(b) If the controller is PI, the disturbance error transfer function is

$$T_w(s) = \frac{-Bs}{s^2(\tau s + 1) + (k_p s + k_I)A}, \tag{4.58}$$

$$n = 1, \tag{4.59}$$

$$K_{n,w} = \frac{Ak_I}{-B}, \tag{4.60}$$

and therefore the system is **type 1** and the error to a unit ramp disturbance input will be

$$e_{ss} = \frac{-B}{Ak_I}. \tag{4.61}$$

---

## 4.3 Control of Dynamic Error: PID Control

We have seen in Section 4.1 basic properties of feedback control, and in Section 4.2 we examined the steady state response of systems to polynomial reference and disturbance input. At the end of Section 4.1 we observed that proportional control changed the time constant of the simple speed-control system. In this section the impact of more sophisticated controls on system characteristic equations is examined in the context of a standard controller structure. The most basic feedback is a constant **P**roportional to error. As we saw in Section 4.2, addition of a term proportional to the **I**ntegral of error has a major influence on the system type and steady-state error to polynomials. The final term in the classical structure term proportional to the **D**erivative of error. Combined, these three terms form the classical **PID** controller, which is widely used in the process and robotics industries.

The PID (proportional-integral-derivative) controller

### 4.3.1 Proportional Control (P)

When the feedback control signal is linearly proportional to the system error, we call the result **Proportional feedback**. This was the case for the feedback used in the controller of speed in Section 4.1, for which the controller transfer function is

$$\frac{U(s)}{E(s)} = D_c(s) = k_p. \tag{4.62}$$

As we saw in Section 4.1.4, the time constant of the feedback system was reduced by a factor $1 + Ak_p$ by proportional control. If the plant is second order, as, for example, is a DC motor with nonnegligible inductance, then the transfer function can be written as

$$G(s) = \frac{A}{s^2 + a_1 s + a_2}. \tag{4.63}$$

In this case, the characteristic equation with proportional control is

$$1 + k_p G(s) = 0, \tag{4.64}$$

$$s^2 + a_1 s + a_2 + k_p = 0. \tag{4.65}$$

The designer can control the constant term and the natural frequency, but not the damping of this equation. If $k_p$ is made large to get adequate steady-state error, the damping may be much too low for satisfactory transient response.

## 4.3.2 Proportional plus Integral Control (PI)

Proportional plus Integral control

Adding an integral term to the controller results in the **Proportional plus Integral (PI)** control equation

$$u(t) = k_p e + k_I \int_{t_0}^{t} e(\tau)\,d\tau, \tag{4.66}$$

for which the $D_c(s)$ in Fig. 4.5 becomes

$$\frac{U(s)}{E(s)} = D_c(s) = k_p + \frac{k_I}{s}. \tag{4.67}$$

This feedback has the primary virtue that, in the steady-state, its control output can be a *nonzero* constant value even when the error signal at its input is *zero*. This comes about because the integral term in the control signal is a summation of all past values of $e(t)$. In fact, the integral term will not stop changing until its input is zero, and therefore if the system reaches a stable steady state, the input signal to the integrator will of necessity be zero. This feature means that a constant disturbance $w$ (see Fig. 4.4) can be canceled by the integrator's output even while the system error is zero.

If PI control is used in the speed example, the transform equation for the controller is

$$U = k_p(\Omega_{\text{ref}} - \Omega_m) + k_I \frac{\Omega_{\text{ref}} - \Omega_m}{s}, \tag{4.68}$$

and the system transform equation with this controller is

$$(\tau s + 1)\Omega_m = A(k_p + \frac{k_I}{s})(\Omega_{\text{ref}} - \Omega_m) + TW. \tag{4.69}$$

If we now multiply by $s$ and collect terms, we obtain

$$(\tau s^2 + (Ak_p + 1)s + Ak_I)\Omega_m = A(k_p s + k_I)\Omega_{\text{ref}} + AsW. \tag{4.70}$$

Because the **PI** controller includes dynamics, use of this controller will change the dynamic response in more complicated ways than the simple speed-up we saw with proportional control. We can understand this by considering the characteristic equation of the speed control with PI control, as seen in Eq. (4.70). The characteristic equation is

$$\tau s^2 + (Ak_p + 1)s + Ak_I = 0. \tag{4.71}$$

The two roots of this equation may be complex and, if so, the natural frequency is $\omega_n = \sqrt{\frac{Ak_I}{\tau}}$, and the damping ratio is $\zeta = \frac{Ak_p+1}{2\tau\omega_n}$. These parameters are both determined by the controller gains. If the plant is second order, then the characteristic equation is

$$1 + \frac{k_p s + k_I}{s} \frac{A}{s^2 + a_1 s + a_2} = 0, \tag{4.72}$$

$$s^3 + a_1 s^2 + a_2 s + Ak_p s + Ak_I = 0. \tag{4.73}$$

In this case, the controller parameters can be used to set two of the coefficients, but not the third. For this we need derivative control.

### 4.3.3 Proportional-Integral-Derivative Control (PID)

The final term in the classical controller is derivative control, D, and the complete three-term controller is described by the transform equation we will use, namely,

$$D_c(s) = \frac{U(s)}{E(s)} = k_p + \frac{k_I}{s} + k_D s, \tag{4.74}$$

or, equivalently, by the equation often used in the process industries, or

$$D_c(s) = k_p[1 + \frac{1}{T_I s} + T_D s], \tag{4.75}$$

where the "reset rate" $T_I$ in seconds, and the "derivative rate," $T_D$, also in seconds, can be given physical meaning to the operator who must select values for them to "tune" the controller. For our purposes, Eq. (4.74) is simpler to use. The effect of the derivative control term depends on the rate of change of the error. As a result, a controller with derivative control exhibits an anticipatory response, as illustrated by the fact that the output of a PD controller having a ramp error $e(t) = t1(t)$ input would *lead* the output of a proportional controller having the same input by $\frac{k_D}{k_p} \triangleq T_D$ seconds, as shown in Fig. 4.11.

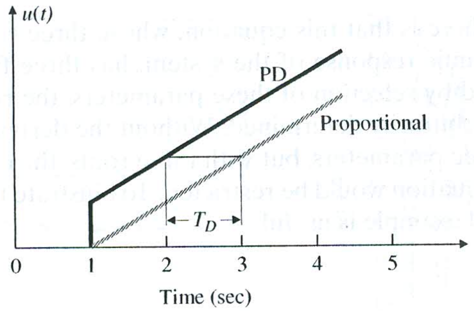


**Figure 4.11** Anticipatory nature of derivative control

Because of the sharp effect of derivative control on suddenly changing signals, the "**D**" term is sometimes introduced into the feedback path as shown in Fig. 4.12(a), which would describe, for example, a tachometer on the shaft of a motor. The closed-loop characteristic equation is the same as if the term were in the forward path, as given by Eq. (4.74) and drawn in Fig. 4.12(b), if the derivative gain is $k_D = k_p k_t$ but the zeros from the reference to the output are different in the two cases. With the derivative in the feedback path, the reference is not differentiated, which may be a desirable result if the reference is subject to sudden changes. With the derivative in the forward path, a step change in the reference input will, in theory, cause an intense initial pulse in the control signal, which may be very undesirable.

To illustrate the effect of a derivative term on PID control, consider speed control, but with the second-order plant. In that case, the characteristic equation is

$$s^2 + a_1 s + a_2 + A(k_p + \frac{k_I}{s} + k_D s) = 0,$$

$$s^3 + a_1 s^2 + a_2 s + A(k_p s + k_I + k_D s^2) = 0. \qquad (4.76)$$

Collecting terms results in

$$s^3 + (a_1 + Ak_D)s^2 + (a_2 + Ak_p)s + Ak_I = 0. \qquad (4.77)$$

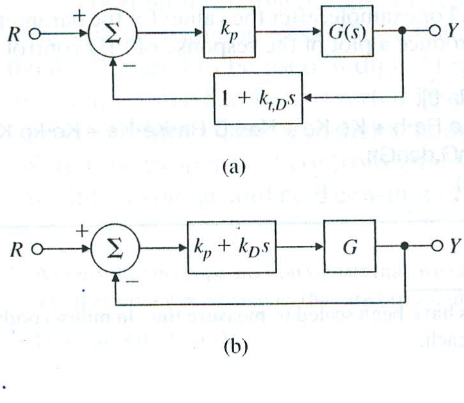


**Figure 4.12** Alternative ways of configuring rate feedback

The point here is that this equation, whose three roots determine the nature of the dynamic response of the system, has three free parameters in $k_P$, $k_I$, and $k_D$, and by selection of these parameters, the roots can be uniquely and, in theory, arbitrarily determined. Without the derivative term, there would be only two free parameters, but with three roots, the choice of roots of the characteristic equation would be restricted. To illustrate the effect more concretely, a numerical example is useful.

## EXAMPLE 4.6 *PID Control of Motor Speed*

Consider the DC motor speed control with parameters[5]

$$J_m = 1.13 \times 10^{-2} \text{ N-m-sec}^2/\text{rad}, \quad b = 0.028 \text{ N-m-sec/rad}, \quad L_a = 10^{-1} \text{ henry},$$
$$R_a = 0.45 \text{ ohms}, \quad K_t = 0.067 \text{ N-m/amp}, \quad K_e = 0.067 \text{ V-sec/rad}. \tag{4.78}$$

Use the controller parameters

$$k_p = 3, \quad k_I = 15 \text{ sec}^{-1}, \quad k_D = 0.3 \text{ sec}. \tag{4.79}$$

Discuss the effects of P, PI, and PID control on the responses of this system to steps in the disturbance and steps in the reference input. Let the unused controller parameters be zero.

**Solution.** Figure 4.13(a) illustrates the effects of P, PI, and PID feedback on the step disturbance response of the system. Note that adding the integral term increases the oscillatory behavior but eliminates the steady-state error, and that adding the derivative term reduces the oscillation while maintaining zero steady-state error. Figure 4.13(b) illustrates the effects of P, PI, and PID feedback on the step reference response, with similar results. The step responses can be computed by forming the numerator and denominator coefficient vectors (in descending powers of $s$) and using the step function in MATLAB. For example, after the values for the parameters are entered, the following commands produce a plot of the response of PID control to a disturbance step:

```
numG = [La Ra 0];
denG = [Jm*La Ra*b + Ke*Ke + Ke*kD Ra*Ke*Ke + Ke*kp Ke*ki];
sysG = tf(numG,denG);
y = step(sysG).
```

---

[5] These values have been scaled to measure time in milliseconds by multiplying the true $L_a$ and $J_m$ by 1000 each.

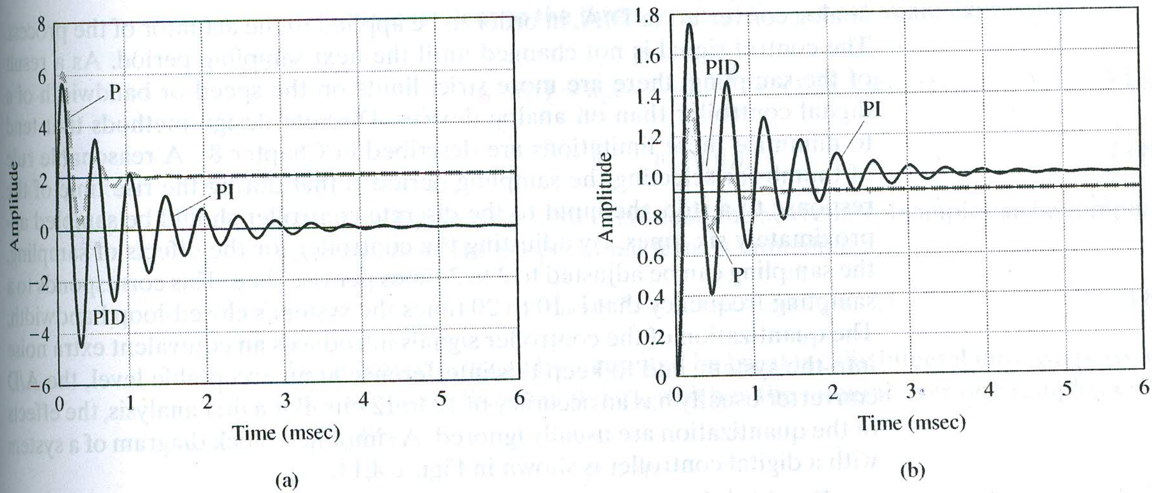


**Figure 4.13** Responses of P, PI, and PID control to (a) step disturbance input and (b) step reference input

## 4.4 Extensions to the Basic Feedback Concepts

### 4.4.1 Digital Implementation of Controllers

As a result of the revolution in the cost-effectiveness of digital computers, there has been an increasing use of digital logic in embedded applications, such as controllers in feedback systems. With the formula for calculating the control signal in software rather than hardware, a digital controller gives the designer much more flexibility in making modifications to the control law after the hardware design is fixed. In many instances, this means that the hardware and software designs can proceed almost independently, saving a great deal of time. Also, it is easy to include binary logic and nonlinear operations as part of the function of a digital controller. Special processors designed for real-time signal processing and known as digital signal processors, or DSPs, are particularly well suited for use as real-time controllers. While, in general, the design of systems to use a digital processor requires sophisticated use of new concepts to be introduced in Chapter 8, such as the *z-transform*, it is quite straightforward to translate a linear continuous analog design into a discrete equivalent. A digital controller differs from an analog controller in that the signals must be **sampled** and **quantized.**[6] A signal to be used in digital logic needs to be sampled first, and then the samples need to be converted by an analog-to-digital converter, or A/D converter,[7] into a quantized digital number. Once the digital computer has calculated the proper next control signal value, this value needs to be converted back into a voltage and held constant or otherwise extrapolated by a digital-to-

[6] A controller that operates on signals that are sampled but *not* quantized is called **discrete**, while one that operates on signals that are both sampled and quantized is called **digital**.

[7] Pronounced "A to D."