Analyzing the Polarization State of Light through the Fourier Series

Danial Imam
Roll no: 2022-12-0009
LUMS School of Science and Engineering

Monday, December, 19, 2022

1 Abstract

In this experiment, we demonstrate how the polarization state of light can be determined using a quarter wave plate and a polarizer, while making use of a mathematical technique called the least squares method. This technique is used to solve an over-determined problem to obtain a best fit solution, where an exact solution does not exist, due to the inherent uncertainty in experimental observations. We compare the obtained results with the predictions made using Jones Calculus, a linear algebra based approach to calculating the effects of optical instruments.

2 Introduction

The polarization state of light can analysed by resolving its oscillations into a horizontal and a vertical component. The time dependent variation in magnitude of these vectors is modelled using two orthogonal simple harmonic oscillators. The phase difference between these two harmonics oscillators ϕ, is an important parameter in describing polarization. It is this quantity which is altered by optical instruments such as half wave plates and quarter wave plates to modify characteristics of light without affecting its frequency or intensity.

In this experiment, we first generated a desired polarization state by passing light emitted from a Helium-Neon laser through a horizontally oriented polarizer, and then through a quarter wave plate (QWP) set at an angle α. We chose values of α

that were as far from 0° and 45° as possible in order to explore the more interesting states of polarization, beyond merely linear and circular.

This light with a known polarization was then the subjected to the experimental procedure involving another QWP and an analyser. Using the data gathered, we worked backwards to calculate the value of α which is equivalent to finding the state of polarization of the light generated by the first QWP. This calculated value was compared with the true value to verify our calculations.

3 Theoretical background

3.1 Jones Calculus

We can represent linearly polarized light a using a normalized column vector as follows:

$$\vec{E} = \frac{1}{\sqrt{E_{0x}^2 + E_{0y}^2}} \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix},$$

where E_{0x} and E_{0y} represent the amplitudes of the horizontal component and vertical component respectively.

In terms of the angle α that the orientation of the oscillation makes with the x-axis, we can write it as[1]:

$$\vec{E} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \tag{1}$$

which is also known as the Jones vector.

We now talk about the matrix representation of the optical instruments used in the experiment.

1. The action of a polarizer at an arbitrary angle θ between its Transmission Axis (TA) and the x-axis is given by the following Jones Matrix[1]:

$$P(\theta) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{2}$$

   For the special case of a horizontal polarizer with $\theta = 0$, P reduces to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

2. The action of a Quarter Wave Plate (QWP) at an arbitrary angle θ between its Fast Axis (FA) and the x-axis is given by the following Jones Matrix[1]:

$$QWP(\theta) = e^{-i\frac{\pi}{4}} \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{pmatrix}. \tag{3}$$

For the special case where $\theta = 0$, QWP reduces to $e^{-i\frac{\pi}{4}} \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$.

3.2 Least Squares Method

if we are given a system of equations where the number of equations - say m - exceeds the number of variables - say n, we are facing what is known as an over-determined problem. In practice, the number of equations corresponds to the number of experimental readings, and the variables correspond to the parameters to be determined.

In the ideal case, where experimental readings give exact measurements with complete precision, we could simply use n number of readings out of the all m readings we obtained to calculate exact values for the unknown parameters. However, in practice, there are always uncertainties in experimental measurements and hence choosing, for example, the first n readings for our calculations would give different results compared to choosing, say, the last n readings.

This is why we need to look for a best fit solution, that approximately satisfies all the available data, but exactly satisifies none of it. The technique we use here is called the least squares method. We can represent our over determined system as follows

$$A_{m\times n}X_{n\times 1} = B_{n\times 1},$$

where A is a rectangular matrix with $m > n$. This matrix is not invertible, hence we multiply it by its transpose, $A^T_{n\times m}$ to otain a square $n \times n$ matrix, the inverse of which is called the pseudo-inverse. The solution is then given by[2]:

$$X_{n\times 1} = (A^T_{n\times m}A_{m\times n})^{-1}A^T_{n\times m}B_{m\times 1}. \tag{4}$$

3.3 Working principle of the Quarter wave plate

In order to fully understand the experiment we must first give a brief description of the quarter wave plate which plays a central role in our experimental setup.

The QWP is made from a birefringent material, which means that the indices of refraction experienced by a beam depends on its angle of polarization. Light whose polarization is aligned with what is known as the Fast Axis (FA) experiences a lower index of refraction, n_f, and travels faster, with a larger wavelength.

On the other hand, light polarized along the Slow Axis (SA) of the QWP experiences a higher index of refraction, n_s and hence travels slower, with a smaller wavelength.

These two axes form an orthonormal basis in the plane perpendicular to the direction of propagation of the wave.

The QWP is used to change the state of polarization of light by introducing a phase difference of 90° between the horizontal component and the vertical component of the oscillations.

If linearly polarized light is incident on a QWP, it becomes elliptically polarized, with the eccentricity depending upon the relative angle between the polarization of the incident beam and the angle of the Fast axis of the QWP. So for example, a relative angle of 0° leaves the incident light unaffected. An angle of 45° results in circular polarization of the transmitted beam (a circle being an ellipse with zero eccentricity), and any other angle results in elliptical polarization.

4 The Experiment

4.1 Experimental Set up

Horizontally polarized light was first passed through a QWP at a fixed angle α to generate a desired state of polarization, and then passed through a second QWP at an angle β which was varied from 0° to 360° in steps of 20°. The transmitted beam was then passed through a polarizer at 0° to isolate the horizontal component of the light beam, whose intensity was then measured at a detector.

This experiment was first conducted for an alpha value of 10°, and then subsequently for alpha values of 20° and 30°.

4.2 Theoretical predictions

4.2.1 Prediction of Intensity of Transmitted Beam

To obtain the Jones vector of the transmitted beam received at the detector, we use the following equation, using the relations from Eq.(2) and Eq.(3):

$$\vec{E}_{trans} = Pol(0^\circ) \times QWP(\beta^\circ) \times QWP(\alpha^\circ) \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

After some tedious algebra, we get the following result:

$$\vec{E}_{trans} = -i \begin{pmatrix} (\cos^2\beta + i\sin^2\beta)(\cos^2\alpha + i\sin^2\alpha) - 2i\sin\beta\cos\beta\sin\alpha\cos\alpha \\ 0 \end{pmatrix},$$

which represents the amplitudes of the x-component and y-component of the transmitted beam.

Since intensity is proportional to the modulus square of the amplitude, the measurement at the detector will be:

$$I = A(\cos^4\beta + \sin^4\beta) + B^2[\sin(2\beta)]^2 - 2B(sin(2\beta)[(Dsin\beta)^2 + (C\cos\beta)^2],$$

where $A = \frac{3+\cos(4\alpha)}{4}$, $B = \frac{\sin(2\alpha)}{2}$, $C = \sin\alpha$ and $D = \cos\alpha$ We can rearrange the above equation to form a Fourier series in terms of angle β as shown:

$$I(\beta) = C_0 + S_2\sin(2\beta) + S_4\sin(4\beta) + C_4\cos(4\beta), \tag{5}$$

where the fourier coefficients are as follows:

$$\begin{aligned} C_0 &= \frac{5+\cos(4\alpha)}{8}, \\ S_2 &= -\frac{\sin(2\alpha)}{2}, \\ S_4 &= \frac{\sin(4\alpha)}{8}, \\ C_4 &= \frac{1+\cos(4\alpha)}{8}. \end{aligned} \tag{6}$$

It is interesting to note that all the information regarding the transmitted beam is encapsulated in just four Fourier coefficients which are all functions of α only.

4.2.2 Prediction of angle α

The above is sufficient to make predictions about the intensity measurements for a given value of α, while rotating the second QWP over a range of angles from $0°$ to $360°$. However, determining the value of α from a set of readings requires the use of the above mentioned least squares method. Since we have 18 values of β with corresponding intensity values, and 4 unknown fourier coefficients, we can express our problem in vectorized form as follows:

$$\begin{pmatrix} 1 & \sin(2\beta_1) & \sin(4\beta_1) & \cos(4\beta_1) \\ 1 & \sin(2\beta_2) & \sin(4\beta_2) & \cos(4\beta_2) \\ . & . & . & . \\ . & . & . & . \\ 1 & \sin(2\beta_{18}) & \sin(4\beta_{18}) & \cos(4\beta_{18}) \end{pmatrix} \begin{pmatrix} C_0 \\ S_2 \\ S_4 \\ C_4 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ . \\ . \\ I_{18} \end{pmatrix},$$

or more compactly as:

$$AX = B.$$

This is solved numerically using Python, making use of Eq (4), and the results presented in the next section for each value of α.

5 Results and Discussion

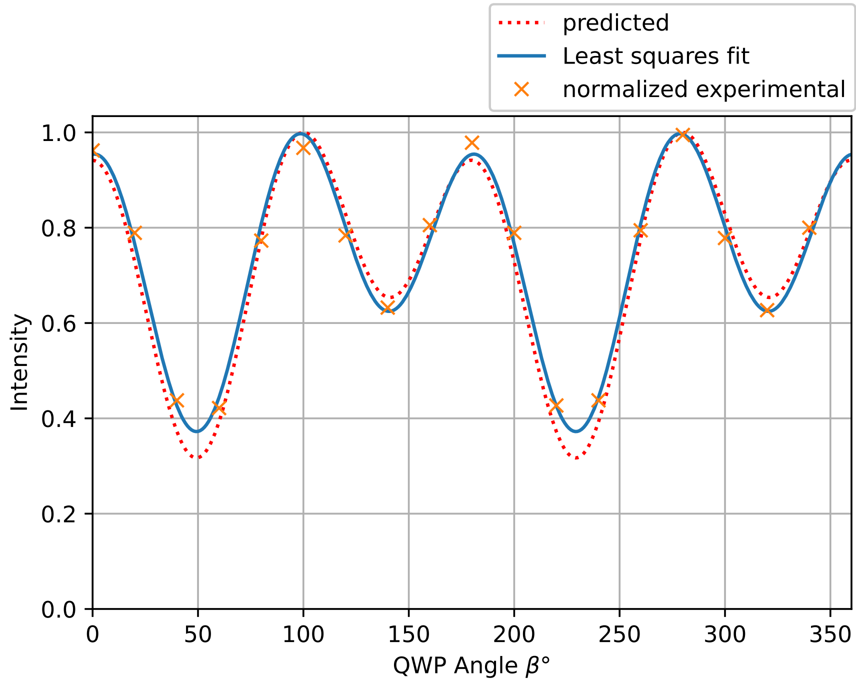

Figure 1: QWP angle $\beta°$ against Intensity for $\alpha = 10°$

The results from the first part of the experiment with α set to $10°$ (reported in Table 2) have been plotted in Fig (1). The numerically calculated values of the Fourier coefficients are: $C_0 = 0.733, S_2 = -0.128, S_4 = 0.078$, and $C_4 = 0.221$. These values have been used in conjunction with Eq (5) to plot the least squares fitted line, which we can see closely follows the experimentally obtained values.

The calculated values of the fourier coefficients can be plugged into the four equations in Eq (6) to obtain four values of α, i.e.: $7.56°, 7.42°, 9.65°$, and $9.96°$. Taking the average, we obtain $8.65°$. This indicates a relative error of 13.5%.

The results from the second part of the experiment with α set to $20°$ (reported in Table 3) have been plotted in Fig (2), and those for α set to $30°$ (reported in Table 4) plotted in Fig (3).

The calculated fourier coefficients and the corresponding calculated α values for all three parts of the experiment are summarized in Table 1

Overall, we can see that the calculated values of α are consistently understated which suggests that the calibration of the QWP may be off by a few degrees, leading to systematic error. To avoid this, all equipment must be checked at the start of the experiment for zero errors. This is made more complicated by the fact

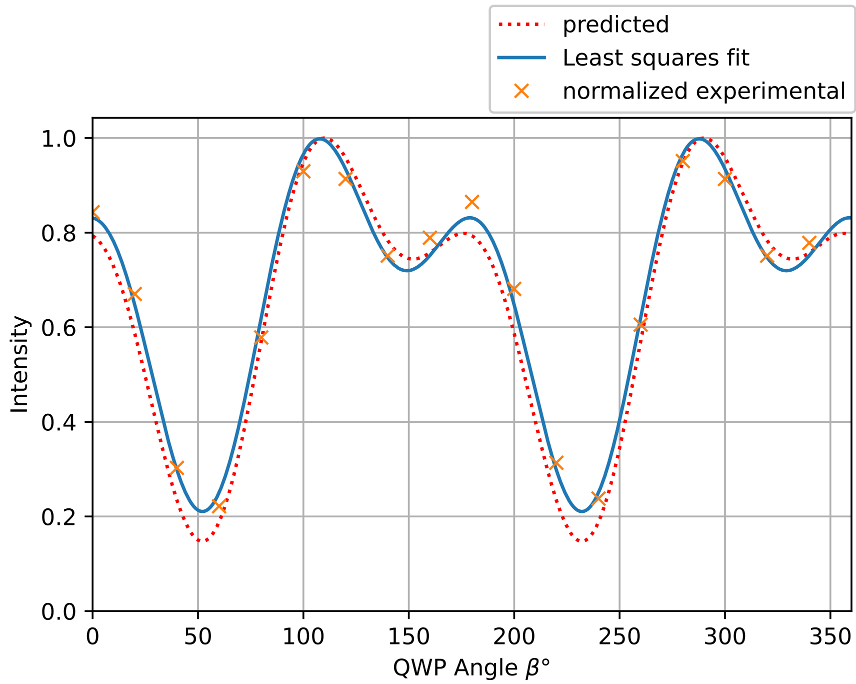

Figure 2: QWP angle $\beta°$ against Intensity for $\alpha = 20°$

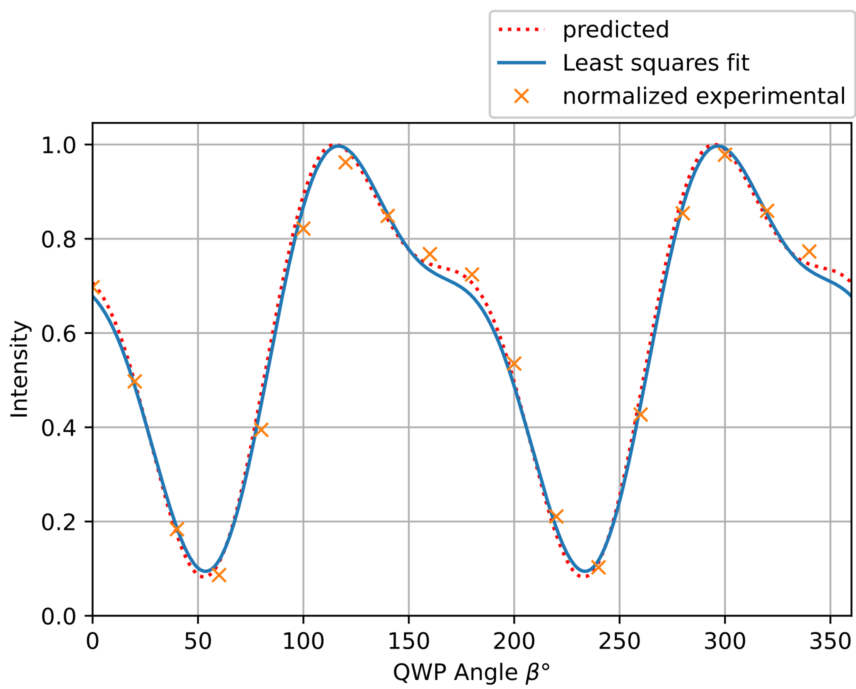

Figure 3: QWP angle $\beta°$ against Intensity for $\alpha = 30°$

	Fourier coefficient	Calculated α	Average of Calculated α	Relative error
		Actual α *= 10*		
C_0	0.733	7.56		
S_2	-0.128	7.42		
S_4	0.078	9.65		
C_4	0.221	9.96	8.6475	13.5%
		Actual α *= 20*		
C_0	0.672	16.98		
S_2	-0.271	16.41		
S_4	0.124	20.69		
C_4	0.159	18.55	18.1575	9.2%
		Actual α *= 30*		
C_0	0.596	25.85		
S_2	-0.381	24.82		
S_4	0.124	24.31		
C_4	0.082	27.53	25.6275	14.6%

Table 1: Results

that the least count of the QWP is 2 degrees, so fine adjustments to the angle are not possible.

6 Conclusions

We demonstrated how to calculate the polarization angle of light using only a quarter wave plate and a polarizer, by applying the applying the least squares technique to find the best fit values of four Fourier coefficients. These coefficients were used to calculate the unknown angle of the first QWP that was used to generate the polarization under study, which is equivalent to finding the polarization state of the incident light.

References

[1] Frank L Pedrotti, Leno M Pedrotti, and Leno S Pedrotti. *Introduction to optics.* Cambridge University Press, 2017.

[2] Gilbert Strang. *Linear algebra and its applications.* Belmont, CA: Thomson, Brooks/Cole, 2006.

A Appendix: Intensity readings from Experiments

QWP angle β° w.r.t. horizontal[1]	**Detector reading (V)**[2]	**Normalized reading**[3]
0	1.78	0.96
20	1.46	0.79
40	0.81	0.44
60	0.78	0.42
80	1.43	0.77
100	1.79	0.97
120	1.45	0.78
140	1.17	0.63
160	1.49	0.81
180	1.81	0.98
200	1.46	0.79
220	0.79	0.43
240	0.81	0.44
260	1.47	0.79
280	1.84	0.99
300	1.44	0.78
320	1.16	0.63
340	1.48	0.80

Table 2: Readings for $\alpha = 10^\circ$

[1] Error in angle reading of $\pm 1^\circ$
[2] Error in multimeter reading of ± 0.005 V
[3] Values normalized by dividing by 1.85

QWP angle β° w.r.t. horizontal[1]	**Detector reading (V)**[2]	**Normalized reading**[3]
0	1.56	0.84
20	1.24	0.67
40	0.56	0.30
60	0.41	0.22
80	1.07	0.58
100	1.72	0.93
120	1.69	0.91
140	1.39	0.75
160	1.46	0.79
180	1.60	0.86
200	1.26	0.68
220	0.58	0.31
240	0.44	0.24
260	1.12	0.61
280	1.76	0.95
300	1.69	0.91
320	1.39	0.75
340	1.44	0.78

Table 3: Readings for $\alpha = 20^\circ$

[1] Error in angle reading of $\pm 1^\circ$
[2] Error in multimeter reading of ± 0.005 V
[3] Values normalized by dividing by 1.85

QWP angle β° w.r.t. horizontal[1]	**Detector reading (V)**[2]	**Normalized reading**[3]
0	1.29	0.74
20	0.92	0.53
40	0.34	0.19
60	0.16	0.09
80	0.73	0.42
100	1.52	0.87
120	1.78	1.02
140	1.57	0.90
160	1.42	0.81
180	1.34	0.77
200	0.99	0.57
220	0.39	0.22
240	0.19	0.11
260	0.79	0.45
280	1.58	0.90
300	1.81	1.03
320	1.59	0.91
340	1.43	0.82

Table 4: Readings for $\alpha = 30^\circ$

[1] Error in angle reading of $\pm 1^\circ$
[2] Error in multimeter reading of ± 0.005 V
[3] Values normalized by dividing by 1.85