10 The thermal diffusion equation

This section assumes familiarity with solving differential equations (see e.g. Boas (1983), Riley *et al.* (2006)). It can be omitted at first reading.

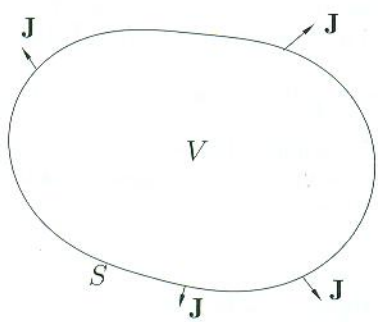

Fig. 10.1 A closed surface S encloses a volume V. The total heat flow out of S is given by $\int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}$.

In the previous chapter, we have seen how the thermal conductivity of a gas can be calculated using kinetic theory. In this chapter, we look at solving problems involving the thermal conductivity of matter using a technique which was developed by mathematicians in the late eighteenth and early nineteenth centuries. The key equation describes thermal diffusion, i.e. how heat appears to 'diffuse' from one place to the other, and most of this chapter introduces techniques for solving this equation.

10.1 Derivation of the thermal diffusion equation

Recall from eqn 9.15 that the heat flux $\boldsymbol{J}$ is given by

$$\boldsymbol{J} = -\kappa \nabla T. \tag{10.1}$$

This equation is very similar mathematically to the equation for particle flux $\boldsymbol{\Phi}$ in eqn 9.26 which is, in three dimensions,

$$\boldsymbol{\Phi} = -D \nabla n, \tag{10.2}$$

where D is the diffusion constant, and also to the flow of electrical current given by the current density $\boldsymbol{J}_\mathrm{e}$ defined by

$$\boldsymbol{J}_\mathrm{e} = \sigma \boldsymbol{E} = -\sigma \nabla \phi, \tag{10.3}$$

where σ is the conductivity, $\boldsymbol{E}$ is the electric field and ϕ here is the electric potential. Because of this mathematical similarity, an equation which is analogous to the diffusion equation (eqn 9.36) holds in each case. We will derive the thermal diffusion equation in this section.

In fact in all these phenomena, there needs to be some account of the fact that you can't destroy energy, or particles, or charge. (We will only treat the thermal case here.) The total heat flow out of a closed surface S is given by the integral

$$\int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}, \tag{10.4}$$

and is a quantity which has the dimension of power. It is therefore equal to the rate which the material inside the surface is losing energy.

This can be expressed as the rate of change of the total thermal energy inside the volume V which is surrounded by the closed surface S. The thermal energy can be written as the volume integral $\int_V CT\,\mathrm{d}V$, where C here is the heat capacity per unit volume (measured in $\mathrm{J\,K^{-1}\,m^{-3}}$) and is equal to ρc, where ρ is the density and c is the heat capacity per unit mass (the specific heat capacity, see Section 2.2). Hence

We haven't worried about what the 'zero' of thermal energy is; there could also be an additive, time-independent, constant in the expression for total thermal energy, but since we are going to differentiate this with respect to time to obtain the rate of change of thermal energy, it doesn't matter.

$$\int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\partial}{\partial t}\int_V CT\,\mathrm{d}V. \tag{10.5}$$

The divergence theorem implies that

$$\int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{J}\,\mathrm{d}V, \tag{10.6}$$

and hence that

$$\nabla \cdot \boldsymbol{J} = -C\frac{\partial T}{\partial t}. \tag{10.7}$$

Substituting in eqn 10.1 then yields the **thermal diffusion equation** which is

$$\boxed{\frac{\partial T}{\partial t} = D\nabla^2 T,} \tag{10.8}$$

where $D = \kappa/C$ is the **thermal diffusivity**. Since κ has units $\mathrm{W\,m^{-1}\,K^{-1}}$ and $C = \rho c$ has units $\mathrm{J\,K^{-1}\,m^{-3}}$, D has units $\mathrm{m^2\,s^{-1}}$.

10.2 The one-dimensional thermal diffusion equation

In one dimension, this equation becomes

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \tag{10.9}$$

and can be solved using conventional methods.

Example 10.1

Solution of the one-dimensional thermal diffusion equation
The one-dimensional thermal diffusion equation looks a bit like a wave equation. Therefore, one method to solve eqn 10.9 is to look for wave-like solutions of the form

$$T(x,t) \propto \exp(\mathrm{i}(kx - \omega t)), \tag{10.10}$$

where $k = 2\pi/\lambda$ is the wave vector, $\omega = 2\pi f$ is the angular frequency, λ is the wavelength and f is the frequency. Substitution of this equation into eqn 10.9 yields

$$-\mathrm{i}\omega = -Dk^2 \tag{10.11}$$

and hence

$$k^2 = \frac{\mathrm{i}\omega}{D} \tag{10.12}$$

so that

$$k = \pm(1+\mathrm{i})\sqrt{\frac{\omega}{2D}}. \tag{10.13}$$

The spatial part of the wave, which looks like $\exp(\mathrm{i}kx)$, can either be of the form

$$\exp\left((\mathrm{i}-1)\sqrt{\frac{\omega}{2D}}x\right), \qquad \text{which blows up as } x \to -\infty\ , \tag{10.14}$$

or

$$\exp\left((-\mathrm{i}+1)\sqrt{\frac{\omega}{2D}}x\right), \qquad \text{which blows up as } x \to \infty\ . \tag{10.15}$$

Let us now solve a problem in which a boundary condition is applied at $x = 0$ and a solution is desired in the region $x > 0$. We don't want solutions which blow up as $x \to \infty$ and pick the first type of solution (i.e. eqn 10.14). Hence our general solution for $x \geq 0$ can be written as

$$T(x,t) = \sum_\omega A(\omega)\exp(-\mathrm{i}\omega t)\exp\left((\mathrm{i}-1)\sqrt{\frac{\omega}{2D}}x\right), \tag{10.16}$$

where we have summed over all possible frequencies. To find which frequencies are needed, we have to be specific about the boundary condition for which we want to solve.

Let us imagine that we want to solve the one-dimensional problem of the propagation of sinusoidal temperature waves into the ground. The waves could be due to the alternation of day and night (for a wave with period 1 day), or winter and summer (for a wave with period 1 year). The boundary condition can be written as

$$T(0,t) = T_0 + \Delta T\cos\Omega t. \tag{10.17}$$

This boundary condition can be rewritten

$$T(0,t) = T_0 + \frac{\Delta T}{2}\mathrm{e}^{\mathrm{i}\Omega t} + \frac{\Delta T}{2}\mathrm{e}^{-\mathrm{i}\Omega t}. \tag{10.18}$$

However, at $x = 0$ the general solution (eqn 10.16) becomes

$$T(0,t) = \sum_\omega A(\omega)\exp(-\mathrm{i}\omega t). \tag{10.19}$$

Comparison of eqns 10.18 and 10.19 implies that the only non-zero values of $A(\omega)$ are

$$A(0) = T_0, \qquad A(-\Omega) = \frac{\Delta T}{2} \qquad \text{and} \qquad A(\Omega) = \frac{\Delta T}{2}. \tag{10.20}$$

Hence the solution to our problem for $x \geq 0$ is

$$T(x,t) = T_0 + \Delta T\,\mathrm{e}^{-x/\delta}\cos\left(\Omega t - \frac{x}{\delta}\right), \tag{10.21}$$

where

$$\delta = \sqrt{\frac{2D}{\Omega}} = \sqrt{\frac{2\kappa}{\Omega C}} \tag{10.22}$$

is known as the **skin depth**. The solution in eqn 10.21 is plotted in Fig. 10.2. [Note that the use of the term skin depth brings out the analogy between this effect and the skin depth which arises when electromagnetic waves are incident on a metal surface, see e.g. Griffiths (2003).]

We note the following important features of this solution:

- T falls off exponentially as $e^{-x/\delta}$.
- There is a phase shift of x/δ radians in the oscillations.
- $\delta \propto \Omega^{-1/2}$ so that faster oscillations fall off faster.

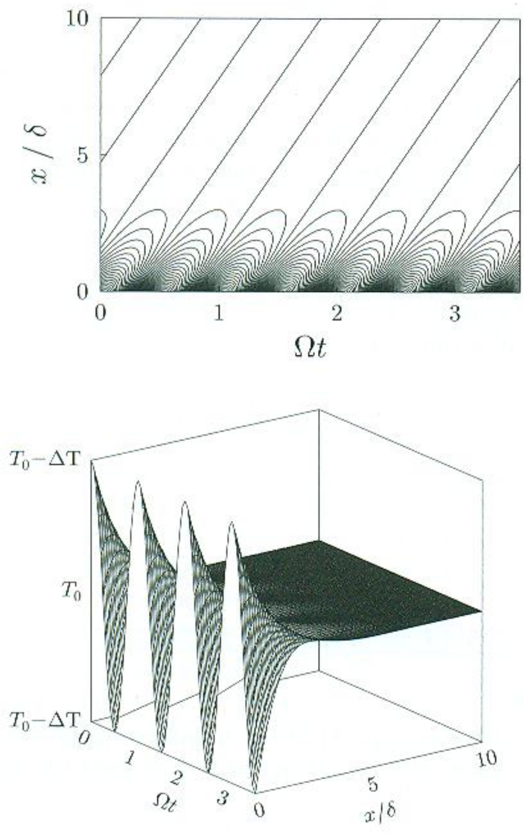

Fig. 10.2 A contour plot and a surface plot of eqn 10.21, showing that the temperature falls off exponentially as $e^{-x/\delta}$. The contour plot shows that there is a phase shift in the oscillations as x increases.

10.3 The steady state

If the system has reached a **steady state**, its properties are not time-dependent. This includes the temperature, so that

$$\frac{\partial T}{\partial t} = 0. \tag{10.23}$$

Hence in this case, the thermal diffusion equation reduces to

$$\nabla^2 T = 0, \tag{10.24}$$

which is Laplace's equation.

10.4 The thermal diffusion equation for a sphere

Very often, heat transfer problems have spherical symmetry (e.g. the cooling of the Earth or the Sun). In this section we will show that one can also solve the (rather forbidding looking) problem of the thermal diffusion equation in a system with spherical symmetry. In spherical polars, we have in general that $\nabla^2 T$ is given by[1]

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}, \tag{10.25}$$

so that if T is not a function of θ or ϕ we can write

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right), \tag{10.26}$$

and hence the diffusion equation becomes

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{C}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right).} \tag{10.27}$$

[1] See Appendix B.

Example 10.2

The thermal diffusion equation for a sphere in the steady state.
In the steady state, $\partial T/\partial t = 0$ and hence we need to solve

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = 0. \tag{10.28}$$

Now if T is independent of r, $\partial T/\partial r = 0$ and this will be a solution. Moreover, if $r^2(\partial T/\partial r)$ is independent of r, this will generate another solution. Now $r^2(\partial T/\partial r) =$ constant implies that $T \propto r^{-1}$. Hence a general solution is

$$T = A + \frac{B}{r}, \tag{10.29}$$

where A and B are constants. This should not surprise us if we know some electromagnetism, as we are solving Laplace's equation in spherical coordinates assuming spherical symmetry, and in electromagnetism the solution for the electric potential in this case is an arbitrary constant plus a Coulomb potential which is proportional to $1/r$.

A practical problem one often needs to solve is cooking a slab of meat. The meat is initially at some cool temperature (the temperature of the kitchen or of the refrigerator) and it is placed into a hot oven. The skill in cooking is getting the inside up to temperature. How long does it take? The next example shows how to calculate this for the (rather artificial) example of a spherical chicken!

Example 10.3

The spherical chicken

A spherical chicken[2] of radius a at initial temperature T_0 is placed into an oven at temperature T_1 at time $t = 0$ (see Fig. 10.3). The boundary conditions are that the oven is at temperature T_1 so that

$$T(a,t) = T_1, \tag{10.30}$$

and the chicken is originally at temperature T_0, so that

$$T(r,0) = T_0. \tag{10.31}$$

We want to obtain the temperature as a function of time at the centre of the chicken, i.e. $T(0,t)$.

Solution: We will show how we can transform this to a one-dimensional diffusion equation. This is accomplished using a substitution

$$T(r,t) = T_1 + \frac{B(r,t)}{r}, \tag{10.32}$$

where $B(r,t)$ is now a function of r and t. This substitution is motivated by the solution to the steady–state problem in eqn 10.29 and of course means that that we can write B as $B = r(T - T_1)$.

We now need to work out some partial differentials:

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial B}{\partial t}, \tag{10.33}$$

$$\frac{\partial T}{\partial r} = -\frac{B}{r^2} + \frac{1}{r}\frac{\partial B}{\partial r}, \tag{10.34}$$

and hence multiplying eqn 10.34 by r^2 we have that

$$r^2\frac{\partial T}{\partial r} = -B + r\frac{\partial B}{\partial r}, \tag{10.35}$$

[2] The methods in this example can also be applied to a spherical nut roast.

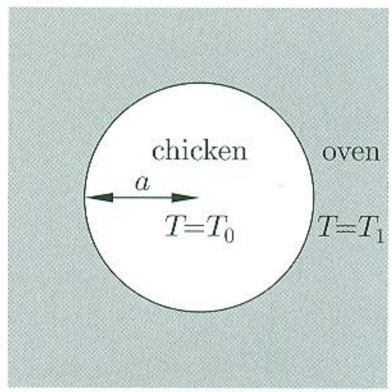

Fig. 10.3 Initial condition of a spherical chicken of radius a at initial temperature T_0 which is placed into an oven at temperature T_1 at time $t = 0$.

and therefore

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right] = r\frac{\partial^2 B}{\partial r^2}, \tag{10.36}$$

which means that eqn 10.27 becomes

$$\frac{\partial B}{\partial t} = D\frac{\partial^2 B}{\partial r^2}, \tag{10.37}$$

where $D = \kappa/C$. This is a one-dimensional diffusion equation and is therefore much easier to solve than the one with which we started.

The new boundary conditions can be rewritten as follows:

(1) because $B = r(T - T_1)$ we have that $B = 0$ when $r = 0$:

$$B(0, t) = 0; \tag{10.38}$$

(2) because $T = T_1$ at $r = a$ we have that:

$$B(a, t) = 0; \tag{10.39}$$

(3) because $T = T_0$ at $t = 0$ we have that:

$$B(r, 0) = r(T_0 - T_1). \tag{10.40}$$

We look for wave-like solutions with these boundary conditions and hence are led to try

$$B = \sin(kr)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{10.41}$$

and substituting this into eqn 10.37 yields

$$\mathrm{i}\omega = Dk^2. \tag{10.42}$$

The relation $ka = n\pi$ where n is an integer fits the first two boundary conditions and hence

$$\mathrm{i}\omega = D\left(\frac{n\pi}{a}\right)^2, \tag{10.43}$$

and hence our general solution is

$$B(r, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi r}{a}\right)\mathrm{e}^{-D\left(\frac{n\pi}{a}\right)^2 t}. \tag{10.44}$$

To find A_n, we need to match this solution at $t = 0$ using our third boundary condition. Hence

$$r(T_0 - T_1) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi r}{a}\right). \tag{10.45}$$

We multiply both sides by $\sin\left(\frac{m\pi r}{a}\right)$ and integrate, so that

Notice that the functions $\sin(n\pi r/a)$ and $\sin(m\pi r/a)$ are orthogonal unless $m = n$.

$$\int_0^a \sin\left(\frac{m\pi r}{a}\right) r(T_0 - T_1)\,\mathrm{d}r = \sum_{n=1}^{\infty} A_n \int_0^a \sin\left(\frac{m\pi r}{a}\right)\sin\left(\frac{n\pi r}{a}\right)\,\mathrm{d}r. \tag{10.46}$$

The right-hand side yields $A_m a/2$ and the left-hand side can be integrated by parts. This yields

$$A_m = \frac{2a}{m\pi}(T_1 - T_0)(-1)^m, \tag{10.47}$$

and hence that

$$B(r,t) = \frac{2a}{\pi}(T_1 - T_0)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi r/a)\mathrm{e}^{-D(n\pi/a)^2 t}, \tag{10.48}$$

so that using eqn 10.32 the temperature $T(r,t)$ inside the chicken ($r \le a$) behaves as

$$T(r,t) = T_1 + \frac{2a}{\pi}(T_1 - T_0)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\sin(n\pi r/a)}{r}\mathrm{e}^{-D(n\pi/a)^2 t}. \tag{10.49}$$

The centre of the chicken has temperature

$$T(0,t) = T_1 + 2(T_1 - T_0)\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-D(n\pi/a)^2 t}, \tag{10.50}$$

using the fact that as $r \to 0$,

$$\frac{1}{r}\sin\left(\frac{n\pi r}{a}\right) \to \frac{n\pi}{a}. \tag{10.51}$$

The expression in eqn 10.50 (see Fig. 10.4) becomes dominated by the first exponential in the sum as time t increases, so that

$$T(0,t) \approx T_1 - 2(T_1 - T_0)\mathrm{e}^{-D(\pi/a)^2 t}, \tag{10.52}$$

for $t \gg a^2/D\pi^2$. Analogous behaviour is of course found for a warm sphere which is cooling in a colder environment. A cooling or warming body thus behaves like a low-pass filter, with the smallest exponent dominating at long times. The smaller the sphere, the shorter the time before it warms or cools according to a simple exponential law.

10.5 Newton's law of cooling

Newton's law of cooling states that the temperature of a cooling body falls exponentially towards the temperature of its surroundings with a rate which is proportional to the area of contact between the body and the environment. The results of the previous section indicate that it is an approximation to reality, as a cooling sphere only cools exponentially at long times.

Newton's law of cooling is often stated as follows: the heat loss of a solid or liquid surface (a hot central heating pipe or the exposed surface of a cup of tea) to the surrounding gas (usually air, which is free

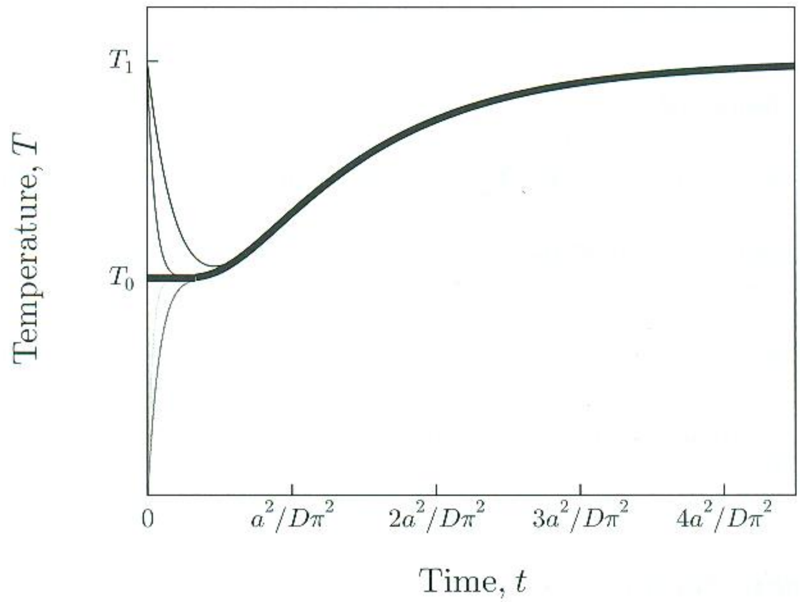

Fig. 10.4 The sum of the first few terms of $T(0,t) = T_1 + 2(T_1 - T_0)\sum_{n=1}^{\infty}(-1)^n e^{-D(n\pi/a)^2 t}$ are shown, together with $T(0,t)$ evaluated from all terms (thick solid line). The sums of only the first few terms fail near $t = 0$ and one needs more and more terms to give an accurate estimate of the temperatures as t gets closer to 0 (although this is the region where one knows what the temperature is anyway!).

to convect the heat away) is proportional to the area of contact multiplied by the temperature difference between the solid/liquid and the gas. Mathematically, this can be expressed as an equation for the heat flux $\boldsymbol{J}$ which is

$$\boldsymbol{J} = \boldsymbol{h}\Delta T, \tag{10.53}$$

where ΔT is the temperature difference between the body and its environment and $\boldsymbol{h}$ is a vector whose direction is normal to the surface of the body and whose magnitude $h = |\boldsymbol{h}|$ is a heat transfer coefficient. In general, h depends on the temperature of the body and its surroundings and varies over the surface, so that Newton's "law" of cooling is more of an empirical relation.

This alternative definition generates an exponential decay of temperature as demonstrated in the following example.

Example 10.4

A polystyrene cup containing tea at temperature T_{hot} at $t = 0$ stands for a while in a room with air temperature T_{air}. The heat loss through the surface area A exposed to the air is, according to Newton's law of cooling, proportional to $A(T(t) - T_{\text{air}})$, where $T(t)$ is the temperature of the tea at time t. Ignoring the heat lost by other means, we have that

$$-C\frac{\partial T}{\partial t} = JA = hA(T - T_{\text{air}}), \tag{10.54}$$

where J is the heat flux, C is the heat capacity of the cup of tea and h is a constant, so that

$$T = T_{\rm air} + (T_{\rm hot} - T_{\rm air})e^{-\lambda t} \tag{10.55}$$

where $\lambda = Ah/C$.

What makes these types of calculations of heat transfer so difficult is that heat transfer from bodies into their surrounding gas or liquid often is dominated by **convection**.[3] Convection can be defined as the transfer of heat by the motion of or within a fluid (i.e. within a liquid or a gas). Convection is often driven by the fact that warmer fluid expands and rises, while colder fluid contracts and sinks; this causes currents in the fluid to be set up which rather efficiently transfer heat. Our analysis of the thermal conductivity in a gas ignores such currents. Convection is a very complicated process and can depend on the precise details of the geometry of the surroundings. A third form of heat transfer is by thermal radiation and this will be the subject of chapter 23.

[3] One can either have **forced convection**, in which fluid is driven past the cooling body by some external input of work (provided by means of a pump, fan, propulsive motion of an aircraft etc.), or **free convection**, in which any external fluid motion is driven only by the temperature difference between the cooling body and the surrounding fluid. Newton's law of cooling is actually only correct for forced convection, while for free convection (which one should probably use for the example of the cooling of a cup of tea in air) the heat transfer coefficient is temperature dependent ($h \propto (\Delta T)^{1/4}$ for laminar flow, $h \propto (\Delta T)^{1/3}$ in the turbulent regime). We examine convection in stars in more detail in Section 35.3.2.

10.6 The Prandtl number

How valid is it to ignore convection? It's clearly fine to ignore it in a solid, but for a fluid we need to know the relative strength of the diffusion of momentum and heat. Convection dominates if momentum diffusion dominates (because convection involves transport of the gas itself) but conduction dominates if heat diffusion dominates. We can express these two diffusivities using the kinematic viscosity $\nu = \eta/c_p$ (with units $\mathrm{m^2s^{-1}}$) and the thermal diffusivity $D = \kappa/\rho c_p$ (also with units $\mathrm{m^2s^{-1}}$), where ρ is the density. To examine their relative magnitudes, we define the **Prandtl number** as the dimensionless ratio $\sigma_{\rm p}$ obtained by dividing ν by D, so that

$$\sigma_{\rm p} = \frac{\nu}{D} = \frac{\eta c_{\rm p}}{\kappa}. \tag{10.56}$$

For an ideal gas, we can use $c_{\rm p}/c_{\rm V} = \gamma = \frac{5}{3}$, and using eqn 9.21 (which states that $\kappa = c_{\rm V}\eta$) we arrive at $\sigma_{\rm p} = \frac{5}{3}$. However, eqn 9.21 resulted from an approximate treatment, and the corrected version is eqn 9.44 (which states that $\kappa = \frac{5}{2}\eta c_V$), and hence we arrive at

$$\sigma_{\rm p} = \frac{2}{3}. \tag{10.57}$$

For many gases, the Prandtl number is found to be around this value. It is between 100 and 40000 for engine oil and around 0.015 for mercury. When $\sigma_{\rm p} \gg 1$ diffusion of momentum (i.e. viscosity) dominates over diffusion of heat (i.e. thermal conductivity), and convection is the dominant mode of heat transport. When $\sigma_{\rm p} \ll 1$ the reverse is true, and thermal conduction dominates the heat transport.

10.7 Sources of heat

If heat is generated at a rate H per unit volume, (so H is measured in $\mathrm{W\,m^{-3}}$), this will add to the divergence of $\boldsymbol{J}$ so that eqn 10.7 becomes

$$\nabla \cdot \boldsymbol{J} = -C\frac{\partial T}{\partial t} + H, \tag{10.58}$$

and hence the thermal diffusion equation becomes

$$\nabla^2 T = \frac{C}{\kappa}\frac{\partial T}{\partial t} - \frac{H}{\kappa}, \tag{10.59}$$

or equivalently

$$\boxed{\frac{\partial T}{\partial t} = D\nabla^2 T + \frac{H}{C}.} \tag{10.60}$$

Example 10.5

A metallic bar of length L with both ends maintained at $T = T_0$ passes a current which generates heat H per unit length of the bar per second. Find the temperature at the centre of the bar in steady state.
Solution: In steady state,

$$\frac{\partial T}{\partial t} = 0, \tag{10.61}$$

and so

$$\frac{\partial^2 T}{\partial x^2} = -\frac{H}{\kappa}. \tag{10.62}$$

Integrating this twice yields

$$T = \alpha x + \beta - \frac{H}{2\kappa}x^2, \tag{10.63}$$

where α and β are constants of integration. The boundary conditions imply that

$$T - T_0 = \frac{H}{2\kappa}x(L - x), \tag{10.64}$$

so that at $x = L/2$ we have that the temperature is

$$T = T_0 + \frac{HL^2}{8\kappa}. \tag{10.65}$$

Chapter summary

- The thermal diffusion equation (in the absence of a heat source) is
$$\boxed{\frac{\partial T}{\partial t} = D\nabla^2 T,} \tag{10.66}$$
where $D = \kappa/C$ is the thermal diffusivity.
- 'Steady state' implies that
$$\frac{\partial}{\partial t}(\text{physical quantity}) = 0. \tag{10.67}$$
- If heat is generated at a rate H per unit volume per unit time, then the thermal diffusion equation becomes
$$\frac{\partial T}{\partial t} = D\nabla^2 T + \frac{H}{C}. \tag{10.68}$$
- Newton's law of cooling states that the heat loss from a solid or liquid surface is proportional to the area of the surface multiplied by the temperature difference between the solid/liquid and the gas.

Exercises

(10.1) One face of a thick uniform layer is subject to sinusoidal temperature variations of angular frequency ω. Show that damped sinusoidal temperature oscillations propagate into the layer and give an expression for the decay length of the oscillation amplitude.

A cellar is built underground and is covered by a ceiling which is 3 m thick and made of limestone. The outside temperature is subject to daily fluctuations of amplitude 10°C and annual fluctuations of 20°C. Estimate the magnitude of the daily and annual temperature variations within the cellar. Assuming that January is the coldest month of the year, when will the cellar's temperature be at its lowest?

[The thermal conductivity of limestone is $1.6\,\mathrm{W m^{-1} K^{-1}}$, and the heat capacity of limestone is $2.5 \times 10^6\,\mathrm{J K^{-1} m^{-3}}$.]

(10.2) (a) A cylindrical wire of thermal conductivity κ, radius a and resistivity ρ uniformly carries a current I. The temperature of its surface is fixed at T_0 using water cooling. Show that the temperature $T(r)$ inside the wire at radius r is given by
$$T(r) = T_0 + \frac{\rho I^2}{4\pi^2 a^4 \kappa}(a^2 - r^2).$$
(b) The wire is now placed in air at temperature T_{air} and the wire loses heat from its surface according to Newton's law of cooling (so that the heat flux from the surface of the wire is given by $\alpha(T(a) - T_{\mathrm{air}})$, where α is a constant). Find the temperature $T(r)$.

(10.3) Show that for the problem of a spherical chicken being cooked in an oven considered in Example 10.3 in this chapter, the temperature T gets 90% of the way from T_0 to T_1 after a time $\sim a^2 \ln 20/\pi^2 D$.

(10.4) A microprocessor has an array of metal fins attached to it, whose purpose is to remove heat generated within the processor. Each fin may be represented by a long thin cylindrical copper rod with one end attached to the processor; heat received by

the rod through this end is lost to the surroundings through its sides.

Show that the temperature $T(x,t)$ at location x along the rod at time t obeys the equation

$$\rho C_p \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{2}{a} R(T),$$

where a is the radius of the rod, and $R(T)$ is the rate of heat loss per unit area of surface at temperature T.

The surroundings of the rod are at temperature T_0. Assume that $R(T)$ has the form of Newton's law of cooling, namely

$$R(T) = A(T - T_0).$$

In the steady state:

(a) obtain an expression for T as a function of x for the case of an infinitely long rod whose hot end has temperature T_{m};

(b) show that the heat that can be transported away by a long rod (with radius a) is proportional to $a^{3/2}$, provided that A is independent of a.

In practice the rod is not infinitely long. What length does it need to have for the results above to be approximately valid? The radius of the rod, a, is 1.5 mm.

[The thermal conductivity of copper is $380\,\mathrm{W\,m^{-1}\,K^{-1}}$. The cooling constant $A = 250\,\mathrm{W\,m^{-2}\,K^{-1}}$.]

(10.5) For oscillations at frequency ω, a viscous penetration depth δ_{v} can be defined by

$$\delta_{\mathrm{v}} = \left(\frac{2\eta}{\rho\omega}\right)^{1/2}, \tag{10.69}$$

analogously to the thermal penetration depth

$$\delta = \left(\frac{2\kappa}{\rho c_p \omega}\right)^{1/2} \tag{10.70}$$

defined in this chapter. Show that

$$\left(\frac{\delta_{\mathrm{v}}}{\delta}\right)^2 = \sigma_{\mathrm{p}}, \tag{10.71}$$

where σ_{p} is the Prandtl number (see eqn 10.56).

(10.6) For thermal waves, calculate the magnitude of the group velocity. This shows that the thermal diffusion equation cannot hold exactly as the velocity of propagation can become larger than that of the carriers. An alternative equation can be derived as follows. Consider the number density n of thermal carriers in a material. In equilibrium, $n = n_0$, so that

$$\left(\frac{\partial n}{\partial t}\right) = -\boldsymbol{v} \cdot \nabla n + \frac{n - n_0}{\tau}, \tag{10.72}$$

where τ is a relaxation time and $\boldsymbol{v}$ is the carrier velocity. Multiply this equation by $\hbar\omega\tau\boldsymbol{v}$, where $\hbar\omega$ is the energy of a carrier, and sum over all $\boldsymbol{k}$ states. Using the fact that $\sum_{\boldsymbol{k}} n_0 \boldsymbol{v} = 0$ and $\boldsymbol{J} = \sum_{\boldsymbol{k}} \hbar\omega n \boldsymbol{v}$, and that $|n - n_0| \ll n_0$ show that

$$\boldsymbol{J} + \tau \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = -\kappa \nabla T, \tag{10.73}$$

and hence the modified thermal diffusion equation becomes

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} = D \nabla^2 T. \tag{10.74}$$

Show that this does not suffer from a group velocity whose magnitude can ever become infinite. Is this modification ever necessary?

(10.7) A series of N large, flat rectangular slabs with thickness Δx_i and thermal conductivity κ_i are placed on top of one another. The top and bottom surfaces are maintained at temperature T_i and T_f respectively. Show that the heat flux J through the slabs is given by $J = (T_i - T_f)/\sum_i R_i$, where $R_i = \Delta x_i / \kappa_i$.

(10.8) The space between two concentric cylinders is filled with material of thermal conductivity κ. The inner (outer) cylinder has radius r_1 (r_2) and is maintained at temperature T_1 (T_2). Derive an expression for the heat flow per unit length between the cylinders.

(10.9) A pipe of radius R is maintained at a uniform temperature T. In order to reduce heat loss from the pipe, it is lagged by an insulating material of thermal conductivity κ. The lagged pipe has radius $r > R$. Assume that all surfaces lose heat according to Newton's law of cooling $\boldsymbol{J} = \boldsymbol{h}\Delta T$, where $h = |\boldsymbol{h}|$ can be taken to be a constant. Show that the heat loss per unit length of pipe is inversely proportional to

$$\frac{1}{hr} + \frac{1}{\kappa} \ln\left(\frac{r}{R}\right), \tag{10.75}$$

and hence show that thin lagging doesn't reduce heat loss if $R < \kappa/h$.

Jean Baptiste Joseph Fourier (1768–1830)

Fourier was born in Auxerre, France, the son of a tailor. He was schooled there in the École Royale Militaire where he showed early mathematical promise. In 1787 he entered a Benedictine abbey to train for the priesthood, but the pull of science was too great and he never followed that vocation, instead becoming a teacher at his old school in Auxerre. He was also interested in politics, and unfortunately there was a lot of it around at the time; Fourier became embroiled in the Revolutionary ferment and in 1794 came close to being guillotined, but following Robespierre's execution by the same means, the political tide turned in Fourier's favour. He was able to study at the École Normale in Paris under such luminaries as Lagrange and Laplace, and in 1795 took up a chair at the École Polytechnique.

Fig. 10.5 J.B.J. Fourier

Fourier joined Napoleon on his invasion of Egypt in 1798, becoming governor of Lower Egypt in the process. There he carried out archaeological explorations and later wrote a book about Egypt (which Napoleon then edited to make the history sections more favourable to himself). Nelson's defeat of the French fleet in late 1798 rendered Fourier isolated there, but he nevertheless set up political institutions. He managed to slink back to France in 1801 to resume his academic post, but Napoleon (a hard man to refuse) sent him back to an administrative position in Grenoble where he ended up on such high-brow activities as supervising the draining of swamps and organizing the construction of a road between Grenoble and Turin. He nevertheless found enough time to work on experiments on the propagation of heat and published, in 1807, his memoir on this subject. Lagrange and Laplace criticized his mathematics (Fourier had been forced to invent new techniques to solve the problem, which we now call Fourier series, and this was fearsomely unfamiliar stuff at the time), while the notoriously difficult Biot (he of the Biot-Savart law fame) claimed that Fourier had ignored his own crucial work on the subject (Fourier had discounted it, as Biot's work on this subject was wrong). Fourier's work won him a prize, but reservations about its importance or correctness remained.

In 1815, Napoleon was exiled to Elba and Fourier managed to avoid Napoleon who was due to pass through Grenoble en route out of France. When Napoleon escaped, he brought an army to Grenoble and Fourier avoided him again, earning Napoleon's displeasure, but he managed to patch things up and got himself made Prefect of Rhône, a position from which he resigned as soon as he could. Following Napoleon's final defeat at Waterloo, Fourier became somewhat out of favour in political circles and was able to continue working on physics and mathematics back in Paris. In 1822 he published his *Théorie analytique de chaleur* (Analytical Theory of Heat) which included all his work on thermal diffusion and the use of Fourier series, a work that was to prove influential with many later thermodynamicists of the nineteenth century.

In 1824, Fourier wrote an essay which pointed towards what we now call the greenhouse effect; he realised that the insulating effect of the atmosphere might increase the Earth's surface temperature. He understood the way planets lose heat via infrared radiation (though he called it "chaleur obscure"). Since so much of his scientific work had been bound up with the nature of heat (even his work on Fourier series was only performed so he could solve heat problems) he became, in his later years, somewhat obsessed by the imagined healing powers of heat. He kept his house overheated, and wore excessively warm clothes, in order to maximize the effect of the supposedly life-giving heat. He died in 1830 after falling down the stairs.