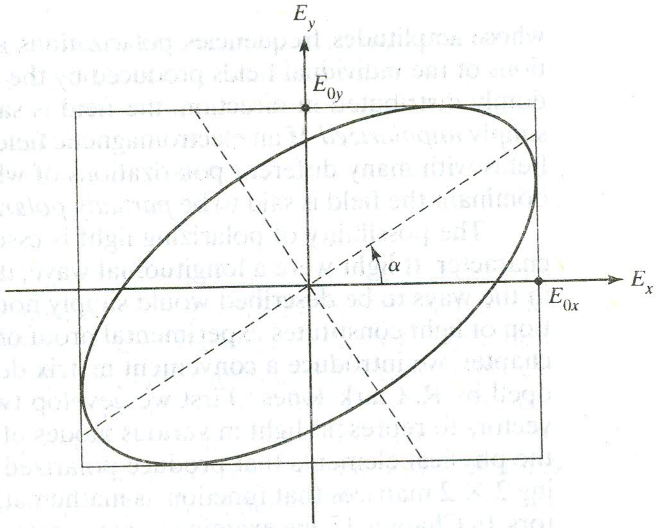

14 Matrix Treatment of Polarization

INTRODUCTION

The polarization of an electromagnetic wave was introduced in Chapter 4. There we noted that the *direction of the electric field vector* $\vec{\mathbf{E}}$ is known as the *polarization* of the electromagnetic wave. In this and the following chapter we extend our discussion of the properties and production of polarized light. As we noted in Chapter 4, the electric field associated with a plane monochromatic electromagnetic wave is perpendicular to the direction of the propagation of the energy carried by the wave. The same can be said of the magnetic field vector, which also maintains an orientation perpendicular to the electric field vector such that the direction of $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ is everywhere the direction of wave propagation. In general, plane monochromatic waves are *elliptically polarized*, in the sense that, over time, the tip of the electric field vector in a given plane perpendicular to the direction of energy propagation traces out an ellipse. Special cases of electromagnetic waves with elliptical polarization include *linearly polarized* waves in which the electric field vector always oscillates back and forth along a given direction in space and *circularly polarized* waves in which, over time, the tip of the electric field vector traces out a circle. These special cases are shown in Figure 4-12 and are worth reviewing. Monochromatic plane waves are idealized models of the electromagnetic waves produced by, for example, laser sources or a distant single-dipole oscillator. Any electromagnetic wave can be regarded as a superposition of plane electromagnetic waves with various frequencies, amplitudes, phases, and polarizations. "Ordinary" light, such as that produced by a hot filament, is typically produced by a number of independent atomic sources whose radiation is not synchronized. The resultant $\vec{\mathbf{E}}$-field vector consists of many components whose amplitudes, frequencies, polarizations, and phases differ. If the polarizations of the individual fields produced by the independent oscillators are randomly distributed in direction, the field is said to be *randomly polarized* or simply *unpolarized*. If an electromagnetic field consists of the superposition of fields with many different polarizations of which one is (or several are) predominant the field is said to be *partially polarized*.

The possibility of polarizing light is essentially related to its transverse character. If light were a longitudinal wave, the production of polarized light in the ways to be described would simply not be possible. Thus, the polarization of light constitutes experimental proof of its transverse character. In this chapter, we introduce a convenient matrix description of polarization developed by R. Clark Jones.[1] First we develop two-element column matrices or vectors to represent light in various modes of polarization. Then we examine the physical elements that produce polarized light and discover corresponding 2×2 matrices that function as mathematical operators on the Jones vectors. In Chapter 15, we examine in more detail the physical processes that are responsible for producing polarized light.

14-1 MATHEMATICAL REPRESENTATION OF POLARIZED LIGHT: JONES VECTORS

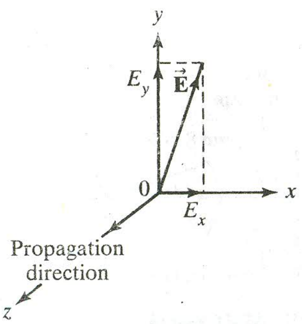

Figure 14-1 Representation of the instantaneous $\vec{\mathbf{E}}$-vector of a light wave traveling in the $+z$-direction.

Consider an electromagnetic wave propagating along the z-direction of the coordinate system shown is Figure 14-1. Let the electric field of this wave, at the origin of the axis system, be represented, at a given time, by the vector $\vec{\mathbf{E}}$ shown. Then, in terms of the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$,

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} \tag{14-1}$$

We write the *complex* field components for waves traveling in the $+z$-direction with amplitudes E_{0x} and E_{0y} and phases φ_x and φ_y as

$$\widetilde{E}_x = E_{0x}e^{i(kz-\omega t+\varphi_x)} \tag{14-2}$$

and

$$\widetilde{E}_y = E_{0y}e^{i(kz-\omega t+\varphi_y)} \tag{14-3}$$

Here, $E_x = \text{Re}\,(\widetilde{E}_x)$ and $E_y = \text{Re}\,(\widetilde{E}_y)$.

Using Eqs. (14-2) and (14-3) in Eq. (14-1) gives, for the complex field $\widetilde{\mathbf{E}}$,

$$\widetilde{\mathbf{E}} = E_{0x}e^{i(kz-\omega t+\varphi_x)}\hat{\mathbf{x}} + E_{0y}e^{i(kz-\omega t+\varphi_y)}\hat{\mathbf{y}}$$

which may also be written

$$\widetilde{\mathbf{E}} = [E_{0x}e^{i\varphi_x}\hat{\mathbf{x}} + E_{0y}e^{i\varphi_y}\hat{\mathbf{y}}]e^{i(kz-\omega t)} = \widetilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \tag{14-4}$$

The bracketed quantity in Eq. (14-4), separated into x- and y-components, is now recognized as the complex amplitude vector $\widetilde{\mathbf{E}}_0$ for the polarized wave. Since the state of polarization of the light is completely determined by the

[1] R. Clark Jones, "A New Calculus for the Treatment of Optical Systems," *Journal of the Optical Society*, Vol. 31 1941; 488.

relative amplitudes and phases of these components, we need concentrate only on the complex amplitude, written as a two-element matrix, or *Jones vector*,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} \tilde{E}_{0x} \\ \tilde{E}_{0y} \end{bmatrix} = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} \tag{14-5}$$

Let us determine the particular forms for Jones vectors that describe *linear*, *circular*, and *elliptical* polarization. In Figure 14-2a, vertically polarized light travels in the $+z$-direction out of the page with its $\vec{\mathbf{E}}$-oscillations along the y-axis. Since $\vec{\mathbf{E}}$ has a sinusoidally varying magnitude as it progresses, the electric field vector varies between, say, $A\hat{\mathbf{y}}$ and $-A\hat{\mathbf{y}}$. We display this behavior by a double-headed arrow, as shown in Figure 14-2a. As time progresses, the tip of the electric field vector traces out positions along the extent of the double-headed arrow. The field depicted in Figure 14-2a is represented by $E_{0x} = 0$ and $E_{0y} = A$. In the absence of an E_x-component, the phase φ_y may be set equal to zero for convenience. Then, by Eq. (14-5), the corresponding Jones vector is

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} 0 \\ A \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{linear polarization along } y$$

Furthermore, when only the mode of polarization is of interest, the amplitude A may be set equal to 1. The Jones vector for vertically linearly polarized light is then simply $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This simplified form is the *normalized* form of the vector. In general, a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is expressed in normalized form when

$$|a|^2 + |b|^2 = 1$$

Similarly, Figure 14-2b represents horizontally polarized light, for which, letting $E_{0y} = 0$, $\varphi_x = 0$, and $E_{0x} = A$,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{linear polarization along } x$$

On the other hand, Figure 14-2c represents linearly polarized light whose vibrations occur along a line making an angle α with respect to the x-axis. Both x- and y-components of $\vec{\mathbf{E}}$ are simultaneously present. Evidently this is a general case of linearly polarized light that reduces to the vertically polarized mode when $\alpha = 90°$ and to the horizontally polarized mode when $\alpha = 0°$. Notice that to produce the resultant vibration shown in Figure 14-3a, the two perpendicular vibrations $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ must be in phase. That is, they must pass through the origin together, increase along their respective positive axes together, reach their maximum values together, and then return together to continue the cycle. Figure 14-3a makes this sequence clear. Accordingly, since we require merely a *relative* phase of zero, we set $\varphi_x = \varphi_y = 0$. For a resultant with amplitude A, the perpendicular component amplitudes are $E_{0x} = A\cos\alpha$ and $E_{0y} = A\sin\alpha$. The Jones vector takes the form

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A\cos\alpha \\ A\sin\alpha \end{bmatrix} = A\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \quad \text{linear polarization at } \alpha \tag{14-6}$$

For the normalized form of the vector, we set $A = 1$, since $\cos^2\alpha + \sin^2\alpha = 1$. Notice that this general form does indeed reduce to the Jones vectors found for

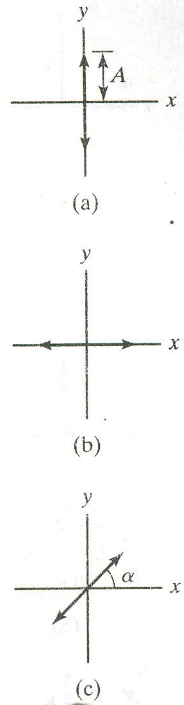

Figure 14-2 Representation of $\vec{\mathbf{E}}$-vectors of linearly polarized light with various orientations. In each case, the light is propagating in the positive z-direction.

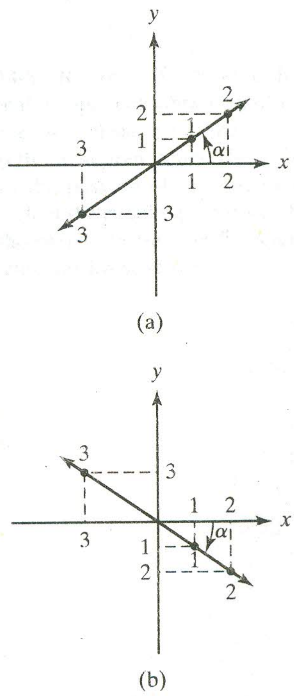

Figure 14-3 (a) Linearly polarized electric field vectors whose x- and y-components are in phase lie in the first and third quadrants. (b) Linearly polarized electric field vectors whose x- and y-components are π out of phase lie in the second and fourth quadrants.

the case $\alpha = 0°$ and $\alpha = 90°$. For other orientations, for example, $\alpha = 60°$,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} \cos(60°) \\ \sin(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

Alternatively, given a vector $\tilde{\mathrm{E}}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$, where a and b are real numbers, the inclination of the corresponding linearly polarized light is given by

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right) \tag{14-7}$$

Generalizing a bit, suppose α were a negative angle, as in Figure 14-3b. In this case, E_{0y} is a negative number, since the sine is an odd function, whereas E_{0x} remains positive. The negative sign ensures that the two vibrations are π out of phase, as needed to produce linearly polarized light with $\vec{\mathbf{E}}$-vectors lying in the second and fourth quadrants. Referring to Figure 14-3b again, this means that if the x-vibration is increasing from the origin along its positive direction, the y-vibration must be increasing from the origin along its negative direction. The resultant vibration takes place along a line with negative slope. Summarizing, a Jones vector $\begin{bmatrix} a \\ b \end{bmatrix}$ with both a and b real numbers, not both zero, represents linearly polarized light at inclination angle $\alpha = \tan^{-1}(b/a)$.

By now it may be apparent that in determining the resultant vibration due to two perpendicular components, we are in fact determining the appropriate *Lissajous figure*. If the phase difference between the vibrations is other than 0 or π, the resultant $\vec{\mathbf{E}}$-vector traces out an *ellipse* rather than a straight line. Of course, the straight line can be considered a special case of the ellipse, as can the circle. Figure 14-4 summarizes the sequence of Lissajous figures as a function of relative phase $\Delta\varphi = \varphi_y - \varphi_x$ for the general case $E_{0x} \neq E_{0y}$. Notice the sense of rotation of the tip of the $\vec{\mathbf{E}}$-vector around the ellipses shown in Figure 14-4, which makes the case $\Delta\varphi = \pi/4$, for example, different from the case $\Delta\varphi = 7\pi/4$. When $E_{0x} = E_{0y}$, the ellipses corresponding to $\Delta\varphi = \pi/2$ or $3\pi/2$ reduce to circles.

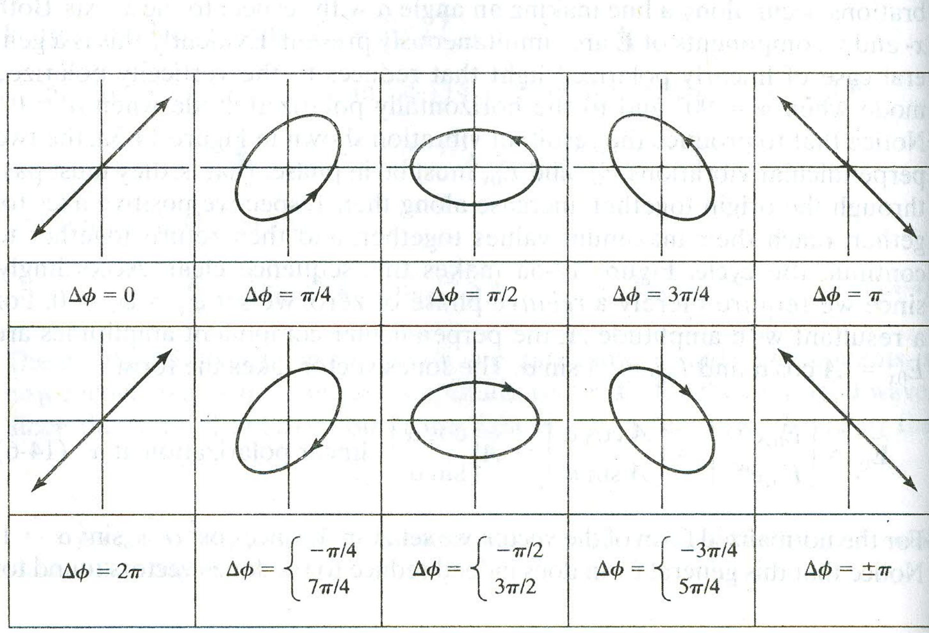

Figure 14-4 Lissajous figures as a function of relative phase for orthogonal vibrations of unequal amplitude. An angle lead greater than π may also be represented as an angle lag of less than π. For all figures we have adopted the phase lag convention $\Delta\varphi = \varphi_y - \varphi_x$.

Now suppose $E_{0x} = E_{0y} = A$ and E_x leads E_y by $\pi/2$. Then at the instant E_x has reached its maximum displacement—$+A$, for example—E_y is zero. A fourth of a period later, E_x is zero and $E_y = +A$, and so on. Figure 14-5 shows a few samples in the process of forming the resultant vibration. For the cases illustrated there, where the x-vibration leads the y-vibration, it is necessary to make $\varphi_y > \varphi_x$. This apparent contradiction results from our choice of phase in the formulation of the $\vec{\mathbf{E}}$-field in Eqs. (14-2) and (14-3), where the time-dependent term in the exponent is negative. To show this, let us observe the wave at $z = 0$ and choose $\varphi_x = 0$ and $\varphi_y = \varepsilon$, so that $\varphi_y > \varphi_x$. Equations (14-2) and (14-3) then become

$$\tilde{E}_x = E_{0x}e^{-i\omega t}$$

$$\tilde{E}_y = E_{0y}e^{-i(\omega t - \varepsilon)}$$

The negative sign before ε indicates a lag ε in the y-vibration relative to the x-vibration. To see that these equations represent the sequence in Figure 14-5, we take their real parts and set $E_{0x} = E_{0y} = A$ and $\varepsilon = \pi/2$, giving

$$E_x = A\cos\omega t$$

$$E_y = A\cos\left(\omega t - \frac{\pi}{2}\right) = A\sin\omega t$$

Recalling that $\omega = 2\pi\nu = 2\pi/T$, each of the cases in Figure 14-5 can be easily verified. Also, since

$$E^2 = E_x^2 + E_y^2 = A^2(\cos^2\omega t + \sin^2\omega t) = A^2$$

the tip of the resultant vector traces out a circle of radius A.

We now deduce the Jones vector for this case—where E_x leads E_y—taking $E_{0x} = E_{0y} = A$, $\varphi_x = 0$, and $\varphi_y = \pi/2$. Then,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ Ae^{i\pi/2} \end{bmatrix} = A\begin{bmatrix} 1 \\ i \end{bmatrix} \tag{14-8}$$

To determine the normalized form of the vector, notice that $1^2 + |i|^2 = 1 + 1 = 2$, so that each element must be divided by $\sqrt{2}$ to produce unity. Thus the Jones vector $(1/\sqrt{2})\begin{bmatrix} 1 \\ i \end{bmatrix}$ represents circularly polarized light when $\vec{\mathbf{E}}$ rotates *counterclockwise*, viewed head-on. This mode is called *left-circularly polarized* (LCP) light. Thus,

$$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{LCP}$$

Similarly, if E_y leads E_x by $\pi/2$, the result will again be circularly polarized light with clockwise rotation leading to *right-circularly polarized* (RCP) light. Replacing $\pi/2$ by $(-\pi/2)$ in Eq. (14-8) gives the normalized Jones vector,

$$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{RCP}$$

Notice that one of the elements in the Jones vector for circularly polarized light is now purely imaginary, and the magnitudes of the elements are the same.

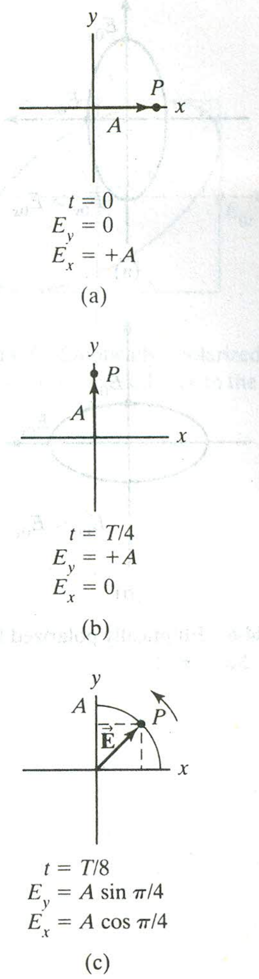

Figure 14-5 Resultant $\vec{\mathbf{E}}$-vibration due to orthogonal component vibrations of equal magnitude and phase difference of $\pi/2$, shown at three different times. The points P represent the position of the resultant. In (c) a sketch of the circular path traced by $\vec{\mathbf{E}}$ is also shown. Notice that the $\vec{\mathbf{E}}$-vector rotates counterclockwise in this case.

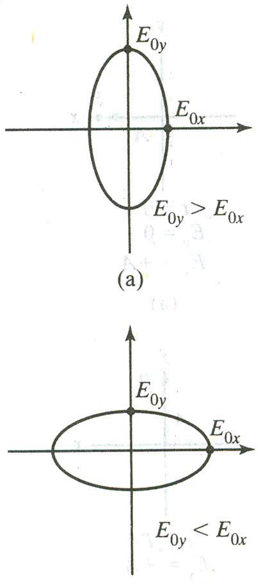

Figure 14-6 Elliptically polarized light for the case $\Delta\varphi = \pi/2$.

Given a particular mathematical form of the vector, the actual character of the light polarization may not always be immediately apparent. For example, the Jones vector $\begin{bmatrix} 2i \\ 2 \end{bmatrix}$ represents right-circularly polarized light since

$$\begin{bmatrix} 2i \\ 2 \end{bmatrix} = 2\begin{bmatrix} i \\ 1 \end{bmatrix} = 2i\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The prefactor of a Jones vector may affect the amplitude and, hence, the irradiance of the light but not the polarization mode. Prefactors such as 2 and $2i$ may therefore be ignored unless information regarding energy is required.

Next suppose that the phase difference between orthogonal vibrations $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ is still $\pi/2$, but $E_{0x} \neq E_{0y}$. In particular, let $E_{0x} = A$ and $E_{0y} = B$, where A and B are positive numbers. In this case, Eq. (14-8) should be modified to give

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ iB \end{bmatrix} \begin{array}{l}\text{counterclockwise} \\ \text{rotation}\end{array} \quad \text{and} \quad \tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ -iB \end{bmatrix} \begin{array}{l}\text{clockwise} \\ \text{rotation}\end{array}$$

These instances of elliptical polarization are illustrated in Figure 14-4 for $\Delta\varphi = \pi/2$ and $\Delta\varphi = 3\pi/2$. Notice that a lag of $\pi/2$ is equivalent to a lead of $3\pi/2$. The ellipse is oriented with its major axis along the x- or y-axis, as in Figure 14-6, depending on the relative magnitudes of E_{0x} and E_{0y}. In addition, either case may produce clockwise rotation of $\vec{\mathbf{E}}$ around the ellipse (when E_y leads E_x) or counterclockwise rotation (when E_x leads E_y). Based on these observations, we conclude that a Jones vector with elements of unequal magnitude, one of which is pure imaginary, represents elliptically polarized light oriented along the x,y-axes. The normalized forms of the Jones vectors now must include a prefactor of $1/\sqrt{A^2 + B^2}$.

It is also possible to produce elliptically polarized light with principal axes inclined to the x,y-axes, as evident in Figure 14-4. This situation occurs when the phase difference $\Delta\varphi$ between $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ is some angle other than $\Delta\varphi = 0, \pm\pi, \pm 2\pi, \pm m\pi$ (linear polarization) or $\Delta\varphi = \pm\pi/2, \pm 3\pi/2, \pm\left(m + \frac{1}{2}\right)\pi$ (circular or elliptical polarization oriented symmetrically about the x,y-axes). Here, $m = 0, \pm 1, \pm 2, \ldots$ For example, consider the case where E_x leads E_y by some positive angle ε, that is, $\varphi_y - \varphi_x = \varepsilon$. Taking $\varphi_x = 0$, $\varphi_y = \varepsilon$, $E_{0x} = A$, and $E_{0y} = b$ (with A and b positive), the Jones vector is

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ be^{i\varepsilon} \end{bmatrix}$$

Using Euler's theorem, we write

$$be^{i\varepsilon} = b(\cos\varepsilon + i\sin\varepsilon) = B + iC$$

The Jones vector for this general case is, then,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ B + iC \end{bmatrix} \quad \text{counterclockwise rotation, general case} \qquad (14\text{-}9)$$

Here the identification of this form with counterclockwise rotation requires that A and C have the same sign. Since multiplying a Jones vector by an overall constant does not change the character of the polarization described by the Jones vector, we shall adopt the convention that A is positive. With that convention a positive imaginary part C of $\tilde{E}_{0y}$ indicates that the Jones vector represents counterclockwise rotation. Note that one of the elements of the Jones vector in Eq. (14-9) is now a complex number having both real and imaginary parts. The normalized form must be divided by $\sqrt{A^2 + B^2 + C^2}$. The Jones vector of Eq. (14-9) represents an electric field vector whose tip travels in a *counterclockwise* direction as it traces out an ellipse whose symmetry axes are inclined at a general angle relative to the x,y-coordinate system. With the help of analytical geometry, it is possible to show that the ellipse whose Jones vector is given by Eq. (14-9) is inclined at an angle α with respect to the x-axis, as shown in Figure 14-7. The angle of inclination is determined from

$$\tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\varepsilon}{E_{0x}^2 - E_{0y}^2} \tag{14-10}$$

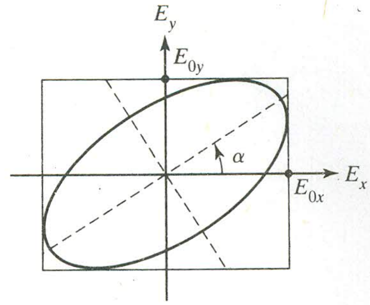

Figure 14-7 Elliptically polarized light oriented at an angle α relative to the x-axis.

The ellipse is situated in a rectangle of sides $2E_{0x}$ and $2E_{0y}$. In terms of the parameters A, B, and C, the derivation of Eq. (14-9) makes clear that

$$E_{0x} = A, \qquad E_{0y} = \sqrt{B^2 + C^2}, \qquad \text{and} \qquad \varepsilon = \tan^{-1}\left(\frac{C}{B}\right) \tag{14-11}$$

Example 14-1

Analyze the Jones vector given by

$$\begin{bmatrix} 3 \\ 2 + i \end{bmatrix}$$

to show that it represents elliptically polarized light.

Solution

The light has relative phase between $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ of $\varphi_y - \varphi_x = \varepsilon = \tan^{-1}\left(\frac{1}{2}\right) = 0.148\pi$. Since $E_{0x} = 3$ and $E_{0y} = \sqrt{2^2 + 1^2} = \sqrt{5}$, the inclination angle of the axis is given by

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{(2)(3)(\sqrt{5})\cos(0.148\pi)}{9 - 5}\right) = 35.8°$$

With this data the ellipse can be sketched as indicated in Figure 14-7. Moreover, from the general equation of an ellipse, we have

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cos\varepsilon = \sin^2\varepsilon \tag{14-12}$$

For this example, the equation of the ellipse is

$$\frac{E_x^2}{9} + \frac{E_y^2}{5} - 0.267E_xE_y = 0.2$$

When E_x lags E_y, the phase angle ε becomes negative and leads to the Jones vector (with A and C positive numbers) *representing* a clockwise rotation instead:

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ B - iC \end{bmatrix} \quad \text{clockwise rotation, general case}$$