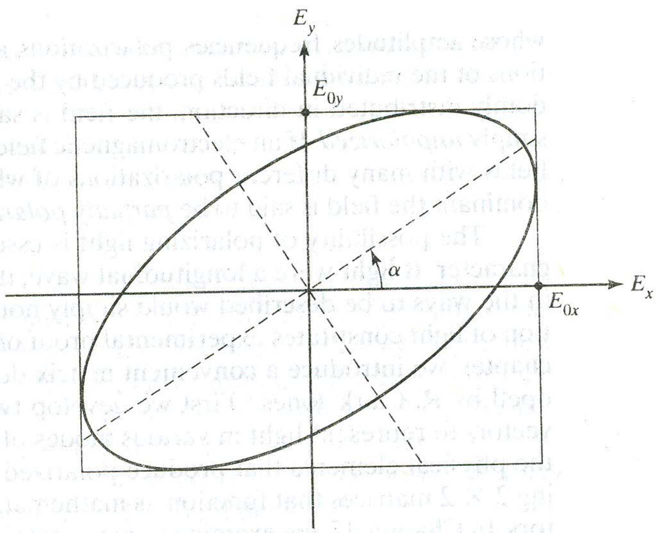


14 Matrix Treatment of Polarization

INTRODUCTION

The polarization of an electromagnetic wave was introduced in Chapter 4. There we noted that the *direction of the electric field vector* $\vec{\mathbf{E}}$ is known as the *polarization* of the electromagnetic wave. In this and the following chapter we extend our discussion of the properties and production of polarized light. As we noted in Chapter 4, the electric field associated with a plane monochromatic electromagnetic wave is perpendicular to the direction of the propagation of the energy carried by the wave. The same can be said of the magnetic field vector, which also maintains an orientation perpendicular to the electric field vector such that the direction of $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ is everywhere the direction of wave propagation. In general, plane monochromatic waves are *elliptically polarized*, in the sense that, over time, the tip of the electric field vector in a given plane perpendicular to the direction of energy propagation traces out an ellipse. Special cases of electromagnetic waves with elliptical polarization include *linearly polarized* waves in which the electric field vector always oscillates back and forth along a given direction in space and *circularly polarized* waves in which, over time, the tip of the electric field vector traces out a circle. These special cases are shown in Figure 4-12 and are worth reviewing. Monochromatic plane waves are idealized models of the electromagnetic waves produced by, for example, laser sources or a distant single-dipole oscillator. Any electromagnetic wave can be regarded as a superposition of plane electromagnetic waves with various frequencies, amplitudes, phases, and polarizations. "Ordinary" light, such as that produced by a hot filament, is typically produced by a number of independent atomic sources whose radiation is not synchronized. The resultant $\vec{\mathbf{E}}$-field vector consists of many components whose amplitudes, frequencies, polarizations, and phases differ. If the polarizations of the individual fields produced by the independent oscillators are randomly distributed in direction, the field is said to be *randomly polarized* or simply *unpolarized*. If an electromagnetic field consists of the superposition of fields with many different polarizations of which one is (or several are) predominant the field is said to be *partially polarized*.

The possibility of polarizing light is essentially related to its transverse character. If light were a longitudinal wave, the production of polarized light in the ways to be described would simply not be possible. Thus, the polarization of light constitutes experimental proof of its transverse character. In this chapter, we introduce a convenient matrix description of polarization developed by R. Clark Jones.[1] First we develop two-element column matrices or vectors to represent light in various modes of polarization. Then we examine the physical elements that produce polarized light and discover corresponding 2×2 matrices that function as mathematical operators on the Jones vectors. In Chapter 15, we examine in more detail the physical processes that are responsible for producing polarized light.

14-1 MATHEMATICAL REPRESENTATION OF POLARIZED LIGHT: JONES VECTORS

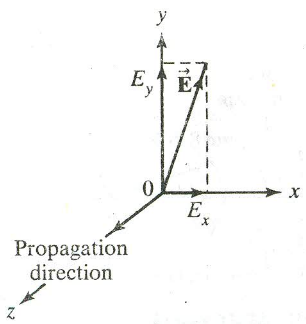


Figure 14-1 Representation of the instantaneous $\vec{\mathbf{E}}$-vector of a light wave traveling in the $+z$-direction.

Consider an electromagnetic wave propagating along the z-direction of the coordinate system shown is Figure 14-1. Let the electric field of this wave, at the origin of the axis system, be represented, at a given time, by the vector $\vec{\mathbf{E}}$ shown. Then, in terms of the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$,

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} \tag{14-1}$$

We write the *complex* field components for waves traveling in the $+z$-direction with amplitudes E_{0x} and E_{0y} and phases φ_x and φ_y as

$$\widetilde{E}_x = E_{0x}e^{i(kz-\omega t+\varphi_x)} \tag{14-2}$$

and

$$\widetilde{E}_y = E_{0y}e^{i(kz-\omega t+\varphi_y)} \tag{14-3}$$

Here, $E_x = \text{Re}\,(\widetilde{E}_x)$ and $E_y = \text{Re}\,(\widetilde{E}_y)$.

Using Eqs. (14-2) and (14-3) in Eq. (14-1) gives, for the complex field $\widetilde{\mathbf{E}}$,

$$\widetilde{\mathbf{E}} = E_{0x}e^{i(kz-\omega t+\varphi_x)}\hat{\mathbf{x}} + E_{0y}e^{i(kz-\omega t+\varphi_y)}\hat{\mathbf{y}}$$

which may also be written

$$\widetilde{\mathbf{E}} = [E_{0x}e^{i\varphi_x}\hat{\mathbf{x}} + E_{0y}e^{i\varphi_y}\hat{\mathbf{y}}]e^{i(kz-\omega t)} = \widetilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \tag{14-4}$$

The bracketed quantity in Eq. (14-4), separated into x- and y-components, is now recognized as the complex amplitude vector $\widetilde{\mathbf{E}}_0$ for the polarized wave. Since the state of polarization of the light is completely determined by the

[1] R. Clark Jones, "A New Calculus for the Treatment of Optical Systems," *Journal of the Optical Society*, Vol. 31 1941; 488.

relative amplitudes and phases of these components, we need concentrate only on the complex amplitude, written as a two-element matrix, or *Jones vector*,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} \tilde{E}_{0x} \\ \tilde{E}_{0y} \end{bmatrix} = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} \tag{14-5}$$

Let us determine the particular forms for Jones vectors that describe *linear*, *circular*, and *elliptical* polarization. In Figure 14-2a, vertically polarized light travels in the $+z$-direction out of the page with its $\vec{\mathbf{E}}$-oscillations along the y-axis. Since $\vec{\mathbf{E}}$ has a sinusoidally varying magnitude as it progresses, the electric field vector varies between, say, $A\hat{\mathbf{y}}$ and $-A\hat{\mathbf{y}}$. We display this behavior by a double-headed arrow, as shown in Figure 14-2a. As time progresses, the tip of the electric field vector traces out positions along the extent of the double-headed arrow. The field depicted in Figure 14-2a is represented by $E_{0x} = 0$ and $E_{0y} = A$. In the absence of an E_x-component, the phase φ_y may be set equal to zero for convenience. Then, by Eq. (14-5), the corresponding Jones vector is

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} 0 \\ A \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{linear polarization along } y$$

Furthermore, when only the mode of polarization is of interest, the amplitude A may be set equal to 1. The Jones vector for vertically linearly polarized light is then simply $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. This simplified form is the *normalized* form of the vector. In general, a vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is expressed in normalized form when

$$|a|^2 + |b|^2 = 1$$

Similarly, Figure 14-2b represents horizontally polarized light, for which, letting $E_{0y} = 0$, $\varphi_x = 0$, and $E_{0x} = A$,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{linear polarization along } x$$

On the other hand, Figure 14-2c represents linearly polarized light whose vibrations occur along a line making an angle α with respect to the x-axis. Both x- and y-components of $\vec{\mathbf{E}}$ are simultaneously present. Evidently this is a general case of linearly polarized light that reduces to the vertically polarized mode when $\alpha = 90°$ and to the horizontally polarized mode when $\alpha = 0°$. Notice that to produce the resultant vibration shown in Figure 14-3a, the two perpendicular vibrations $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ must be in phase. That is, they must pass through the origin together, increase along their respective positive axes together, reach their maximum values together, and then return together to continue the cycle. Figure 14-3a makes this sequence clear. Accordingly, since we require merely a *relative* phase of zero, we set $\varphi_x = \varphi_y = 0$. For a resultant with amplitude A, the perpendicular component amplitudes are $E_{0x} = A\cos\alpha$ and $E_{0y} = A\sin\alpha$. The Jones vector takes the form

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A\cos\alpha \\ A\sin\alpha \end{bmatrix} = A\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \quad \text{linear polarization at } \alpha \tag{14-6}$$

For the normalized form of the vector, we set $A = 1$, since $\cos^2\alpha + \sin^2\alpha = 1$. Notice that this general form does indeed reduce to the Jones vectors found for

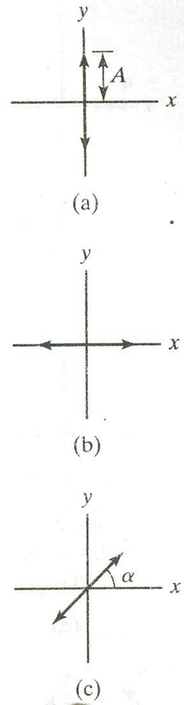


Figure 14-2 Representation of $\vec{\mathbf{E}}$-vectors of linearly polarized light with various orientations. In each case, the light is propagating in the positive z-direction.

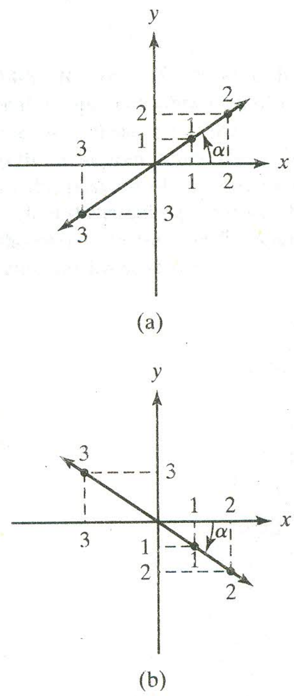


Figure 14-3 (a) Linearly polarized electric field vectors whose x- and y-components are in phase lie in the first and third quadrants. (b) Linearly polarized electric field vectors whose x- and y-components are π out of phase lie in the second and fourth quadrants.

the case $\alpha = 0°$ and $\alpha = 90°$. For other orientations, for example, $\alpha = 60°$,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} \cos(60°) \\ \sin(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

Alternatively, given a vector $\tilde{\mathrm{E}}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$, where a and b are real numbers, the inclination of the corresponding linearly polarized light is given by

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right) \tag{14-7}$$

Generalizing a bit, suppose α were a negative angle, as in Figure 14-3b. In this case, E_{0y} is a negative number, since the sine is an odd function, whereas E_{0x} remains positive. The negative sign ensures that the two vibrations are π out of phase, as needed to produce linearly polarized light with $\vec{\mathbf{E}}$-vectors lying in the second and fourth quadrants. Referring to Figure 14-3b again, this means that if the x-vibration is increasing from the origin along its positive direction, the y-vibration must be increasing from the origin along its negative direction. The resultant vibration takes place along a line with negative slope. Summarizing, a Jones vector $\begin{bmatrix} a \\ b \end{bmatrix}$ with both a and b real numbers, not both zero, represents linearly polarized light at inclination angle $\alpha = \tan^{-1}(b/a)$.

By now it may be apparent that in determining the resultant vibration due to two perpendicular components, we are in fact determining the appropriate *Lissajous figure*. If the phase difference between the vibrations is other than 0 or π, the resultant $\vec{\mathbf{E}}$-vector traces out an *ellipse* rather than a straight line. Of course, the straight line can be considered a special case of the ellipse, as can the circle. Figure 14-4 summarizes the sequence of Lissajous figures as a function of relative phase $\Delta\varphi = \varphi_y - \varphi_x$ for the general case $E_{0x} \neq E_{0y}$. Notice the sense of rotation of the tip of the $\vec{\mathbf{E}}$-vector around the ellipses shown in Figure 14-4, which makes the case $\Delta\varphi = \pi/4$, for example, different from the case $\Delta\varphi = 7\pi/4$. When $E_{0x} = E_{0y}$, the ellipses corresponding to $\Delta\varphi = \pi/2$ or $3\pi/2$ reduce to circles.

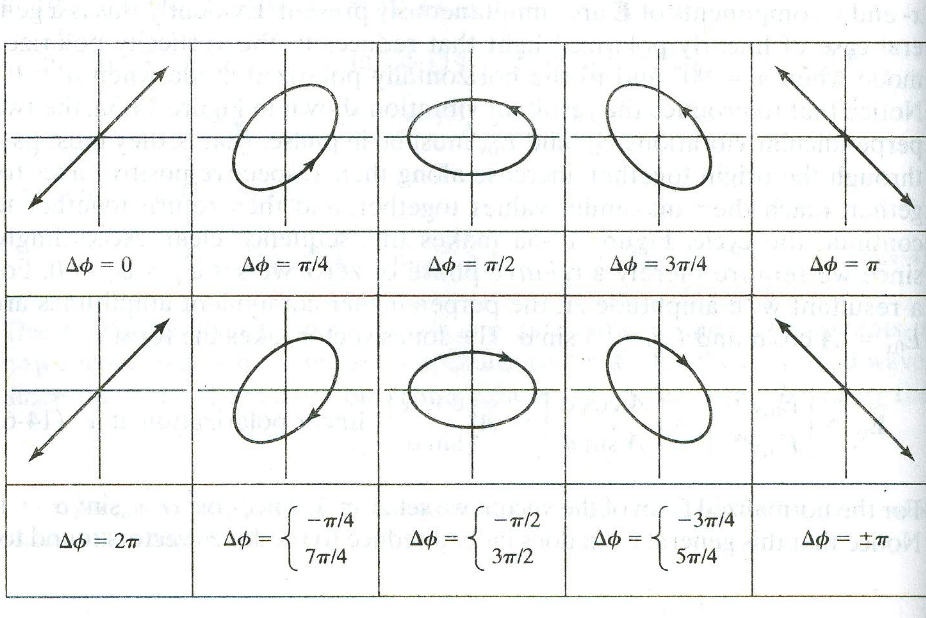


Figure 14-4 Lissajous figures as a function of relative phase for orthogonal vibrations of unequal amplitude. An angle lead greater than π may also be represented as an angle lag of less than π. For all figures we have adopted the phase lag convention $\Delta\varphi = \varphi_y - \varphi_x$.

Now suppose $E_{0x} = E_{0y} = A$ and E_x leads E_y by $\pi/2$. Then at the instant E_x has reached its maximum displacement—$+A$, for example—E_y is zero. A fourth of a period later, E_x is zero and $E_y = +A$, and so on. Figure 14-5 shows a few samples in the process of forming the resultant vibration. For the cases illustrated there, where the x-vibration leads the y-vibration, it is necessary to make $\varphi_y > \varphi_x$. This apparent contradiction results from our choice of phase in the formulation of the $\vec{\mathbf{E}}$-field in Eqs. (14-2) and (14-3), where the time-dependent term in the exponent is negative. To show this, let us observe the wave at $z = 0$ and choose $\varphi_x = 0$ and $\varphi_y = \varepsilon$, so that $\varphi_y > \varphi_x$. Equations (14-2) and (14-3) then become

$$\widetilde{E}_x = E_{0x}e^{-i\omega t}$$

$$\widetilde{E}_y = E_{0y}e^{-i(\omega t-\varepsilon)}$$

The negative sign before ε indicates a lag ε in the y-vibration relative to the x-vibration. To see that these equations represent the sequence in Figure 14-5, we take their real parts and set $E_{0x} = E_{0y} = A$ and $\varepsilon = \pi/2$, giving

$$E_x = A\cos\omega t$$

$$E_y = A\cos\left(\omega t - \frac{\pi}{2}\right) = A\sin\omega t$$

Recalling that $\omega = 2\pi\nu = 2\pi/T$, each of the cases in Figure 14-5 can be easily verified. Also, since

$$E^2 = E_x^2 + E_y^2 = A^2(\cos^2\omega t + \sin^2\omega t) = A^2$$

the tip of the resultant vector traces out a circle of radius A.

We now deduce the Jones vector for this case—where E_x leads E_y—taking $E_{0x} = E_{0y} = A$, $\varphi_x = 0$, and $\varphi_y = \pi/2$. Then,

$$\widetilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ Ae^{i\pi/2} \end{bmatrix} = A\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad (14\text{-}8)$$

To determine the normalized form of the vector, notice that $1^2 + |i|^2 = 1 + 1 = 2$, so that each element must be divided by $\sqrt{2}$ to produce unity. Thus the Jones vector $(1/\sqrt{2})\left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right]$ represents circularly polarized light when $\vec{\mathbf{E}}$ rotates *counterclockwise*, viewed head-on. This mode is called *left-circularly polarized* (LCP) light. Thus,

$$\widetilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{LCP}$$

Similarly, if E_y leads E_x by $\pi/2$, the result will again be circularly polarized light with clockwise rotation leading to *right-circularly polarized* (RCP) light. Replacing $\pi/2$ by $(-\pi/2)$ in Eq. (14-8) gives the normalized Jones vector,

$$\widetilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{RCP}$$

Notice that one of the elements in the Jones vector for circularly polarized light is now purely imaginary, and the magnitudes of the elements are the same.

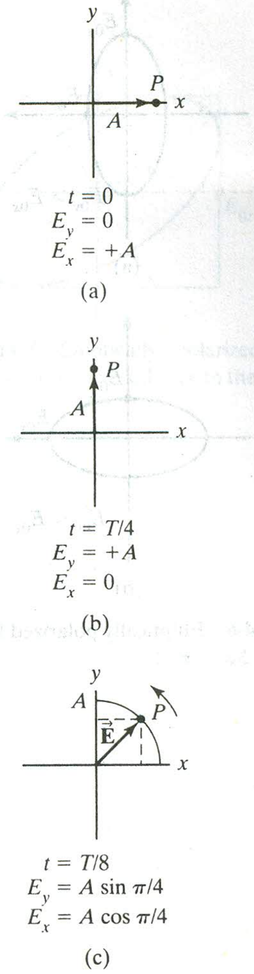


Figure 14-5 Resultant $\vec{\mathbf{E}}$-vibration due to orthogonal component vibrations of equal magnitude and phase difference of $\pi/2$, shown at three different times. The points P represent the position of the resultant. In (c) a sketch of the circular path traced by $\vec{\mathbf{E}}$ is also shown. Notice that the $\vec{\mathbf{E}}$-vector rotates counterclockwise in this case.

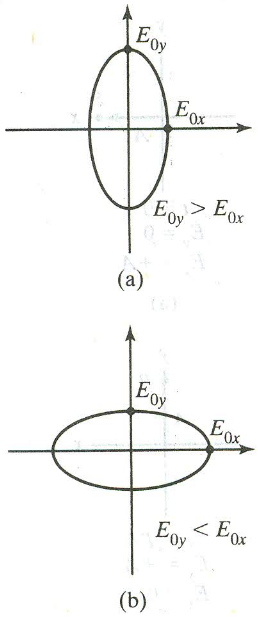


Figure 14-6 Elliptically polarized light for the case $\Delta\varphi = \pi/2$.

Given a particular mathematical form of the vector, the actual character of the light polarization may not always be immediately apparent. For example, the Jones vector $\begin{bmatrix} 2i \\ 2 \end{bmatrix}$ represents right-circularly polarized light since

$$\begin{bmatrix} 2i \\ 2 \end{bmatrix} = 2\begin{bmatrix} i \\ 1 \end{bmatrix} = 2i\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The prefactor of a Jones vector may affect the amplitude and, hence, the irradiance of the light but not the polarization mode. Prefactors such as 2 and $2i$ may therefore be ignored unless information regarding energy is required.

Next suppose that the phase difference between orthogonal vibrations $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ is still $\pi/2$, but $E_{0x} \neq E_{0y}$. In particular, let $E_{0x} = A$ and $E_{0y} = B$, where A and B are positive numbers. In this case, Eq. (14-8) should be modified to give

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ iB \end{bmatrix} \begin{array}{l} \text{counterclockwise} \\ \text{rotation} \end{array} \quad \text{and} \quad \tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ -iB \end{bmatrix} \begin{array}{l} \text{clockwise} \\ \text{rotation} \end{array}$$

These instances of elliptical polarization are illustrated in Figure 14-4 for $\Delta\varphi = \pi/2$ and $\Delta\varphi = 3\pi/2$. Notice that a lag of $\pi/2$ is equivalent to a lead of $3\pi/2$. The ellipse is oriented with its major axis along the x- or y-axis, as in Figure 14-6, depending on the relative magnitudes of E_{0x} and E_{0y}. In addition, either case may produce clockwise rotation of $\vec{\mathbf{E}}$ around the ellipse (when E_y leads E_x) or counterclockwise rotation (when E_x leads E_y). Based on these observations, we conclude that a Jones vector with elements of unequal magnitude, one of which is pure imaginary, represents elliptically polarized light oriented along the x,y-axes. The normalized forms of the Jones vectors now must include a prefactor of $1/\sqrt{A^2 + B^2}$.

It is also possible to produce elliptically polarized light with principal axes inclined to the x,y-axes, as evident in Figure 14-4. This situation occurs when the phase difference $\Delta\varphi$ between $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ is some angle other than $\Delta\varphi = 0, \pm\pi, \pm 2\pi, \pm m\pi$ (linear polarization) or $\Delta\varphi = \pm\pi/2, \pm 3\pi/2, \pm\left(m + \frac{1}{2}\right)\pi$ (circular or elliptical polarization oriented symmetrically about the x,y-axes). Here, $m = 0, \pm 1, \pm 2, \ldots$ For example, consider the case where E_x leads E_y by some positive angle ε, that is, $\varphi_y - \varphi_x = \varepsilon$. Taking $\varphi_x = 0$, $\varphi_y = \varepsilon$, $E_{0x} = A$, and $E_{0y} = b$ (with A and b positive), the Jones vector is

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ be^{i\varepsilon} \end{bmatrix}$$

Using Euler's theorem, we write

$$be^{i\varepsilon} = b(\cos\varepsilon + i\sin\varepsilon) = B + iC$$

The Jones vector for this general case is, then,

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ B + iC \end{bmatrix} \quad \text{counterclockwise rotation, general case} \qquad (14\text{-}9)$$

Here the identification of this form with counterclockwise rotation requires that A and C have the same sign. Since multiplying a Jones vector by an overall constant does not change the character of the polarization described by the Jones vector, we shall adopt the convention that A is positive. With that convention a positive imaginary part C of $\tilde{E}_{0y}$ indicates that the Jones vector represents counterclockwise rotation. Note that one of the elements of the Jones vector in Eq. (14-9) is now a complex number having both real and imaginary parts. The normalized form must be divided by $\sqrt{A^2 + B^2 + C^2}$. The Jones vector of Eq. (14-9) represents an electric field vector whose tip travels in a *counterclockwise* direction as it traces out an ellipse whose symmetry axes are inclined at a general angle relative to the x,y-coordinate system. With the help of analytical geometry, it is possible to show that the ellipse whose Jones vector is given by Eq. (14-9) is inclined at an angle α with respect to the x-axis, as shown in Figure 14-7. The angle of inclination is determined from

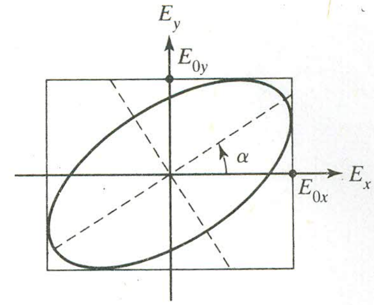


Figure 14-7 Elliptically polarized light oriented at an angle α relative to the x-axis.

$$\tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\varepsilon}{E_{0x}^2 - E_{0y}^2} \tag{14-10}$$

The ellipse is situated in a rectangle of sides $2E_{0x}$ and $2E_{0y}$. In terms of the parameters A, B, and C, the derivation of Eq. (14-9) makes clear that

$$E_{0x} = A, \qquad E_{0y} = \sqrt{B^2 + C^2}, \qquad \text{and} \qquad \varepsilon = \tan^{-1}\left(\frac{C}{B}\right) \tag{14-11}$$

Example 14-1

Analyze the Jones vector given by

$$\begin{bmatrix} 3 \\ 2 + i \end{bmatrix}$$

to show that it represents elliptically polarized light.

Solution

The light has relative phase between $\tilde{E}_{0x}$ and $\tilde{E}_{0y}$ of $\varphi_y - \varphi_x = \varepsilon = \tan^{-1}\left(\frac{1}{2}\right) = 0.148\pi$. Since $E_{0x} = 3$ and $E_{0y} = \sqrt{2^2 + 1^2} = \sqrt{5}$, the inclination angle of the axis is given by

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{(2)(3)\left(\sqrt{5}\right)\cos(0.148\pi)}{9 - 5}\right) = 35.8°$$

With this data the ellipse can be sketched as indicated in Figure 14-7. Moreover, from the general equation of an ellipse, we have

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cos\varepsilon = \sin^2\varepsilon \tag{14-12}$$

For this example, the equation of the ellipse is

$$\frac{E_x^2}{9} + \frac{E_y^2}{5} - 0.267E_xE_y = 0.2$$

When E_x lags E_y, the phase angle ε becomes negative and leads to the Jones vector (with A and C positive numbers) *representing* a clockwise rotation instead:

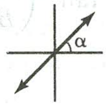
$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} A \\ B - iC \end{bmatrix} \quad \text{clockwise rotation, general case}$$

This form, together with the form representing counterclockwise rotation given in Eq. (14-9), are the most general forms of the Jones vector, including all those discussed previously as special cases.

Table 14-1 provides a convenient summary of the most common Jones vectors in their normalized forms. It should be emphasized that the forms given in Table 14-1 are not unique. First, any Jones vector may be multiplied by a real constant, changing amplitude but not polarization mode. Vectors in Table 14-1 have all been multiplied by prefactors, when necessary, to put them in normalized form. Thus, for example, the vector $\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right] = 2\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and so represents linearly polarized light making an angle of 45° with the x-axis and with amplitude of $2\sqrt{2}$. Second, each of the vectors in Table 14-1 can be multiplied by a factor of the form $e^{i\varphi}$, which has the effect of promoting the

TABLE 14-1 SUMMARY OF JONES VECTORS $\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix}$

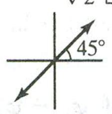
I. Linear Polarization ($\Delta\varphi = m\pi$)

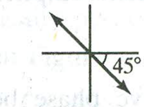
General: 

$\tilde{\mathbf{E}}_0 = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

Vertical: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Horizontal: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$


At + 45°: $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ At −45°: $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



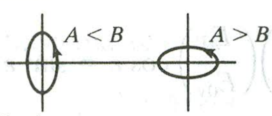


II. Circular Polarization $\left(\Delta\phi = \dfrac{\pi}{2}\right)$

Left:  $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

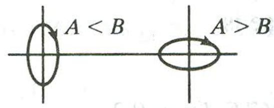
Right:  $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

III. Elliptical Polarization

Left: 

$\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ iB \end{bmatrix}$ $A > 0, B > 0$

$(\Delta\phi = (m + 1/2)\,\pi)$

Right: 

$\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ -iB \end{bmatrix}$ $A > 0, B > 0$

Left:  $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B + iC \end{bmatrix}$ $A > 0, C > 0$

$\left(\Delta\phi \neq \begin{cases} m\pi \\ (m + 1/2)\pi \end{cases}\right)$

Right:  $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B - iC \end{bmatrix}$ $A > 0, C > 0$

phase of each element by φ, that is, $\varphi_x \rightarrow \varphi_x + \varphi$ and $\varphi_y \rightarrow \varphi_y + \varphi$. Since the phase difference is unchanged in this process, the new vector represents the same polarization mode. Recall that the vectors in Table 14-1 were formulated by choosing, somewhat arbitrarily, $\varphi_x = 0$. Thus, for example, multiplying the vector representing left-circularly polarized light by $e^{i\pi/2} = i$,

$$i\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

produces an alternate form of the vector. Clearly, given the second form, one could deduce the standard form in Table 14-1 by extracting the factor i.

The usefulness of these Jones vectors will be demonstrated after Jones matrices representing polarizing elements are also developed. However, at this point it is already possible to calculate the result of the superposition of two or more polarized modes by adding their Jones vectors. The addition of left- and right-circularly polarized light, for example, gives

$$\begin{bmatrix} 1 \\ i \end{bmatrix} + \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1+1 \\ i-i \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

or linearly polarized light of twice the amplitude. We conclude that linearly polarized light can be regarded as being made up of left- and right-circularly polarized light in equal proportions. As another example, consider the superposition of vertically and horizontally linearly polarized light in phase:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The result is linearly polarized light at an inclination of 45°. Notice that the addition of orthogonal components of linearly polarized light is *not* unpolarized light, even though unpolarized light is often symbolized by such components. There is no Jones vector representing unpolarized or partially polarized light.[2]

14-2 MATHEMATICAL REPRESENTATION OF POLARIZERS: JONES MATRICES

Various devices can serve as optical elements that transmit light but modify the state of polarization. The physical mechanisms underlying their operation will be discussed in the next chapter. These polarizers can be generally described by 2×2 Jones matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where the matrix elements a, b, c, and d determine the manner in which the polarizers modify the polarization of the light that they transmit. Here, we will categorize such polarizers in terms of their effects, which are basically three in number.

[2]A matrix approach that handles partially polarized light, using 1×4 *Stokes vectors* and 4×4 *Mueller matrices* can be found in M. J. Walker, "Matrix Calculus and the Stokes Parameters of Polarized Radiation," *American Journal of Physics,* Vol. 22, 1954: 170 and W. A. Shurcliff, *Polarized Light: Production and Use* (Cambridge, Mass.: Harvard University Press, 1962).

Linear Polarizer

The linear polarizer selectively removes all or most of the $\vec{\mathbf{E}}$-vibrations in a given direction, while allowing vibrations in the perpendicular direction to be transmitted. In most cases, the selectivity is not 100% efficient, so that the transmitted light is partially polarized. Figure 14-8 illustrates the operation schematically. Unpolarized light traveling in the $+z$-direction passes through a linear polarizer, whose preferential axis of transmission, or transmission axis (TA), is vertical. The unpolarized light is represented by two perpendicular (x and y) vibrations, since any direction of vibration present can be resolved into components along these directions. The light transmitted includes components only along the TA direction and is therefore linearly polarized in the vertical, or y, direction. The horizontal components of the original light have been removed by absorption. In the figure, the process is assumed to be 100% efficient.

Phase Retarder

The phase retarder does not remove either of the orthogonal components of the $\vec{\mathbf{E}}$-vibrations, but rather introduces a phase difference between them. If light corresponding to each orthogonal vibration travels with a different speed through such a *retardation plate*, there will be a cumulative phase difference, $\Delta\varphi$, between the two waves as they emerge.

Symbolically, Figure 14-9 shows the effect of a retardation plate on unpolarized light in a case where the vertical component travels through the plate faster than the horizontal component. This is suggested by the schematic separation of the two components on the optical axis, although of course both

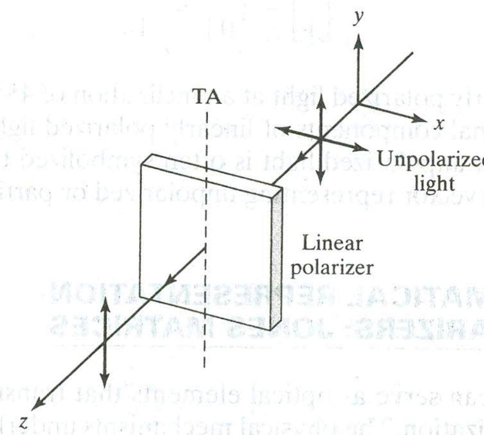


Figure 14-8 Operation of a linear polarizer.

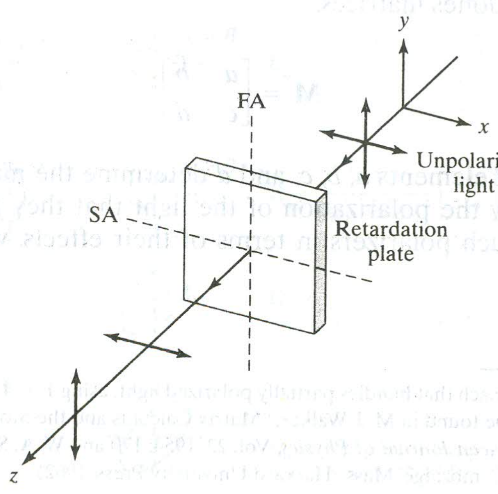


Figure 14-9 Operation of a phase retarder.

waves are simultaneously present at each point along the axis. The fast axis (FA) and slow axis (SA) directions of the plate are also indicated. When the net phase difference $\Delta\varphi = \pi/2$, the retardation plate is called a *quarter-wave plate*; when it is π, it is called a *half-wave plate*.

Rotator

The rotator has the effect of rotating the direction of linearly polarized light incident on it by some particular angle. Vertical linearly polarized light is shown incident on a rotator in Figure 14-10. The effect of the rotator element is to transmit linearly polarized light whose direction of vibration has been, in this case, rotated counterclockwise by an angle θ.

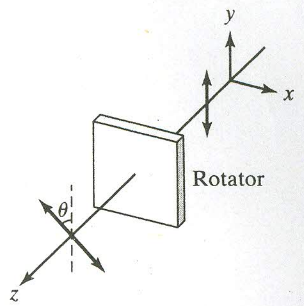


Figure 14-10 Operation of a rotator.

We desire now to create a set of matrices corresponding to these three types of polarizers so that just as the optical element alters the polarization mode of the actual light beam, an element matrix operating on a Jones vector will produce the same result mathematically. We adopt a pragmatic point of view in formulating appropriate matrices. For example, consider a linear polarizer with a transmission axis along the vertical, as in Figure 14-8. Let a 2×2 matrix representing the polarizer operate on vertically polarized light, and let the elements of the matrix to be determined be represented by letters *a, b, c,* and *d*. The resultant transmitted or product light in this case must again be vertically linearly polarized light. Symbolically,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This matrix equation—according to the rules of matrix multiplication—is equivalent to the algebraic equations

$$a(0) + b(1) = 0$$
$$c(0) + d(1) = 1$$

from which we conclude $b = 0$ and $d = 1$. To determine elements a and c, let the same polarizer operate on horizontally polarized light. In this case, no light is transmitted, or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The corresponding algebraic equations are now

$$a(1) + b(0) = 0$$
$$c(1) + d(0) = 0$$

from which $a = 0$ and $c = 0$. We conclude here without further proof, then, that the appropriate matrix is

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{linear polarizer, TA vertical} \qquad (14\text{-}13)$$

The matrix for a linear polarizer, TA horizontal, can be obtained in a similar manner and is included in Table 14-2, near the end of this chapter. Suppose next that the linear polarizer has a TA inclined at 45° to the x-axis. To keep matters as simple as possible we consider, in turn, the action of the polarizer on light linearly polarized in the same direction as—and perpendicular to—the TA of the polarizer. Light polarized along the same direction as the TA is represented by the Jones vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and light with a polarization direction that is perpendicular to the TA is represented by the Jones vector $\begin{bmatrix}1\\-1\end{bmatrix}$. Then, following the approach used earlier,

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Equivalently,

$$a+b=1$$
$$c+d=1$$
$$a-b=0$$
$$c-d=0$$

or $a=b=c=d=\frac{1}{2}$. Thus, the correct matrix is

$$\mathbf{M}=\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \quad \text{linear polarizer, TA at 45°} \tag{14-14}$$

In the same way, a general matrix representing a linear polarizer with TA at angle θ can be determined. This is left as an exercise for the student. The result is

$$\boxed{\mathbf{M}=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix} \quad \text{linear polarizer, TA at } \theta} \tag{14-15}$$

which includes Eqs. (14-13) and (14-14) as special cases, with $\theta = 90°$ and $\theta = 45°$, respectively.

Proceeding to the case of a phase retarder, we desire a matrix that will transform the elements

$$E_{0x}e^{i\varphi_x} \quad \text{into} \quad E_{0x}e^{i(\varphi_x+\varepsilon_x)}$$

and

$$E_{0y}e^{i\varphi_y} \quad \text{into} \quad E_{0y}e^{i(\varphi_y+\varepsilon_y)}$$

where ε_x and ε_y represent the advance in phase of the E_x- and E_y-components of the incident light. Of course, ε_x and ε_y may be negative quantities. Inspection is sufficient to show that this is accomplished by the matrix operation

$$\begin{bmatrix}e^{i\varepsilon_x} & 0\\0 & e^{i\varepsilon_y}\end{bmatrix}\begin{bmatrix}E_{0x}e^{i\varphi_x}\\E_{0y}e^{i\varphi_y}\end{bmatrix}=\begin{bmatrix}E_{0x}e^{i(\varphi_x+\varepsilon_x)}\\E_{0y}e^{i(\varphi_y+\varepsilon_y)}\end{bmatrix}$$

Thus, the general form of a matrix representing a phase retarder is

$$\mathbf{M}=\begin{bmatrix}e^{i\varepsilon_x} & 0\\0 & e^{i\varepsilon_y}\end{bmatrix} \quad \text{phase retarder} \tag{14-16}$$

As a special case, consider a quarter-wave plate (QWP) for which $|\varepsilon_x-\varepsilon_y|=\pi/2$. We distinguish the case for which $\varepsilon_y-\varepsilon_x=\pi/2$ (SA vertical) from the case for which $\varepsilon_x-\varepsilon_y=\pi/2$ (SA horizontal). In the former case, then, let $\varepsilon_x=-\pi/4$ and $\varepsilon_y=+\pi/4$. Obviously, other choices—an infinite number of them—are possible, so that Jones matrices, like Jones vectors, are not unique.

This particular choice, however, leads to a common form of the matrix, due to its symmetrical form:

$$\mathbf{M} = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad \text{QWP, SA vertical} \qquad (14\text{-}17)$$

In arriving at the last 2×2 matrix in Eq. (14-17), we used the relationship $e^{i\pi/4} = e^{-i\pi/4}e^{i\pi/2}$ and the indentity $i = e^{i\pi/2}$. Similarly, when $\varepsilon_x > \varepsilon_y$,

$$\mathbf{M} = e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \quad \text{QWP, SA horizontal} \qquad (14\text{-}18)$$

Corresponding matrices for half-wave plates (HWP), where $|\varepsilon_x - \varepsilon_y| = \pi$, are given by

$$\mathbf{M} = \begin{bmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{HWP, SA vertical} \qquad (14\text{-}19)$$

$$\mathbf{M} = \begin{bmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix} = e^{i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{HWP, SA horizontal} \qquad (14\text{-}20)$$

The elements of the matrices are identical in this case, since advancement of phase by π is physically equivalent to retardation by π. The only difference lies in the prefactors that modify the phases of all the elements of the Jones vector in the same way and hence do not affect interpretation of the results.

The requirement for a rotator of angle β is that an $\vec{\mathbf{E}}$-vector, oscillating linearly at angle θ and with normalized components $\cos\theta$ and $\sin\theta$, be converted to one that oscillates linearly at angle $(\theta + \beta)$. That is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta + \beta) \\ \sin(\theta + \beta) \end{bmatrix}$$

Thus, the matrix elements must satisfy

$$a\cos\theta + b\sin\theta = \cos(\theta + \beta)$$

$$c\cos\theta + d\sin\theta = \sin(\theta + \beta)$$

From the trigonometric identities for the sine and cosine of the sum of two angles,

$$\cos(\theta + \beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$$

$$\sin(\theta + \beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$$

it follows that

$$a = \cos\beta \qquad b = -\sin\beta$$

$$c = \sin\beta \qquad d = \cos\beta$$

so that the desired rotator matrix is

$$\mathbf{M} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \quad \text{rotator through angle } +\beta \qquad (14\text{-}21)$$

The Jones matrices derived in this chapter are summarized in Table 14-2.

TABLE 14-2 SUMMARY OF JONES MATRICES

I. Linear polarizers

TA horizontal	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	TA vertical	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	TA at 45° to horizontal	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

II. Phase retarders

		General	$\boxed{\begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix}}$
QWP, SA vertical	$e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$	QWP, SA horizontal	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$
HWP, SA vertical	$e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	HWP, SA horizontal	$e^{i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

III. Rotator

Rotator	$(\theta \rightarrow \theta + \beta)$	$\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

As an important example, consider the production of circularly polarized light by combining a linear polarizer with a QWP. Let the linear polarizer (LP) produce light vibrating at an angle of 45°, as in Figure 14-11, which is then transmitted by a QWP with SA horizontal. In this arrangement, the light incident on the QWP is divided equally between fast and slow axes. On emerging, a phase difference of $\pi/2$ results in circularly polarized light. With the Jones calculus, this process is equivalent to allowing the QWP matrix to operate on the Jones vector for the linearly polarized light,

$$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{i\pi/4}\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

giving right-circularly polarized light (see Table 14-1). If the fast and slow axes of the QWP are interchanged, a similar calculation shows that the result is left-circularly polarized instead.

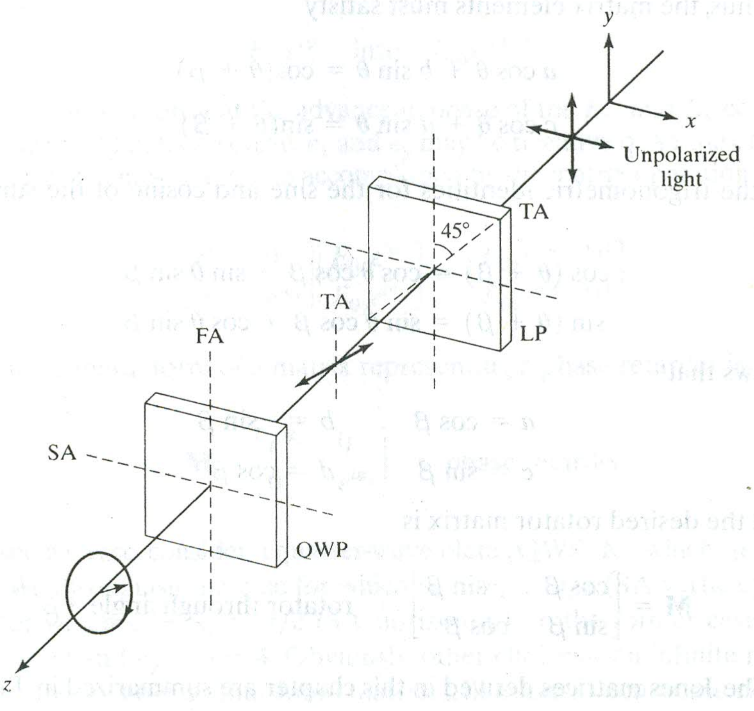


Figure 14-11 Production of right circularly polarized light.

Example 14-2

Consider the result of allowing left-circularly polarized light to pass through an eighth-wave plate.

Solution

We first need a matrix that represents the eighth-wave plate, that is, a phase retarder that introduces a relative phase of $2\pi/8 = \pi/4$. Thus, letting $\varepsilon_x = 0$,

$$\mathbf{M} = \begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

This matrix then operates on the Jones vector representing the left-circularly polarized light:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ ie^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{i3\pi/4} \end{bmatrix}$$

The resultant Jones vector indicates that the light is elliptically polarized, and the components are out of phase by $3\pi/4$. Using Euler's equation to expand $e^{i3\pi/4}$, we obtain

$$e^{i3\pi/4} = -\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)$$

and using our standard notation for this case, we have

$$\begin{bmatrix} \tilde{E}_{0x} \\ \tilde{E}_{0y} \end{bmatrix} = \begin{bmatrix} A \\ B + iC \end{bmatrix}, \quad \text{where } A = 1, B = -\frac{1}{\sqrt{2}}, \text{ and } C = \frac{1}{\sqrt{2}}$$

Since A and C have the same sign, the output field vector represents elliptically polarized light with counterclockwise rotation. Comparing this matrix with the general form in Eq. (14-5), we determine that $E_{0x} = A = 1$ and $E_{0y} = \sqrt{B^2 + C^2} = 1$. Making use of Eq. (14-10), we also determine that $\alpha = -45°$.

Of course, the Jones calculus can handle a case where polarized light is transmitted by a series of polarizing elements, since the product of element matrices can represent an overall *system matrix*. If light represented by Jones vector $\mathbf{V}$ passes sequentially through a series of polarizers represented by $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \ldots, \mathbf{M}_m$, so that $(\mathbf{M}_m \ldots \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1)\mathbf{V} = \mathbf{M}_s\mathbf{V}$, then the system matrix is given by $\mathbf{M}_s = \mathbf{M}_m \ldots \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$.

PROBLEMS

14-1 Derive the Jones matrix, Eq. (14-15), representing a linear polarizer whose transmission axis is at an arbitrary angle θ with respect to the horizontal.

14-2 Write the normalized Jones vectors for each of the following waves, and describe completely the state of polarization of each.

a. $\vec{\mathbf{E}} = E_0 \cos(kz - \omega t)\hat{\mathbf{x}} - E_0 \cos(kz - \omega t)\hat{\mathbf{y}}$

b. $\vec{\mathbf{E}} = E_0 \sin 2\pi\left(\frac{z}{\lambda} - vt\right)\hat{\mathbf{x}} + E_0 \sin 2\pi\left(\frac{z}{\lambda} - vt\right)\hat{\mathbf{y}}$

c. $\vec{\mathbf{E}} = E_0 \sin(kz - \omega t)\hat{\mathbf{x}} + E_0 \sin\left(kz - \omega t - \frac{\pi}{4}\right)\hat{\mathbf{y}}$

d. $\vec{\mathbf{E}} = E_0 \cos(kz - \omega t)\hat{\mathbf{x}} + E_0 \cos\left(kz - \omega t + \frac{\pi}{2}\right)\hat{\mathbf{y}}$

14-3 Describe as completely as possible amplitude, wave direction, and the state of polarization of each of the following waves.

a. $\vec{\mathbf{E}} = 2E_0\hat{\mathbf{x}}e^{i(kz-\omega t)}$

b. $\vec{\mathbf{E}} = E_0(3\hat{\mathbf{x}} + 4\hat{\mathbf{y}})e^{i(kz-\omega t)}$

c. $\vec{\mathbf{E}} = 5E_0(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i(kz+\omega t)}$

14-4 Two linearly polarized beams are given by

$$\vec{\mathbf{E}}_1 = E_{01}(\hat{\mathbf{x}} - \hat{\mathbf{y}})\cos(kz - \omega t) \quad \text{and}$$
$$\vec{\mathbf{E}}_2 = E_{02}\left(\sqrt{3}\hat{\mathbf{x}} + \hat{\mathbf{y}}\right)\cos(kz - \omega t)$$

Determine the angle between their directions of polarization by (a) forming their Jones vectors and finding the vibration direction of each and (b) forming the dot product of their vector amplitudes.

14-5 Find the character of polarized light after passing in turn through (a) a half-wave plate with slow axis at 45°; (b) a linear polarizer with transmission axis at 45°; (c) a quarter-wave plate with slow axis horizontal. Assume the original light to be linearly polarized vertically. Use the matrix approach and analyze the final Jones vector to describe the product light. (*Hint:* First find the effect of the HWP alone on the incident light.)

14-6 Write the equations for the electric fields of the following waves in exponential form:

a. A linearly polarized wave traveling in the x-direction. The $\vec{\mathbf{E}}$-vector makes an angle of 30° relative to the y-axis.

b. A right-elliptically polarized wave traveling in the y-direction. The major axis of the ellipse is in the z-direction and is twice the minor axis.

c. A linearly polarized wave traveling in the x,y-plane in a direction making an angle of 45° relative to the x-axis. The direction of polarization is the z-direction.

14-7 Determine the conditions on the elements A, B, and C of the general Jones vector (Eq. 14-9), representing polarized light, that lead to the following special cases: (a) linearly polarized light; (b) elliptically polarized light with major axis aligned along a coordinate axis; (c) circularly polarized light. In each case, from the meanings of A, B, C, deduce the possible values of phase difference between component vibrations.

14-8 Write a computer program that will determine E_y-values of elliptically polarized light from the equation for the ellipse, Eq. (14-12), with input constants A, B, and C and variable input parameter E_x. Plot the ellipse for the example given in the text,

$$\widetilde{\mathbf{E}}_0 = \begin{bmatrix} 3 \\ 2 + i \end{bmatrix}$$

14-9 Specify the polarization mode for each of the following Jones vectors:

a. $\begin{bmatrix} 3i \\ i \end{bmatrix}$ **b.** $\begin{bmatrix} i \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} 4i \\ 5 \end{bmatrix}$ **d.** $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$

e. $\begin{bmatrix} 2 \\ 2i \end{bmatrix}$ **f.** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

g. $\begin{bmatrix} 2 \\ 6 + 8i \end{bmatrix}$

14-10 Linearly polarized light with an electric field $\vec{\mathbf{E}}$ is inclined at $+30°$ relative to the x-axis and is transmitted by a QWP with SA horizontal. Describe the polarization mode of the product light.

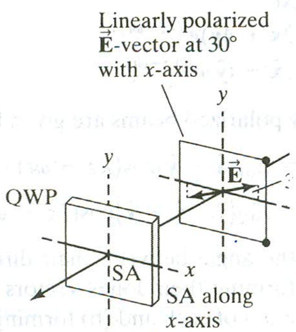


Figure 14-12 Problem 14-10.

14-11 Using the Jones calculus, show that the effect of a HWP on light linearly polarized at inclination angle α is to rotate the polarization through an angle of 2α. The HWP may be used in this way as a "laser-line rotator," allowing the polarization of a laser beam to be rotated without having to rotate the laser.

14-12 An important application of the QWP is its use in an "isolator." For example, to prevent feedback from interferometers into lasers by front-surface, back reflections, the beam is first allowed to pass through a combination of linear polarizer and QWP, with OA of the QWP at 45° to the TA of the polarizer. Consider what happens to such light after reflection from a plane surface and transmission back through this optical device.

14-13 Light linearly polarized with a horizontal transmission axis is sent through another linear polarizer with TA at 45° and then through a QWP with SA horizontal. Use the Jones matrix technique to determine and describe the product light.

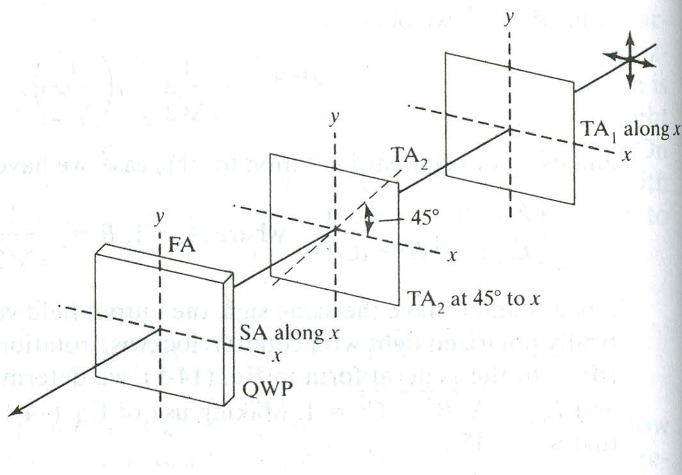


Figure 14-13 Problem 14-13.

14-14 A light beam passes consecutively through (1) a linear polarizer with TA at 45° clockwise from vertical, (2) a QWP with SA vertical, (3) a linear polarizer with TA horizontal, (4) a HWP with FA horizontal, (5) a linear polarizer with TA vertical. What is the nature of the product light?

14-15 Unpolarized light passes through a linear polarizer with TA at 60° from the vertical, then through a QWP with SA horizontal, and finally through another linear polarizer with TA vertical. Determine, using Jones matrices, the character of the light after passing through (a) the QWP and (b) the final linear polarizer.

14-16 Determine the state of polarization of circularly polarized light after it is passed normally through (a) a QWP; (b) an eighth-wave plate. Use the matrix method to support your answer.

14-17 Show that the matrix $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ represents a right-circular polarizer, converting any incident polarized light into right circularly-polarized light. What is the proper matrix to represent a left-circular polarizer?

14-18 Show that elliptical polarization can be regarded as a combination of circular and linear polarizations.

14-19 Derive the equation of the ellipse for polarized light given in Eq. (14-12). (*Hint:* Combine the E_x and E_y equations for the general case of elliptical polarization, eliminating the space and time dependence between them.)

14-20 **a.** Identify the state of polarization corresponding to the Jones vector

$$\begin{bmatrix} 2 \\ 3e^{i\pi/3} \end{bmatrix}$$

and write it in the standard, normalized form of Table 14-1.

b. Let this light be transmitted through an element that rotates linearly polarized light by +30°. Find the new, normalized form and describe the result.

14-21 Determine the nature of the polarization that results from Eq. (14-12) when (a) $\varepsilon = \pi/2$; (b) $E_{0x} = E_{0y} = E_0$; (c) both (a) and (b); (d) $\varepsilon = 0$.

14-22 A quarter-wave plate is placed between crossed polarizers such that the angle between the polarizer TA of the first polarizer and the QWP fast axis is θ. How does the polarization of the emergent light vary as a function of θ?