Chapter 10

Diffraction

In the 1600's, Christiaan Huygens developed a wave description for light. Unfortunately, his ideas were largely overlooked at the time because Sir Isaac Newton promoted a competing theory. Newton proposed that light should be thought of as many tiny bullets, or *corpuscles*, as he called them. Newton's ideas prevailed for more than a century, perhaps because he was right on so many other things, until 1807 when Thomas Young performed his famous two-slit experiment, conclusively demonstrating the wave nature of light. Even then, Young's conclusions were accepted only gradually by others, a notable exception being a young Frenchman named Augustin Fresnel. The two formed a close friendship through correspondence, and it was Fresnel that followed up on Young's conclusions and dedicated his life to a study of light.

Christiaan Huygens (1629–1695, Dutch) was born in The Hague, Netherlands. His father was friends with the mathematician René Descartes, which probably influenced his upbringing. Huygens studied law and mathematics at the University of Leiden, which preceded a very productive career as a scientist and mathematician. During mid career, Huygens held a position in the French Academy of Sciences in Paris for 15 years, but he spent the majority of his life in The Hague. Huygens was the first to advocate the wave theory of light. He was able to explain birefringence in terms of his wave theory assuming a refractive index that varied with direction. Huygens constructed a telescope with which he discovered Saturn's moon Titan. He also made the first detailed observations of the Orion nebula. Huygens made significant advancements in clock-making technology and wrote a book on probability theory. Huygens was one of the earliest science-fiction writers and speculated that life exists on other planets in his book Cosmotheoros. (Wikipedia)

Fresnel's skill as a mathematician allowed him to transform physical intuition into powerful and concise ideas. Perhaps Fresnel's greatest accomplishment was the adaptation of *Huygens' principle* of wavelet superposition into a mathematical formula. Ironically, he used Newton's calculus to achieve this. Huygens' principle asserts that a wave front can be thought of as many wavelets, which propagate and interfere to form new wave fronts. This is illustrated in Fig. 10.1. The phenomenon of diffraction is then understood as the spilling of wavelets around obstructions in the path of light.

After formulating Huygens' principle as a *diffraction integral*, Fresnel made an approximation to his own formula, called the *Fresnel approximation*, for the sake of making the integration easier to perform. As far as approximations go, the Fresnel approximation is surprisingly accurate in describing the light field in the region downstream from an aperture. The diffraction pattern can evolve in complicated ways as the distance from an aperture increases. At distances far downstream from an aperture, the diffraction pattern acquires a final form that no longer evolves, other than to grow in proportion to distance. This *far-field* limit is often of interest, and it turns out that the Fresnel diffraction formula can be simplified further in this case. The far-field limit of the Fresnel diffraction formula is called the Fraunhofer approximation.

From the modern perspective, Fresnel's diffraction formula needs justifica-

tion starting from Maxwell's equation. The diffraction formula is based on *scalar diffraction* theory, which ignores polarization effects. In some situations, ignoring polarization is benign, but in other situations, ignoring polarization effects produces significant errors. These issues as well as the approximations leading to scalar diffraction theory are discussed in section 10.2.

10.1 Huygens' Principle as Formulated by Fresnel

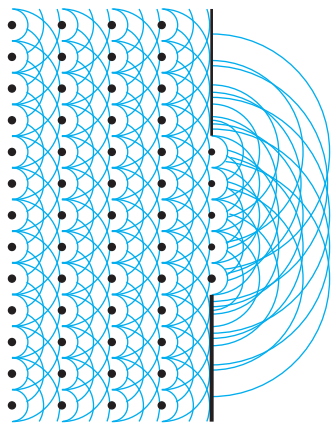

Figure 10.1 Wave fronts depicted as a series of Huygens' wavelets.

In this section we discuss the calculus of summing up the contributions from the many wavelets originating in an aperture illuminated by a light field. Each point in the aperture is thought of as a source of a *spherical wavelet*.[1] In our modern notation, such a spherical wave can be written as proportional to e^{ikR}/R, where R is the distance from the source. As a spherical wave propagates, its strength falls off in proportion to the distance traveled and the phase is related to the distance propagated, similar to the phase of a plane wave. It should be noted that by choosing k, we consider only a single wavelength of light (i.e. one frequency).

A spherical wave of the form e^{ikR}/R technically does not satisfy Maxwell's equations (see P10.4). For one thing, it utterly fails near $R = 0$. However, if R is large compared to a wavelength, this spherical wave starts to resemble actual solutions to Maxwell's equations, as will be examined in the next section. It is within this regime that the diffraction formula derived here is successful.

Consider an aperture or opening in an opaque screen located at the plane $z = 0$. Let the aperture be illuminated with a light field distribution $E(x', y', z = 0)$ within the aperture. Then for a point (x, y, z) lying somewhere after the aperture $(z > 0)$, the net field is given by adding together the contribution of wavelets emitted from each point in the aperture.

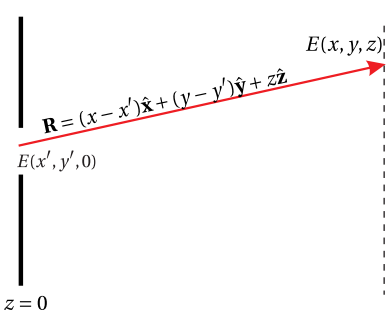

Figure 10.2

Each spherical wavelet is assigned the strength and phase of the field at the point where it originates. Mathematically, this summation takes the form

$$E(x, y, z) = -\frac{i}{\lambda} \iint_{\text{aperture}} E(x', y', 0) \frac{e^{ikR}}{R} dx' dy' \tag{10.1}$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + z^2} \tag{10.2}$$

is the radius of each wavelet as it individually intersects the point (x, y, z). We will call (10.1) the Huygens-Fresnel[2] diffraction formula, although Fresnel is credited with this integral formulation. The factor $-i/\lambda$ in front of the integral in (10.1) ensures the right phase and field strength (not to mention correct units). Justification for this factor is given in section 10.3 and in appendix 10.A. To summarize,

[1] For simplicity, we use the term 'spherical wave' in this book to refer to waves of the type imagined by Huygens (i.e. of the form e^{ikR}/R). There is a different family of waves based on spherical harmonics that are also sometimes referred to as spherical waves. These waves have angular as well as radial dependence, and they *are* solutions to Maxwell's equations. See J. D. Jackson, *Classical Electrodynamics*, 3rd ed., pp. 429–432 (New York: John Wiley, 1999).

[2] M. Born and E. Wolf, *Principles of Optics*, 7th ed., p. 414 (Cambridge University Press, 1999).

(10.1) tells us how to compute the field downstream given knowledge of the field in an aperture. The field at each point (x', y') in the aperture, which may vary with strength and phase, is treated as the source for a spherical wave. The integral in (10.1) sums the contributions from all of these wavelets.

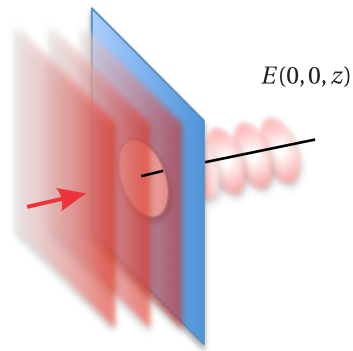

Figure 10.3 Circular aperture illuminated by a plane wave.

Example 10.1

Find the on-axis[3] (i.e. $x, y = 0$) intensity following a circular aperture of *diameter* D illuminated by a uniform plane wave.

Solution: The diffraction integral (10.1) takes the form

$$E(0,0,z) = -\frac{i}{\lambda} \iint_{\text{aperture}} E(x', y', 0) \frac{e^{ik\sqrt{x'^2+y'^2+z^2}}}{\sqrt{x'^2+y'^2+z^2}} \, dx' dy'$$

The circular hole encourages a change to cylindrical coordinates: $x' = \rho' \cos\phi'$ and $y' = \rho' \sin\phi'$; $dx'dy' \to \rho' d\rho' d\phi'$. In this case, the limits of integration define the geometry of the aperture, and the integration is accomplished as follows:

$$\begin{aligned} E(0,0,z) &= -\frac{iE_0}{\lambda} \int_0^{2\pi} d\phi' \int_0^{D/2} \frac{e^{ik\sqrt{\rho'^2+z^2}}}{\sqrt{\rho'^2+z^2}} \rho' \, d\rho' \\ &= -\frac{iE_0}{\lambda} 2\pi \left. \frac{e^{ik\sqrt{\rho'^2+z^2}}}{ik} \right|_0^{D/2} = -E_0 \left(e^{ik\sqrt{(D/2)^2+z^2}} - e^{ikz} \right) \end{aligned}$$

The on-axis intensity is then proportional to

$$\begin{aligned} E(0,0,z)\, E^*(0,0,z) &= |E_0|^2 \left(e^{ik\sqrt{(D/2)^2+z^2}} - e^{ikz} \right) \left(e^{-ik\sqrt{(D/2)^2+z^2}} - e^{-ikz} \right) \\ &= 2|E_0|^2 \left[1 - \cos\left(k\sqrt{(D/2)^2+z^2} - kz \right) \right] \end{aligned} \tag{10.3}$$

A graph of this function is shown in Fig. 10.4.

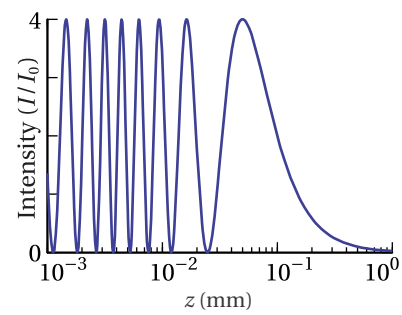

Figure 10.4 Intensity on axis following a circular aperture with $D = 20\lambda$ and wavelength $\lambda = 500$ nm.

When an aperture has a complicated shape, it may be convenient to break up the diffraction integral (10.1) into several pieces. As an example, suppose that we have an aperture consisting of a circular obstruction within a square opening as depicted in Fig. 10.5. Thus, the light transmits through the region between the circle and the square. One can evaluate the overall diffraction pattern by first evaluating the diffraction integral for the entire square (ignoring the circular block) and then subtracting the diffraction integral for a circular opening having the shape of the block. This removes the unwanted part of the previous integration and yields the overall result. When doing this, it is important to add and subtract the integrals (i.e. fields), not their squares (i.e. intensity).

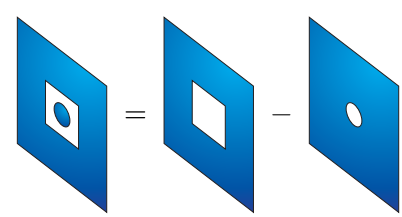

Figure 10.5 Aperture comprised of the region between a circle and a square.

It may be less obvious at first that you can use the above superposition technique to handle diffraction from finite obstructions that interrupt an infinitely

[3] An analytical solution is not possible off axis.

wide plane wave. One simply computes the diffraction of the blocked portion of the field as though it came from an opening in a mask. The result is then subtracted from the plane wave (no integration needed for the plane wave), as depicted in Fig. 10.6. This is known as *Babinet's principle*.

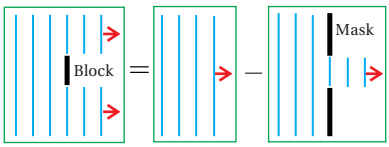

Figure 10.6 Side view of a circular block in a plane wave giving rise to diffraction in the geometric shadow.

When Fresnel first presented his diffraction formula to the French Academy of Sciences, a certain judge of scientific papers named Siméon Poisson noticed that Fresnel's formula predicted that there should be light in the center of the geometric shadow behind a circular obstruction. This seemed so absurd to Poisson that he initially disbelieved the theory, until the spot was shortly thereafter experimentally confirmed, much to Poisson's chagrin. Needless to say, Fresnel's paper was then awarded first prize, and this spot appearing behind circular blocks has since been known as *Poisson's spot*.

Example 10.2

Find the on-axis (i.e. $x, y = 0$) intensity behind a circular block of *diameter D* placed in a uniform plane wave.

Solution: From Example 10.1, the on-axis field behind a circular aperture is $E_0\left(e^{ikz} - e^{ik\sqrt{(D/2)^2+z^2}}\right)$. Babinet's principle says to subtract this result from a plane wave to obtain the field behind the circular block. The situation is depicted in Fig. 10.6. The on-axis field is then

$$E(0,0,z) = E_0 e^{ikz} - E_0\left(e^{ikz} - e^{ik\sqrt{(D/2)^2+z^2}}\right) = E_0 e^{ik\sqrt{(D/2)^2+z^2}}$$

The on axis intensity becomes

$$I(0,0,z) \propto E(0,0,z)\,E^*(0,0,z) = |E_0|^2\, e^{ik\sqrt{(D/2)^2+z^2}}\, e^{-ik\sqrt{(D/2)^2+z^2}} = |E_0|^2$$

In the exact center of the shadow behind the circular obstruction, the intensity is the same as the illuminating plane wave for all distance z. A spot of light in the center forms right away; no wonder Poisson was astonished!

10.2 Scalar Diffraction Theory

In this section we provide the background motivation for Huygen's principle and Fresnel's formulation of it. Consider a light field with a single frequency ω. The light field can be represented by $\mathbf{E}(\mathbf{r})\,e^{-i\omega t}$, and the time derivative in the wave equation (2.13) can be easily performed. It reduces to

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = 0 \tag{10.4}$$

where $k \equiv n\omega/c$ is the magnitude of the usual wave vector (see also (9.2)). Equation (10.4) is called the *Helmholtz equation*. Again, it is merely the wave equation

written for the case of a single frequency, where the trivial time dependence has been removed. To obtain the full wave solution, just append the factor $e^{-i\omega t}$ to the solution of (10.4).

At this point we take an *egregious* step: We ignore the vectorial nature of $\mathbf{E}(\mathbf{r})$ and write (10.4) using only the magnitude $E(\mathbf{r})$. When using scalar diffraction theory, we must keep in mind that it is based on this serious step. Under the scalar approximation, the vector Helmholtz equation (10.4) becomes the *scalar Helmholtz equation*:

$$\nabla^2 E(\mathbf{r}) + k^2 E(\mathbf{r}) = 0 \tag{10.5}$$

This equation of course is consistent with (10.4) in the case of a plane wave. However, we are interested in spherical waves of the form $E(r) = E_0 r_0 e^{ikr}/r$. It turns out that such spherical waves are exact solutions to the scalar Helmholtz equation (10.5). The proof is left as an exercise (see P10.3). Nevertheless, spherical waves of this form only approximately satisfy the vector Helmholtz equation (10.4). We can get away with this sleight of hand if the radius r is large compared to a wavelength (i.e., $kr \gg 1$) and if we restrict $\mathbf{r}$ to a narrow range perpendicular to the polarization.

Francois Jean Dominique Arago (1786-1853, French) was born in Catalan France, where his father was the Treasurer of the Mint. As a teenager, Arago was sent to a municipal college in Perpignan where he developed a deep interest in mathematics. In 1803, he entered the Ecole Polytechnique in Paris, where he purportedly was disappointed that he was not presented with new knowledge at a higher rate. He associated with famous French mathematicians Siméon Poisson and Pierre-Simmon Laplace. He later worked with Jean-Baptiste Biot to measure the meridian arch to determine the exact length of the meter. This work took him to the Balearic Islands, Spain, where he was imprisoned as a spy, being suspected because of lighting fires atop a mountain as part of his surveying efforts. After a heroic prison escape and a subsequent string of misfortunes, he eventually made it back to France where he took a strong interest in optics and the wave theory of light. Arago and Fresnel established a fruitful collaboration that extended for many years. It was Arago who demonstrated Poisson's spot (sometimes called Arago's spot). Arago also invented the first polarizing filter. In later life, he served a brief stint as the French prime minister. (Wikipedia)

Significance of the Scalar Wave Approximation

The solution of the scalar Helmholtz equation is not completely unassociated with the solution to the vector Helmholtz equation. In fact, if $E_{\text{scalar}}(\mathbf{r})$ obeys the scalar Helmholtz equation (10.5), then

$$\mathbf{E}(\mathbf{r}) = \mathbf{r} \times \nabla E_{\text{scalar}}(\mathbf{r}) \tag{10.6}$$

obeys the vector Helmholtz equation (10.4).

Consider a spherical wave, which is a solution to the scalar Helmholtz equation:

$$E_{\text{scalar}}(\mathbf{r}) = E_0 r_0 e^{ikr}/r \tag{10.7}$$

Remarkably, when this expression is placed into (10.6) the result is zero. Although zero is in fact a solution to the vector Helmholtz equation, it is not very interesting. A more interesting solution to the scalar Helmholtz equation is

$$E_{\text{scalar}}(\mathbf{r}) = r_0 E_0 \left(1 - \frac{i}{kr}\right) \frac{e^{ikr}}{r} \cos\theta \tag{10.8}$$

which is one of an infinite number of unique 'spherical' solutions that exist. Notice that in the limit of large r, this expression looks similar to (10.7), aside from the factor $\cos\theta$. The vector form of this field according to (10.6) is

$$\mathbf{E}(\mathbf{r}) = -\hat{\phi} r_0 E_0 \left(1 - \frac{i}{kr}\right) \frac{e^{ikr}}{r} \sin\theta \tag{10.9}$$

This field looks approximately like the scalar spherical wave solution (10.7) in the limit of large r if the angle is chosen to lie near $\theta \cong \pi/2$ (spherical coordinates). Since our use of the scalar Helmholtz equation is in connection with this spherical wave under these conditions, the results are close to those obtained from the vector Helmholtz equation.

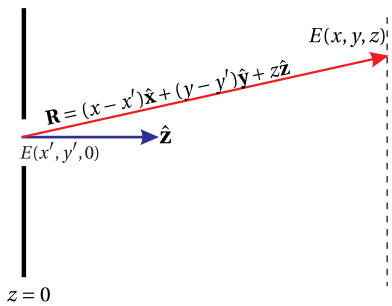

Figure 10.7

Fresnel developed his diffraction formula (10.1) a half century before Maxwell assembled the equations of electromagnetic theory. In 1887, Gustav Kirchhoff demonstrated that Fresnel's diffraction formula satisfies the scalar Helmholtz equation. In doing this he clearly showed the approximations implicit in the theory, and made a slight revision to the formula:

$$E(x,y,z) = -\frac{i}{\lambda} \iint_{\text{aperture}} E(x',y',0)\,\frac{e^{ikR}}{R}\left[\frac{1+\cos(\mathbf{R},\hat{\mathbf{z}})}{2}\right] dx'\,dy' \tag{10.10}$$

The factor in square brackets, Kirchhoff's revision, is known as the *obliquity factor*. Here, $\cos(\mathbf{R},\hat{\mathbf{z}})$ indicates the cosine of the angle between $\mathbf{R}$ and $\hat{\mathbf{z}}$. Notice that this factor is approximately equal to one when the point (x,y,z) is chosen to be in the forward direction; we usually study diffraction under this circumstance. On the other hand, the obliquity factor equals zero for fields traveling in the reverse direction (i.e. in the $-\hat{\mathbf{z}}$ direction). This fixes a problem with Fresnel's version of the formula (10.1) based on Huygens' wavelets, which suggested that light could as easily diffract in the reverse direction as in the forward direction

In honor of Kirchhoff's work, (10.10) is referred to as the Fresnel-Kirchhoff diffraction formula. The details of Kirchhoff's more rigorous derivation, including how the factor $-i/\lambda$ naturally arises, are given in appendix 10.A. Since the Fresnel-Kirchhoff formula can be understood as a superposition of spherical waves, it is not surprising that it satisfies the scalar Helmholtz equation (10.5).

10.3 Fresnel Approximation

Although the Fresnel-Kirchhoff integral looks innocent enough, it is actually quite difficult to evaluate analytically. Even the Huygens-Fresnel version (10.1) where the obliquity factor $(1+\cos(\mathbf{r},\hat{\mathbf{z}}))/2$ is approximated as one (i.e. far forward direction) is challenging. The integration can be challenging even if we choose a field $E(x',y',0)$ that is uniform across the aperture (i.e. a constant).

Fresnel introduced an approximation[4] to his diffraction formula that makes the integration somewhat easier to perform. The approximation is analogous to the paraxial approximation made for rays in chapter 9.

Besides letting the obliquity factor be one, Fresnel approximated R by the distance z in the denominator of (10.10) . Then the denominator can be brought out in front of the integral since it no longer depends on x' and y'. This is valid to the extent that we restrict ourselves to small angles:

$$R \cong z \qquad \text{(denominator only; Fresnel approximation)} \tag{10.11}$$

The above approximation is wholly inappropriate in the exponent of (10.10) since small changes in R can result in dramatic variations in the periodic function e^{ikR}.

[4] J. W. Goodman, *Introduction to Fourier Optics*, Sect. 4-1 (New York: McGraw-Hill, 1968).

To approximate R in the exponent, we must proceed with caution. To this end we expand (10.2) under the assumption $z^2 \gg (x-x')^2 + (y-y')^2$. Again, this is consistent with the idea of restricting ourselves to relatively small angles. The expansion of (10.2) is written as

$$R = z\sqrt{1 + \frac{(x-x')^2 + (y-y')^2}{z^2}} \cong z\left[1 + \frac{(x-x')^2 + (y-y')^2}{2z^2} + \cdots\right]$$

(exponent; Fresnel approximation) (10.12)

Substitution of (10.11) and (10.12) into the Huygens-Fresnel diffraction formula (10.1) yields

$$E(x,y,z) \cong -\frac{i e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z} \iint\limits_{\text{aperture}} E(x',y',0)\, e^{i\frac{k}{2z}(x'^2+y'^2)} e^{-i\frac{k}{z}(xx'+yy')}\, dx'\, dy'$$

(Fresnel approximation) (10.13)

This approximation may look a bit messier than before, but in terms of being able to make progress on integration our chances are somewhat improved.

Example 10.3

Compute the Fresnel diffraction field following a rectangular aperture (dimensions Δx by Δy) illuminated by a uniform plane wave.

Solution: According to (10.13), the field downstream is

$$E(x,y,z) = -iE_0 \frac{e^{ikz}}{\lambda z} e^{i\frac{k}{2z}(x^2+y^2)} \int\limits_{-\Delta x/2}^{\Delta x/2} dx'\, e^{i\frac{k}{2z}x'^2} e^{-i\frac{kx}{z}x'} \int\limits_{-\Delta y/2}^{\Delta y/2} dy'\, e^{i\frac{k}{2z}y'^2} e^{-i\frac{ky}{z}y'}$$

Unfortunately, the integration in the preceding example must be performed numerically. This is often the case for diffraction integrals in the Fresnel approximation, but at least numerical fast Fourier transforms can aid in the process. Figure 10.8 shows the result of integration for a rectangular aperture with a height twice its width.

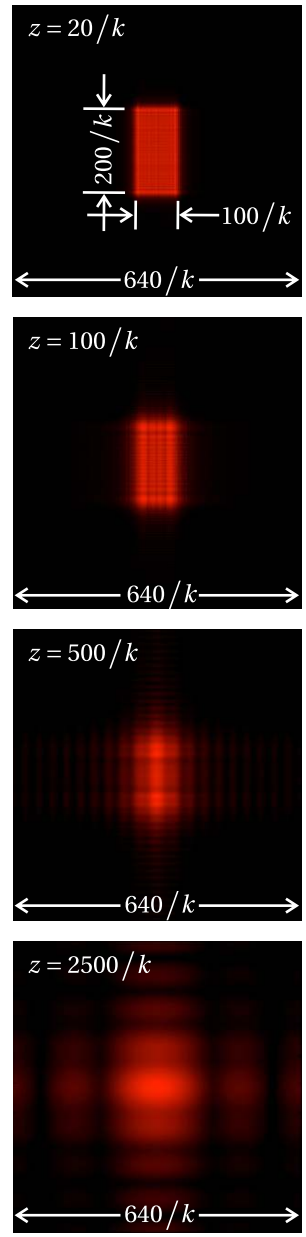

Figure 10.8 Field amplitude following a rectangular aperture computed in the Fresnel approximation.

Paraxial Wave Equation

If we assume that the light coming through the aperture is highly directional, such that it propagates mainly in the z-direction, we are motivated to write the field as $E(x,y,z) = \tilde{E}(x,y,z)e^{ikz}$. Upon substitution of this into the scalar Helmholtz equation (10.5), we arrive at

$$\frac{\partial^2 \tilde{E}}{\partial x^2} + \frac{\partial^2 \tilde{E}}{\partial y^2} + 2ik\frac{\partial \tilde{E}}{\partial z} + \frac{\partial^2 \tilde{E}}{\partial z^2} = 0 \tag{10.14}$$

At this point we make the paraxial wave approximation,[5] which is $|2k\frac{\partial \tilde{E}}{\partial z}| \gg |\frac{\partial^2 \tilde{E}}{\partial z^2}|$. That is, we assume that the amplitude of the field varies slowly in the z-direction

[5] P. W. Milonni and J. H. Eberly, *Laser*, Sect. 14.4 (New York: Wiley, 1988).

such that the wave looks much like a plane wave. We permit the amplitude to change as the wave propagates in the z-direction as long as it does so on a scale much longer than a wavelength. This leads to the *paraxial wave equation*:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+2ik\frac{\partial}{\partial z}\right)\tilde{E}(x,y,z)\cong 0 \qquad \text{(paraxial wave equation)} \quad (10.15)$$

It turns out that the Fresnel approximation (10.13) is an *exact* solution to the paraxial wave equation (see P10.5). That is, (10.15) is satisfied by

$$\tilde{E}(x,y,z)\cong -\frac{i}{\lambda z}\iint_{-\infty}^{\infty}\tilde{E}(x',y',0)e^{i\frac{k}{2z}\left[(x-x')^2+(y-y')^2\right]}dx'dy' \quad (10.16)$$

When the factor e^{ikz} is appended, this field is identical to (10.13).

Joseph von Fraunhofer (1787–1826, German) was born in Straubing, Bavaria. He was orphaned at age 11, whereupon he was apprenticed to a glassmaker. The workshop collapsed, trapping him in the rubble. The Prince of Bavaria directed the rescue efforts and thereafter took an interest in Fraunhofer's education. The prince required the glassmaker to allow young Joseph time to study, and he naturally took an interest in optics. Fraunhofer later worked at the Optical Institute at Benediktbeuern, where he learned techniques for making the finest optical glass in his day. Fraunhofer developed numerous glass recipes and was expert at creating optical devices. Fraunhofer was the inventor of the spectroscope, making it possible to do quantitative spectroscopy. Using his spectroscope, Fraunhofer was the first to observe and document hundreds of absorption lines in the sun's spectrum. He also noticed that these varied for different stars, thus establishing the field of stellar spectroscopy. He was also the inventor of the diffraction grating. In 1822, he was granted an honorary doctorate from the University of Erlangen. Fraunhofer passed away at age 39, perhaps due to heavy-metal poisoning from glass blowing. (Wikipedia)

10.4 Fraunhofer Approximation

An additional approximation to the diffraction integral was made famous by Joseph von Fraunhofer. The *Fraunhofer approximation* is the limiting case of the Fresnel approximation when the field is observed at a distance far after the aperture (called the *far field*).[6] A diffraction pattern continuously evolves along the z-direction, as described by the Fresnel approximation. Eventually it evolves into a final diffraction pattern that maintains itself as it continues to propogate (although it increases its size in proportion to distance). It is this far-away diffraction pattern that is obtained from the Fraunhofer approximation. Since the Fresnel approximation requires the angles to be small (i.e. the paraxial approximation), so does the Fraunhofer approximation.

To obtain the diffraction pattern at a distance very far from the aperture, we make the following approximation:[7]

$$e^{i\frac{k}{2z}(x'^2+y'^2)}\cong 1 \qquad \text{(far field)} \quad (10.17)$$

The validity of this approximation depends on a comparison of the size of the aperture to the distance z where the diffraction pattern is observed. We need

$$z \gg \frac{k}{2}\left(\text{aperture radius}\right)^2 \qquad \text{(condition for far field)} \quad (10.18)$$

By removing the factor (10.17) from (10.13), we obtain the Fraunhofer diffraction formula:

(Fraunhofer approximation)

$$E\left(x,y,z\right)\cong -\frac{ie^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\iint_{\text{aperture}}E\left(x',y',0\right)e^{-i\frac{k}{z}(xx'+yy')}dx'dy' \quad (10.19)$$

[6] Since the Fraunhofer approximation is easier to use, many textbooks present it before the Fresnel approximation.

[7] J. W. Goodman, *Introduction to Fourier Optics*, p. 61 (New York: McGraw-Hill, 1968).

Obviously, the removal of $e^{i\frac{k}{2z}(x'^2+y'^2)}$ from the integrand improves our chances of being able to perform the integration analytically. In fact the integral can be interpreted as a two-dimensional (inverse) Fourier transform on the aperture field $E(x', y', 0)$, where kx/z and ky/z can be thought of as 'spatial frequencies'.

Once we are in the regime where the Fraunhofer approximation is valid, a change in z is not very interesting since it appears within the integral only in the combination x/z or y/z. At a larger distance z, the same diffraction pattern is obtained with a proportionately larger values of x or y. The Fraunhofer diffraction pattern thus preserves itself indefinitely as the field propagates. It grows in size as the distance z increases, but the *angular* size defined by x/z or y/z remains the same.

Example 10.4

Compute the Fraunhofer diffraction pattern following a rectangular aperture (dimensions Δx by Δy) illuminated by a uniform plane wave.

Solution: According to (10.19), the field downstream is

$$E(x,y,z) = -iE_0 \frac{e^{ikz}}{\lambda z} e^{i\frac{k}{2z}(x^2+y^2)} \int_{-\Delta x/2}^{\Delta x/2} dx' e^{-i\frac{kx}{z}x'} \int_{-\Delta y/2}^{\Delta y/2} dy' e^{-i\frac{ky}{z}y'}$$

It is left as an exercise (see P10.6) to perform the integration and compute the intensity. The result turns out to be

$$I(x,y,z) = I_0 \frac{\Delta x^2 \Delta y^2}{\lambda^2 z^2} \operatorname{sinc}^2\left(\frac{\pi \Delta x}{\lambda z} x\right) \operatorname{sinc}^2\left(\frac{\pi \Delta y}{\lambda z} y\right) \tag{10.20}$$

where $\operatorname{sinc}(\xi) \equiv \sin\xi/\xi$. Note that $\lim_{\xi\to 0} \operatorname{sinc}(\xi) = 1$.

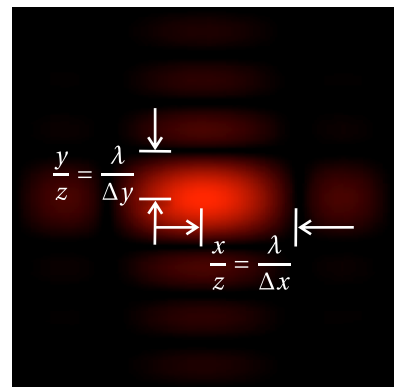

Figure 10.9 Fraunhofer diffraction pattern (field amplitude) generated by a uniformly illuminated rectangular aperture with a height twice the width.

10.5 Diffraction with Cylindrical Symmetry

Sometimes the field transmitted by an aperture is cylindrically symmetric. In this case, the field at the aperture can be written as

$$E(x', y', z=0) = E(\rho', z=0) \tag{10.21}$$

where $\rho \equiv \sqrt{x^2+y^2}$. Under cylindrical symmetry, the two-dimensional integration over x' and y' in (10.13) or (10.19) can be reduced to a single-dimensional integral over a cylindrical coordinate ρ'. With the coordinate transformation

$$x \equiv \rho\cos\phi \qquad y \equiv \rho\sin\phi \qquad x' \equiv \rho'\cos\phi' \qquad y' \equiv \rho'\sin\phi' \tag{10.22}$$

the Fresnel diffraction integral (10.13) becomes

$$E(\rho,z) = -\frac{ie^{ikz}e^{i\frac{k\rho^2}{2z}}}{\lambda z} \int_0^{2\pi} d\phi' \int_{\text{aperture}} \rho' d\rho' E(\rho',0)\, e^{i\frac{k\rho'^2}{2z}} e^{-i\frac{k}{z}(\rho\rho'\cos\phi\cos\phi' + \rho\rho'\sin\phi\sin\phi')} \tag{10.23}$$

Notice that in the exponent of (10.23) we can write

$$\rho'\rho\left(\cos\phi'\cos\phi+\sin\phi'\sin\phi\right)=\rho'\rho\cos\left(\phi'-\phi\right) \tag{10.24}$$

With this simplification, the diffraction formula (10.23) can be written as

$$E\left(\rho,z\right)=-\frac{ie^{ikz}e^{i\frac{k\rho^2}{2z}}}{\lambda z}\int\limits_{\text{aperture}}\rho' d\rho' E\left(\rho',0\right)e^{i\frac{k\rho'^2}{2z}}\int\limits_0^{2\pi}d\phi' e^{-i\frac{k\rho\rho'}{z}\cos(\phi-\phi')} \tag{10.25}$$

We are able to perform the integration over ϕ' with the help of the formula (0.57):

$$\int\limits_0^{2\pi}e^{-i\frac{k\rho\rho'}{z}\cos(\phi-\phi')}d\phi'=2\pi J_0\left(\frac{k\rho\rho'}{z}\right) \tag{10.26}$$

J_0 is called the zero-order *Bessel function*. Equation (10.25) then reduces to

$$E\left(\rho,z\right)=-\frac{2\pi ie^{ikz}e^{i\frac{k\rho^2}{2z}}}{\lambda z}\int\limits_{\text{aperture}}\rho' d\rho' E\left(\rho',0\right)e^{i\frac{k\rho'^2}{2z}}J_0\left(\frac{k\rho\rho'}{z}\right)$$

(Fresnel approximation with cylindrical symmetry) (10.27)

The integral in (10.27) is called a *Hankel transform* on $E\left(\rho',0\right)e^{i\frac{k\rho'^2}{2z}}$.

In the case of the Fraunhofer approximation, the diffraction integral becomes a Hankel transform on just the field $E\left(\rho',z=0\right)$ since $\exp\left(i\frac{k\rho'^2}{2z}\right)$ goes to one. Under cylindrical symmetry, the Fraunhofer approximation is

$$E\left(\rho,z\right)=-\frac{2\pi ie^{ikz}e^{i\frac{k\rho^2}{2z}}}{\lambda z}\int\limits_{\text{aperture}}\rho' d\rho' E\left(\rho',0\right)J_0\left(\frac{k\rho\rho'}{z}\right)$$

(Fraunhofer approximation with cylindrical symmetry) (10.28)

Just as fast Fourier transform algorithms aid in the numerical evaluation of diffraction integrals in Cartesian coordinates, fast Hankel transforms also exist and can be used with cylindrically symmetric diffraction integrals.

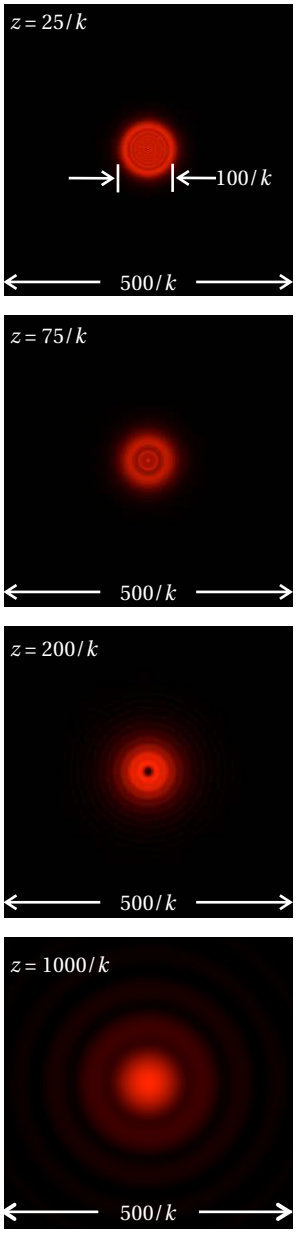

Figure 10.10 Field amplitude following a circular aperture computed in the Fresnel approximation.

Example 10.5

Compute the Fresnel and Fraunhofer diffraction patterns following a circular aperture (diameter D) illuminated by a uniform plane wave.

Solution: According to (10.27), the field downstream is

$$E\left(\rho,z\right)=-iE_0\frac{2\pi e^{ikz}e^{i\frac{k\rho^2}{2z}}}{\lambda z}\int\limits_0^{D/2}\rho' d\rho' e^{i\frac{k\rho'^2}{2z}}J_0\left(\frac{k\rho\rho'}{z}\right)$$

Unfortunately, this Fresnel integral must be performed numerically. The result of the calculation for a uniform field illuminating a circular aperture is shown in Fig. 10.10.

On the other hand, the field in the Fraunhofer limit (10.28) is

$$E(\rho,z) = -iE_0 \frac{2\pi e^{ikz} e^{i\frac{k\rho^2}{2z}}}{\lambda z} \int_0^{D/2} \rho' d\rho' J_0\left(\frac{k\rho\rho'}{z}\right)$$

which *can* be integrated analytically (with the aid of (0.58)). It is left as an exercise to perform the integration and to show that the intensity of the Fraunhofer pattern is

$$I(\rho,z) = I_0 \left(\frac{\pi D^2}{4\lambda z}\right)^2 \left[2\frac{J_1(kD\rho/2z)}{(kD\rho/2z)}\right]^2 \tag{10.29}$$

The function $\frac{2J_1(\xi)}{\xi}$, which we will call the *jinc* function,[8] looks similar to the sinc function (see Example 10.4) except that its first zero is at $\xi = 1.22\pi$ rather than at π. Note that $\lim_{\xi\to 0} \frac{2J_1(\xi)}{\xi} = 1$.

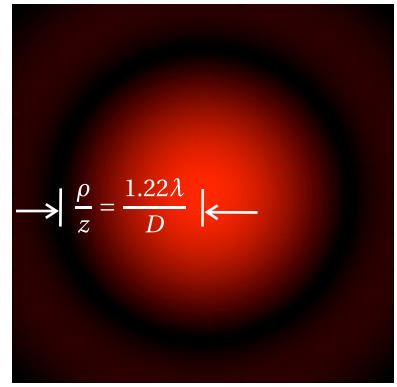

Figure 10.11 Fraunhofer diffraction pattern (field amplitude) generated for a uniformly illuminated circular aperture.

Appendix 10.A Fresnel-Kirchhoff Diffraction Formula

To begin the derivation of the Fresnel-Kirchhoff diffraction formula,[9] we employ Green's theorem (proven in appendix 10.B):

$$\oint_S \left[U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right] da = \int_V \left[U\nabla^2 V - V\nabla^2 U\right] dv \tag{10.30}$$

The notation $\partial/\partial n$ implies a derivative in the direction normal to the surface. We choose the following functions:

$$\begin{aligned} V &\equiv e^{ikr}/r \\ U &\equiv E(\mathbf{r}) \end{aligned} \tag{10.31}$$

where $E(\mathbf{r})$ is assumed to satisfy the scalar Helmholtz equation, (10.5). When these functions are used in Green's theorem (10.30), we obtain

$$\oint_S \left[E\frac{\partial}{\partial n}\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\frac{\partial E}{\partial n}\right] da = \int_V \left[E\nabla^2\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\nabla^2 E\right] dv \tag{10.32}$$

The right-hand side of this equation vanishes[10] since we have

$$E\nabla^2\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\nabla^2 E = -k^2 E\frac{e^{ikr}}{r} + \frac{e^{ikr}}{r}k^2 E = 0 \tag{10.33}$$

[8] Most authors define the jinc without the factor of 2, which gives the inconvenient normalization $\lim_{\xi\to 0} \text{jinc}\xi = 1/2$.

[9] See J. W. Goodman, *Introduction to Fourier Optics*, Sect. 3-3 (New York: McGraw-Hill, 1968).

[10] We exclude the point $r = 0$; see P0.4 and P0.5.

where we have taken advantage of the fact that $E(\mathbf{r})$ and e^{ikr}/r both satisfy (10.5). This is exactly the reason for our judicious choices of the functions V and U since with them we were able to make half of (10.30) disappear. We are left with

$$\oint_S \left[E\frac{\partial}{\partial n}\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\frac{\partial E}{\partial n}\right] da = 0 \tag{10.34}$$

Now consider a volume between a small sphere of radius ϵ at the origin and an outer surface of arbitrary shape. The total surface that encloses the volume is comprised of two parts (i.e. $S = S_1 + S_2$ as depicted in Fig. 10.12).

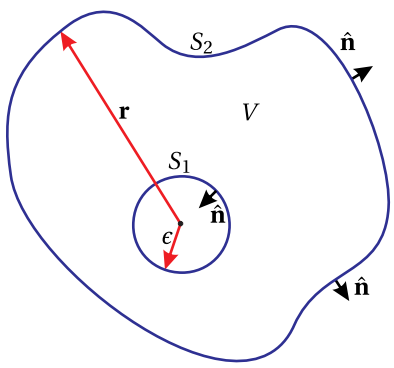

Figure 10.12 A two-part surface enclosing volume V.

When we apply (10.34) to the surface in Fig. 10.12, we have

$$\oint_{S_2} \left[E\frac{\partial}{\partial n}\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\frac{\partial E}{\partial n}\right] da = -\oint_{S_1} \left[E\frac{\partial}{\partial n}\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\frac{\partial E}{\partial n}\right] da \tag{10.35}$$

This geometry with multiple surfaces is motivated by the hope of finding the field at the origin (inside the little sphere) from knowledge of the field on the outside surface. To this end, we assume that ϵ is so small that $E(\mathbf{r})$ is approximately the same everywhere on the surface S_1. Then the integral over S_1 becomes

$$\oint_{S_1} \left[E\frac{\partial}{\partial n}\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\frac{\partial E}{\partial n}\right] da = \lim_{r=\epsilon\to 0}\int_0^{2\pi} d\phi \int_0^{\pi} \left[E\left(\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right)\frac{\partial r}{\partial n} - \frac{e^{ikr}}{r}\left(\frac{\partial E}{\partial r}\right)\frac{\partial r}{\partial n}\right] r^2 \sin\theta d\theta \tag{10.36}$$

where we have used spherical coordinates. Notice that we have employed the chain rule to execute the normal derivative $\partial/\partial n$. Since r always points opposite to the direction of the surface normal $\hat{\mathbf{n}}$, the normal derivative $\partial r/\partial n$ is always equal to -1.[11] We can perform the angular integration in (10.36) as well as take the limit $\epsilon \to 0$:

$$\begin{aligned}\lim_{\epsilon\to 0}\oint_{S_1} \left[E\frac{\partial}{\partial n}\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\frac{\partial E}{\partial n}\right] da &= -4\pi \lim_{\epsilon\to 0}\left[r^2\left(-\frac{e^{ikr}}{r^2} + ik\frac{e^{ikr}}{r}\right)E - r^2\frac{e^{ikr}}{r}\left(\frac{\partial E}{\partial r}\right)\right]_{r=\epsilon} \\ &= -4\pi \lim_{\epsilon\to 0}\left[\left(-e^{ik\epsilon} + ik\epsilon e^{ik\epsilon}\right)E - e^{ik\epsilon}\epsilon\left(\frac{\partial E}{\partial r}\right)_{r=\epsilon}\right] \\ &= 4\pi E(0)\end{aligned} \tag{10.37}$$

With the aid of (10.37), Green's theorem applied to our specific geometry reduces to

$$E(0) = \frac{1}{4\pi}\oint_{S_2}\left[\frac{e^{ikr}}{r}\frac{\partial E}{\partial n} - E\frac{\partial}{\partial n}\frac{e^{ikr}}{r}\right] da \tag{10.38}$$

[11] From the definition of the normal derivative we have $\partial r/\partial n \equiv \nabla r \cdot \hat{\mathbf{n}} = -\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = -1$.

If we know E everywhere on the outer surface S_2, this equation allows us to predict the field $E(0)$ at the origin.

Now let us choose a specific surface S_2. Consider an infinite mask with a finite aperture connected to a hemisphere of infinite radius $R \to \infty$. In the end, we will suppose that light enters through the mask and propagates to our origin (among other points). In our present coordinate system, the vectors $\mathbf{r}$ and $\hat{\mathbf{n}}$ point opposite to the incoming light.

We must evaluate (10.38) on the surface depicted in the figure. For the portion of S_2 that is on the hemisphere, the integrand tends to zero as R becomes large. To argue this, it is necessary to recognize the fact that at large distances the field takes on a form proportional to e^{ikr}/r so that the two terms in the integrand cancel. On the mask, we assume, as did Kirchhoff, that both $\partial E/\partial n$ and E are zero.[12] Thus, we are left with only the integration over the open aperture:

$$E(0) = \frac{1}{4\pi} \iint_{\text{aperture}} \left[\frac{e^{ikr}}{r} \frac{\partial E}{\partial n} - E \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \right] da \tag{10.39}$$

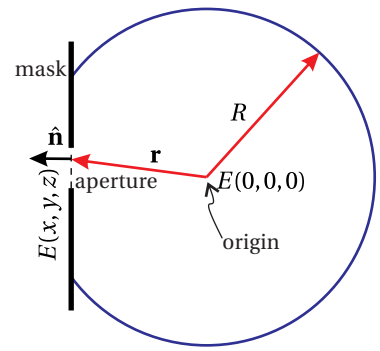

Figure 10.13 Surface S_2 depicted as a mask and a large hemisphere.

We have essentially arrived at the result that we are seeking. The field coming through the aperture is integrated to find the field at the origin, which is located beyond the aperture. Let us manipulate the formula a little further. The second term in the integral of (10.39) can be rewritten as follows:

$$\frac{\partial}{\partial n} \frac{e^{ikr}}{r} = \left(\frac{\partial}{\partial r} \frac{e^{ikr}}{r} \right) \frac{\partial r}{\partial n} = \left(\frac{ik}{r} - \frac{1}{r^2} \right) e^{ikr} \cos(\mathbf{r}, \hat{\mathbf{n}}) \underset{r \gg \lambda}{\to} \frac{ike^{ikr}}{r} \cos(\mathbf{r}, \hat{\mathbf{n}}) \tag{10.40}$$

where $\partial r/\partial n = \cos(\mathbf{r}, \hat{\mathbf{n}})$ indicates the cosine of the angle between $\mathbf{r}$ and $\hat{\mathbf{n}}$. We have also assumed that the distance r is much larger than a wavelength in order to drop a term. Next, we assume that the field illuminating the aperture can be written as $E \cong \tilde{E}(x, y) e^{ikz}$. This represents a plane-wave field traveling through the aperture from left to right. Then, we have

$$\frac{\partial E}{\partial n} = \frac{\partial E}{\partial z} \frac{\partial z}{\partial n} = ik\tilde{E}(x, y) e^{ikz} (-1) = -ikE \tag{10.41}$$

Substituting (10.40) and (10.41) into (10.39) yields

$$E(0) = -\frac{i}{\lambda} \iint_{\text{aperture}} E \frac{e^{ikr}}{r} \left[\frac{1 + \cos(\mathbf{r}, \hat{\mathbf{n}})}{2} \right] da \tag{10.42}$$

Finally, we wish to rearrange our coordinate system to that depicted in Fig. 10.2. In our derivation, it was less cumbersome to place the origin at a point of interest

[12] Later Sommerfeld noticed that these two assumptions actually contradict each other, and he revised Kirchhoff's work to be more accurate. In practice this revision makes only a tiny difference as light spills onto the back of the aperture, over a length scale of a wavelength. We will ignore this effect and go with Kirchhoff's (slightly flawed) assumption. For further discussion see J. W. Goodman, *Introduction to Fourier Optics*, Sect. 3-4 (New York: McGraw-Hill, 1968).

after the aperture. Now that we have completed our mathematics, we can switch around the coordinate system and place the origin in the plane of the aperture as in Fig. 10.2:

$$E(x,y,z) = -\frac{i}{\lambda} \iint_{\text{aperture}} E(x',y',0)\,\frac{e^{ikR}}{R}\left[\frac{1+\cos(\mathbf{r},\hat{\mathbf{z}})}{2}\right] dx'\,dy' \tag{10.43}$$

where

$$R = \sqrt{(x-x')^2 + (y-y')^2 + z^2} \tag{10.44}$$

which brings us to the Fresnel-Kirchhoff diffraction formula (10.10).

Appendix 10.B Green's Theorem

To derive Green's theorem, we begin with the divergence theorem (see (0.11)):

$$\oint_S \mathbf{f}\cdot\hat{\mathbf{n}}\,da = \int_V \nabla\cdot\mathbf{f}\,dv \tag{10.45}$$

The unit vector $\hat{\mathbf{n}}$ always points normal to the surface of volume V over which the integral is taken. Let the vector function $\mathbf{f}$ be $U\nabla V$, where U and V are both analytical functions of the position coordinate $\mathbf{r}$. Then (10.45) becomes

$$\oint_S (U\nabla V)\cdot\hat{\mathbf{n}}\,da = \int_V \nabla\cdot(U\nabla V)\;dv \tag{10.46}$$

We recognize $\nabla V\cdot\hat{\mathbf{n}}$ as the directional derivative of V, directed along the surface normal $\hat{\mathbf{n}}$. This is often represented in shorthand notation as

$$\nabla V\cdot\hat{\mathbf{n}} \equiv \frac{\partial V}{\partial n} \tag{10.47}$$

The integrand on the right-hand side of (10.46) can be expanded with the product rule:

$$\nabla\cdot(U\nabla V) = \nabla U\cdot\nabla V + U\nabla^2 V \tag{10.48}$$

With these substitutions, (10.46) becomes

$$\oint_S U\frac{\partial V}{\partial n}\,da = \int_V \left[\nabla U\cdot\nabla V + U\nabla^2 V\right] dv \tag{10.49}$$

So far we haven't done much. Equation (10.49) is nothing more than the divergence theorem applied to the vector function $U\nabla V$. We can also write an equation similar to (10.49) where U and V are interchanged:

$$\oint_S V\frac{\partial U}{\partial n}\,da = \int_V \left[\nabla V\cdot\nabla U + V\nabla^2 U\right] dv \tag{10.50}$$

We subtract (10.50) from (10.49), which leads to (10.30) known as Green's theorem.

Exercises

Exercises for 10.1 Huygens' Principle as Formulated by Fresnel

P10.1 Huygens' principle can be used to describe refraction. Use a drawing program or a ruler and compass to produce a picture similar to Fig. 10.14, which shows that the graphical prediction of refracted angle from the Huygens' principle. Verify that the Huygens picture matches the numerical prediction from Snell's Law for an incident angle of your choice. Use $n_i = 1$ and $n_t = 2$.

HINT: Draw the wavefronts hitting the interface at an angle and treat each point where the wavefronts strike the interface as the source of circular waves propagating into the $n = 2$ material. The wavelength of the circular waves must be exactly half the wavelength of the incident light since $\lambda = \lambda_{vac}/n$. Use at least four point sources and connect the matching wavefronts by drawing tangent lines as in the figure.

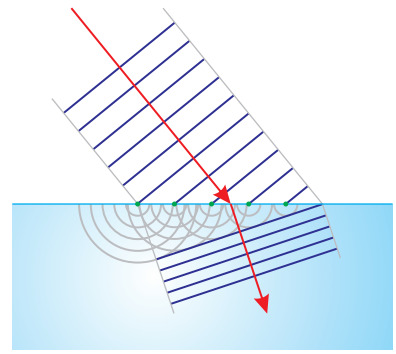

Figure 10.14

L10.2 (a) Why does the on-axis intensity behind a circular opening fluctuate (see Example 10.1) whereas the on-axis intensity behind a circular obstruction remains constant (see Example 10.2)?

(b) Create a collimated laser beam several centimeters wide. Observe the on-axis intensity on a movable screen (e.g. a hand-held card) behind a small circular aperture and behind a small circular obstruction placed in the beam. (video)

(c) In the case of the circular aperture, measure the distance to several on-axis minima and check that it agrees with (10.3).

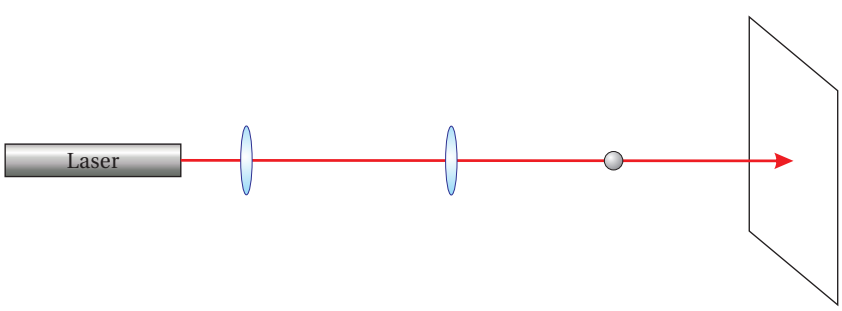

Figure 10.15

Exercises for 10.2 Scalar Diffraction Theory

P10.3 Show that $E(r) = E_0 r_0 e^{ikr}/r$ is a solution to the scalar Helmholtz equation (10.5).

HINT: In spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\psi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

P10.4 (a) A vector field is needed to satisfy Maxwell's equations instead of the scalar field in P10.3, whose real part after appending $e^{-i\omega t}$ is

$$E(r) = \frac{A}{r}\cos\left(kr - \omega t\right)$$

Let's attempt to create a vector field from this scalar field in the simplest way possible. From experience, we expect a transverse wave, which we take to oscillate in the $\hat{\phi}$ direction:

$$\mathbf{E}(r) = \frac{A}{r}\cos\left(kr - \omega t\right)\hat{\phi}$$

(i) Show that **E** satisfies Gauss's Law (1.1). (ii) Compute the curl of **E** in Faraday's Law (1.3) to deduce **B**. (iii) Show that this **B** satisfies Gauss' Law for magnetism (1.2). (iv) Finally, show that the above **E** and **B** do *not* satisfy Ampere's law (1.4).

HINT: In spherical coordinates

$$\nabla\cdot\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{E} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) - \frac{\partial E_\theta}{\partial\phi}\right] + \hat{\theta}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rE_\phi\right)\right]$$
$$+\hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{\partial E_r}{\partial\theta}\right]$$

(b) The following somewhat more complicated 'spherical' wave

$$\mathbf{E}(r,\theta) = \frac{A\sin\theta}{r}\left[\cos\left(kr - \omega t\right) - \frac{1}{kr}\sin\left(kr - \omega t\right)\right]\hat{\phi}$$

(i.e. the real part of (10.9) with time dependence appended) does satisfy Maxwell's equations. Describe how this wave behaves as a function of r and θ. What conditions need to be satisfied for this equation to be well approximated by the spherical wave in part (a)?

Exercises for 10.3 Fresnel Approximation

P10.5 By direct substitution, show that (10.16) satisfies the paraxial wave equation (10.15).

Exercises for 10.4 Fraunhofer Approximation

P10.6 Calculate the Fraunhofer diffraction *field* and *intensity* patterns for a rectangular aperture (dimensions Δx by Δy) illuminated by a plane wave E_0. In other words, derive (10.20).

P10.7 A single narrow slit has a mask placed over it so the aperture function is not a square profile but rather a cosine: $E(x', y', 0) = E_0 \cos(\pi x'/L)$ for $-L/2 < x' < L/2$ and $E(x', y', 0) = 0$ otherwise. Calculate the far-field (Fraunhofer) diffraction pattern. Make a plot of intensity as a function of $kLx/2z$; qualitatively compare the pattern to that of a regular single slit. Do not perform any integration in the y dimension. Write the intensity as being proportional to an x-dependent expression.

Exercises for 10.5 Diffraction with Cylindrical Symmetry

P10.8 (a) Repeat Example 10.1 to find the on-axis intensity (i.e. $\rho = 0$) after a circular aperture in both the Fresnel approximation (10.27) and the Fraunhofer approximation (10.28).

(b) Make suitable approximations directly to (10.3) to obtain the same answers as in part (a).

(c) Check how well the Fresnel and Fraunhofer approximations work by graphing the Fresnel- and Fraunhofer-approximation results together with (10.3) on a single plot as a function of z. Take $D = 10\ \mu$m and $\lambda = 500$ nm. To see the result better, use a log scale on the z-axis.

Answer:

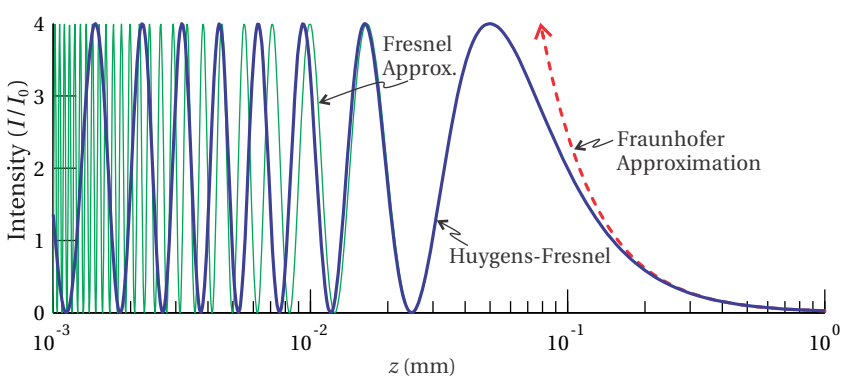

Figure 10.17 On-axis intensity behind a circular aperture calculated using the Fresnel diffraction formula (10.1), the Fresnel approximation (10.27), and the Fraunhofer approximation (10.28).

P10.9 Calculate the Fraunhofer diffraction intensity pattern (10.29) for a circular aperture (diameter D) illuminated by a plane wave E_0. That is, repeat example 10.5 while filling in the integration step. For added benefit, try to do it without peeking.

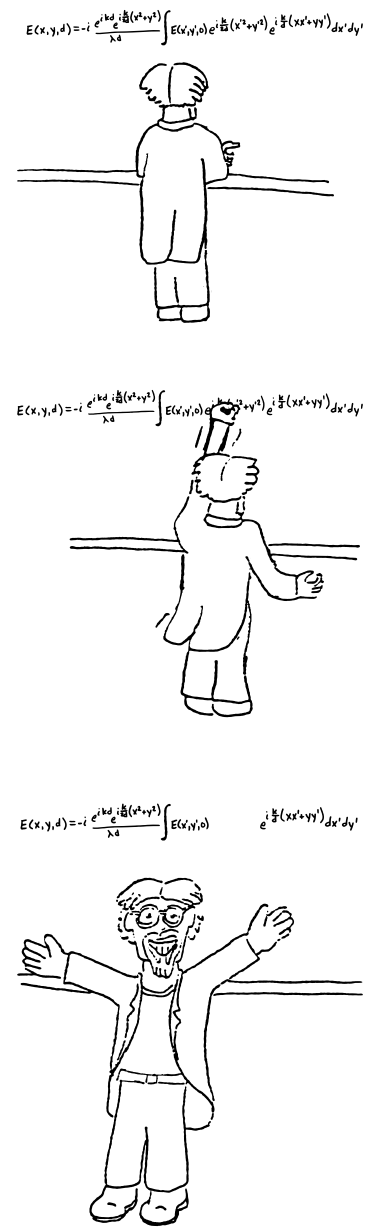

Figure 10.16 "The Fraunhofer Approximation" by Sterling Cornaby

Exercises for 10.A Fresnel-Kirchhoff Diffraction Formula

P10.10 Learn by heart the derivation of the Fresnel-Kirchhoff diffraction formula (outlined in Appendix 10.A). Indicate the percentage of how well you understand the derivation. If you write 100% percent, it means that you can reproduce the derivation without peeking.