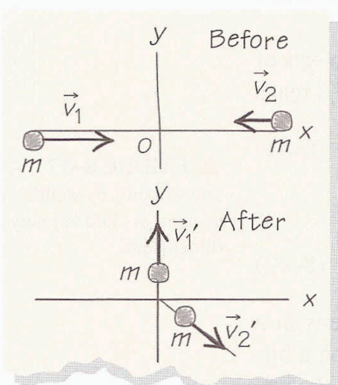
$$v_1^2 = v_1'^2 + v_2'^2. \tag{8–44}$$

Comparing Eqs. (8–43) and (8–44), we conclude that $\vec{v}_1' \cdot \vec{v}_2' = 0$. In other words, the final velocity vectors $\vec{v}_1'$ and $\vec{v}_2'$ are perpendicular to one another. The fact that the angle between the velocities of the outgoing cue ball and the recoiling target ball is a right angle is a fact well known to billiards players (Fig. 8–18). (If spins come in, then this is no longer necessarily true.) It is also worthwhile noting what Eq. (8–44) tells us: The final velocities form the sides of a right triangle whose hypotenuse has magnitude $|v_1|$.

CONCEPTUAL EXAMPLE 8–9 There are two near extremes that describe billiard shots. In the first, the cue ball strikes the target ball nearly head on and the target ball moves rapidly in nearly the original direction of the cue ball. In the second, the cue ball barely grazes the target ball and the cue ball continues with nearly the same speed in its original direction. Qualitatively, what happens to the cue ball in the first case and what happens to the target ball in the second case?

Answer Carried to the limit, these are one-dimensional collisions. In the limit of the first collision, the cue ball stops dead, and in the limit of the second collision the cue ball just misses the target ball, which therefore does not move. The answer to our question must be close to these cases. The billiard balls have equal masses, and the target ball is originally at rest, so that we also have "right angle rules," in which the final velocities obey a right triangle rule [Eq. (8–44)] and the billiard ball motions after the collision make a right angle. Thus in the first case the cue ball moves off slowly at nearly a right angle to the original direction of motion. In the second case the target ball moves slowly off to the side at nearly a right angle to the original direction of motion.

EXAMPLE 8–10 Two billiard balls of equal mass m approach each other along the x-axis; one is moving to the right with a speed of $v_1 = 10\ \text{m/s}$ and the other is moving to the left with a speed of $v_2 = 5\ \text{m/s}$. After the collision, which is elastic, one of the balls moves in the direction of the y-axis (Fig. 8–19). What are the velocities $\vec{v}_1'$ and $\vec{v}_2'$ of the balls after the collision?



▲ **FIGURE 8–19** Two billiard balls approach, collide, and change directions.

Setting It Up Figure 8–19 shows the situation together with the x- and y-axes. The initial speeds $\vec{v}_1$ and $\vec{v}_2$ are specified in the problem, as is the direction of the final velocity $\vec{v}_1' = v_1'\hat{j}$ of one of the balls—this ball has only one final velocity component. We do *not* know the value of the mass m, which will in fact cancel. We want the final velocities of the two balls, $\vec{v}_1'$ and $\vec{v}_2'$.

Strategy If we use the momentum conservation equation and divide by the common factor of m, we deal with a conservation of velocities. Because the collision is elastic, we can also use the conservation of kinetic energy, from which a common factor of m also cancels. The unknown mass m will cancel from the problem. We will want to count equations to make sure we have enough information to find the unknowns, which are the three final velocity components, one for ball 1 and two for ball 2. Assuming there is enough information, we can algebraically solve for the velocity components.

Working It Out With the cancellation of the mass factors, the momentum and energy conservation equations read

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_1' + \vec{v}_2', \qquad v_1^2 + v_2^2 = v_1'^2 + v_2'^2.$$

The balls move in the xy-plane, so the first equation stands for two component equations and the second equation provides a third. There are three unknowns, the single component of $\vec{v}_1'$ and the two components of $\vec{v}_2'$. The problem is thus solvable. Writing $\vec{v}_2' = v_{2x}'\hat{i} + v_{2y}'\hat{j}$, momentum conservation reads

$$(10\ \text{m/s})\hat{i} + (-5\ \text{m/s})\hat{i} = v_1'\hat{j} + v_{2x}'\hat{i} + v_{2y}'\hat{j}.$$

We now separately equate the coefficients of $\hat{i}$ and of $\hat{j}$; that is, we use the fact that momentum conservation holds in the x-direction and in the y-direction separately. We find

$$v_{2x}' = (10\ \text{m/s}) + (-5\ \text{m/s}) = 5\ \text{m/s} \quad \text{and} \quad v_{2y}' = -v_1'.$$

The energy conservation equation reads similarly

$$(10\ \text{m/s})^2 + (-5\ \text{m/s})^2 = v_1'^2 + (v_{2x}'^2 + v_{2y}'^2),$$
$$100\ \text{m}^2/\text{s}^2 + 25\ \text{m}^2/\text{s}^2 = v_1'^2 + 25\ \text{m}^2/\text{s}^2 + v_1'^2.$$

Thus we have $100\ \text{m}^2/\text{s}^2 = 2v_1'^2$, or $v_1' = \sqrt{50}\ \text{m/s}$. We also have $v_{2y}' = -\sqrt{50}\ \text{m/s}$.

Although the collisions of real billiard balls are quite elastic, this is not the case for many other real collisions. A high-speed photograph of a baseball meeting a baseball bat (Fig. 8–20) shows that the ball undergoes significant deformation, which, even though the deformation is not permanent, usually means that there is inelasticity, that is, some energy is lost in the collision.