At wavelengths for which $\lambda \ll \lambda_i$ the ith term becomes approximately proportional to λ^2, and for $\lambda \gg \lambda_i$ it becomes approximately constant. As an example, the dispersion in fused silica, illustrated in Example 5.6-1, is well described by three resonances. For some materials the Sellmeier equation is conveniently approximated by a power series.

D. Optics of Conductive Media

Conductive materials such as metals, semiconductors, doped dielectrics, and ionized gases have free electric charges and an associated electric current density $\mathcal{J}$. In such media, the first of the source-free Maxwell's equations, (5.1-7), must be modified by including the current density $\mathcal{J}$ along with the displacement current density $\partial\mathcal{D}/\partial t$, so that

$$\nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}. \tag{5.5-29}$$

The other three Maxwell's equations remain the same. For a monochromatic wave of angular frequency ω, this equation takes the form

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J}, \tag{5.5-30}$$

which is a modified version of (5.3-2).

For a medium with linear dielectric properties, $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_o(1 + \chi)\mathbf{E}$. Similarly, for a medium with linear conductive properties and **conductivity** σ, the electric current density is proportional to the electric field,

$$\mathbf{J} = \sigma\mathbf{E}, \tag{5.5-31}$$

which is a form of Ohm's law [see (18.1-13) and (18.1-14)]. The right-hand side of (5.5-30) then becomes $(j\omega\epsilon + \sigma)\mathbf{E} = j\omega(\epsilon + \sigma/j\omega)\mathbf{E}$, so that

$$\nabla \times \mathbf{H} = j\omega\epsilon_e\mathbf{E}, \tag{5.5-32}$$

where the effective electric permittivity ϵ_e is

$$\boxed{\epsilon_e = \epsilon + \frac{\sigma}{j\omega}.} \tag{5.5-33}$$

Effective Permittivity

The effective permittivity ϵ_e is a complex frequency-dependent parameter that represents a combination of the dielectric and conductive properties of the medium. Since the second term in (5.5-33) varies inversely with frequency, the contribution of the conductive component diminishes as the frequency increases.

Moreover, since (5.5-32) takes the same form as the analogous equation for a dielectric medium, the laws of wave propagation derived in Secs. 5.3–5.5 are applicable even in the presence of conductivity. Thus, the wavenumber in (5.5-2) and (5.5-3) becomes $k = \beta - j\frac{1}{2}\alpha = \omega\sqrt{\epsilon_e\mu_o}$, and the impedance in (5.5-6) becomes $\eta = \sqrt{\mu_o/\epsilon_e}$, while the refractive index n and the attenuation coefficient α in (5.5-5) are determined from the complex equation $n - j\alpha/2k_o = \sqrt{\epsilon_e/\epsilon_o}$. When $\sigma/\omega \gg \epsilon$, conductive effects dominate and $\epsilon_e \approx \sigma/j\omega$. We then have $n - j\alpha/2k_o \approx \sqrt{\sigma/j\omega\epsilon_o}$ and $\eta \approx \sqrt{j\omega\mu_o/\sigma}$, from which we obtain

$$n \approx \sqrt{\sigma/2\omega\epsilon_o} \tag{5.5-34}$$

$$\alpha \approx \sqrt{2\omega\mu_o\sigma} \tag{5.5-35}$$

$$\eta \approx (1+j)\sqrt{\omega\mu_o/2\sigma}\,, \tag{5.5-36}$$

Conductive Medium

where we have made use of the relation $k_o = \omega/c_o = \omega\sqrt{\epsilon_o\mu_o}$. The optical intensity is attenuated by a factor e^{-1} at a depth $d_p = 1/\alpha = 1/\sqrt{2\omega\mu_o\sigma}$, which is known as the **penetration depth** or **skin depth**. † Both d_p and n vary as $1/\sqrt{\omega}$.

For metals, σ is very large and therefore so is α, indicating that optical waves are significantly attenuated as they cross the surface of the material. However, the impedance η is very small, so these materials are highly reflective (see Exercise 6.2-2).

EXAMPLE 5.5-1. ***Penetration Depth and Refractive Index of Copper.*** Copper has a conductivity of $\sigma = 0.58 \times 10^8$ $(\Omega\text{-m})^{-1}$, so that the penetration depth is a scant $d_p = 1.9$ nm at a wavelength $\lambda_o = 1$ μm. In accordance with (5.5-34) and (5.5-35), the refractive index is related to the penetration depth via $n = \sigma\eta_o d_p$, which, for the case at hand, turns out to be $n = 41.6$.

The Drude Model

Since the relation between $\mathcal{J}$ and $\mathcal{E}$ is dynamic, the conductivity σ must be frequency dependent with a finite bandwidth. Treating the conduction electrons as independent particles in an ideal gas that move freely between scattering events, the Drude model prescribes a frequency-dependent conductivity

$$\sigma = \frac{\sigma_0}{1 + j\omega\tau}, \tag{5.5-37}$$

where σ_0 is the low-frequency conductivity and τ is a relaxation time. It then follows from (5.5-33) that

$$\epsilon_e = \epsilon + \frac{\sigma_0}{j\omega(1 + j\omega\tau)}. \tag{5.5-38}$$

For $\omega \gg 1/\tau$, (5.5-38) provides $\epsilon_e \approx \epsilon - \sigma_0/\omega^2\tau$. It is apparent that the conductivity then reduces the real part of the permittivity of the medium, acting like a negative contribution to the dielectric constant, with a functional form that is inversely proportional to the square of the frequency. In particular, if the medium has free-space-like dielectric properties with $\epsilon = \epsilon_0$, the effective permittivity can be written as

$$\epsilon_e = \epsilon_o\left(1 - \frac{\omega_p^2}{\omega^2}\right), \tag{5.5-39}$$

where $\omega_p = \sqrt{\sigma_0/\epsilon_o\tau}$ is known as the **plasma frequency**.

A simple classical microscopic theory provides an underlying rationale for the results of the Drude model. The construct is similar to that of the Lorentz model; since the

† The penetration depth is sometimes defined as the distance over which the field, rather than the intensity, is attenuated by a factor e^{-1}.

electrons of interest in a conductive medium are free rather than bound, however, the restoring force is excluded ($\kappa = 0$) as is the damping ($\sigma = 0$). Under these conditions, the Lorentz equation of motion (5.5-16) becomes $m\,d^2x/dt^2 = -e\mathcal{E}$, so that the corresponding polarization density $\mathcal{P} = -Nex$ obeys the simple equation $d^2\mathcal{P}/dt^2 = (Ne^2/m)\mathcal{E}$, where N is electron density of the medium. For a field oscillating at an angular frequency ω, this gives rise to $-\omega^2 P = (Ne^2/m)E$, which is equivalent to a conductivity-related reduction of the dielectric constant of magnitude $P/\epsilon_o E = -(Ne^2/\epsilon_o m)/\omega^2$. This is consistent with (5.5-39), with a plasma frequency given by

$$\omega_p = \sqrt{\frac{Ne^2}{\epsilon_o m}}. \tag{5.5-40}$$

Combining (5.5-40) with the Drude result $\omega_p = \sqrt{\sigma_0/\epsilon_o\tau}$ yields $\sigma_0 = Ne^2\tau/m$, which accords with (18.1-13).

It is apparent from (5.5-39) that wave propagation in a medium described by the Drude model assumes distinctly different behavior below, at, and above the plasma frequency, as illustrated in Fig. 5.5-10.

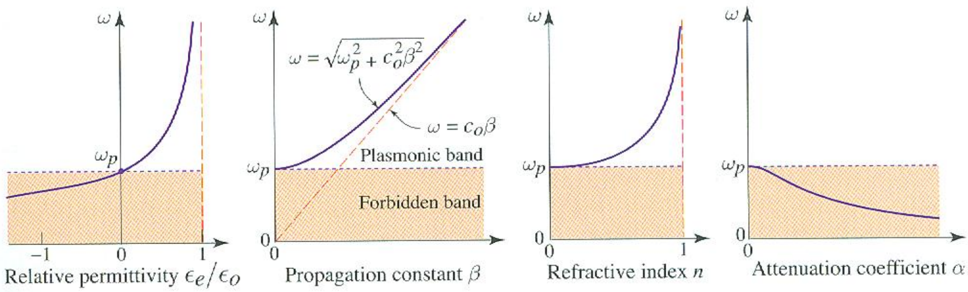

Figure 5.5-10 Frequency dependence of the relative permittivity ϵ_e/ϵ_o, propagation constant β, refractive index n, and attenuation coefficient α of a medium described by the Drude model.

- At frequencies below the plasma frequency ($\omega < \omega_p$), the effective permittivity is negative, so that $k = \omega\sqrt{\epsilon_e\mu_o}$ is imaginary, corresponding to attenuation without propagation. This spectral band may therefore be regarded as a **forbidden band**. The attenuation coefficient $\alpha = 2k_o(\omega_p^2/\omega^2 - 1)^{1/2}$ decreases monotonically with increasing frequency and vanishes at the plasma frequency. A negative permittivity also corresponds to an imaginary impedance. Therefore, at the boundary between an ordinary medium with real impedance and a conductive medium with imaginary impedance, the light is fully reflected (see Sec. 6.2) so that the interface serves as a perfect mirror.
- At frequencies above the plasma frequency ($\omega > \omega_p$), the effective permittivity is positive and real so that the conductive medium behaves like a lossless dielectric, albeit with unique dispersion characteristics. The propagation constant becomes $\beta = (\omega^2 - \omega_p^2)^{1/2}/c_o$ while the refractive index $n = (1 - \omega_p^2/\omega^2)^{1/2}$ lies below unity and is very small near the plasma frequency. This spectral band is referred to as the **plasmonic band**.
- At the plasma frequency, $\omega = \omega_p$, the propagation constant $\beta = 0$ so that the wave does not travel in the conductive medium. However, the electric current density oscillates and the free electrons undergo longitudinal oscillations; the

quantum quasi-particle associated with these oscillations is called a **plasmon** (much as a photon is associated with an optical field, as discussed in Chapter 12).

In most metals, the plasma frequency lies in the ultraviolet so that they are reflective and shiny in the visible band. Some metals, such as copper, have a plasma frequency in the visible band so that they reflect only a portion of the visible spectrum and therefore have a distinct color. In doped semiconductors, the plasma frequency is usually in the infrared.

5.6 PULSE PROPAGATION IN DISPERSIVE MEDIA

The propagation of pulses of light in dispersive media is important in many applications including optical fiber communication systems, as will be discussed in detail in Chapters 9 and 24. As indicated above, a dispersive medium is characterized by a frequency-dependent refractive index and absorption coefficient, so that monochromatic waves of different frequencies travel through the medium with different velocities and undergo different attenuations. Since a pulse of light comprises a sum of many monochromatic waves, each of which is modified differently, the pulse is delayed and broadened (dispersed in time); in general its shape is also altered. In this section we provide a simplified analysis of these effects; a detailed description is deferred to Chapter 22.

Group Velocity

Consider a pulsed plane wave traveling in the z direction through a lossless dispersive medium with refractive index $n(\omega)$. Following the example set forth in Sec. 2.6, assume that the initial complex wavefunction at $z = 0$ is $U(0, t) = \mathcal{A}(t)\exp(j\omega_0 t)$, where ω_0 is the central angular frequency and $\mathcal{A}(t)$ is the complex envelope of the wave. It will be shown below that if the dispersion is weak, i.e., if n varies slowly within the spectral bandwidth of the wave, then the complex wavefunction at a distance z is approximately $U(z, t) = \mathcal{A}(t - z/v)\exp[j\omega_0(t - z/c)]$, where $c = c_o/n(\omega_0)$ is the speed of light in the medium at the central frequency, and v is the velocity at which the envelope travels (see Fig. 5.6-1). The parameter v, called the **group velocity**, is given by

$$\boxed{\frac{1}{v} = \beta' = \frac{d\beta}{d\omega},} \tag{5.6-1}$$

Group Velocity

where $\beta = \omega n(\omega)/c_o$ is the frequency-dependent propagation constant and the derivative in (5.6-1), which is often denoted β', is evaluated at the central frequency ω_0. The group velocity is a characteristic of the dispersive medium, and generally varies with the central frequency. The corresponding time delay $\tau_d = z/v$ is called the **group delay**.

Since the phase factor $\exp[j\omega_0(t - z/c)]$ is a function of $t - z/c$, the speed of light c, given by $1/c = \beta(\omega_0)/\omega_0$, is often called the **phase velocity**. In an ideal (nondispersive) medium, $\beta(\omega) = \omega/c$ whereupon $v = c$ and the group and phase velocities are identical.

☐ **Derivation of the Formula for the Group Velocity.** The proof of (5.6-1) relies on a Fourier decomposition of the envelope $\mathcal{A}(t)$ into its constituent harmonic functions. A component of frequency Ω, assumed to have a Fourier amplitude $A(\Omega)$, corresponds to a monochromatic wave of frequency