Since $\mathrm{S}_1^2 + \mathrm{S}_2^2 + \mathrm{S}_3^2 = \mathrm{S}_0^2$, only three of the four components of the Stokes vector are independent; they completely define the intensity and the state of polarization of the light. A generalization of the Stokes parameters suitable for describing partially coherent light is presented in Sec. 11.4.

We conclude that there are a number of equivalent representations for describing the state of polarization of an optical field: (1) the polarization ellipse, (2) the Poincaré sphere, and (3) the Stokes vector. Yet another equivalent representation, the Jones vector, is introduced in the following section.

B. Matrix Representation

The Jones Vector

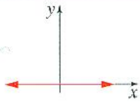
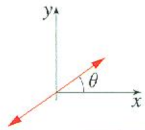
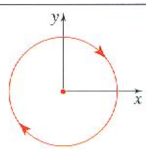
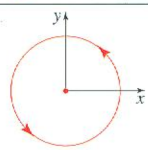
As indicated above, a monochromatic plane wave of frequency ν traveling in the z direction is completely characterized by the complex envelopes $A_x = \mathrm{a}_x \exp(j\varphi_x)$ and $A_y = \mathrm{a}_y \exp(j\varphi_y)$ of the x and y components of the electric-field vector. These complex quantities may be written in the form of a column matrix known as the **Jones vector**:

$$\mathbf{J} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}. \qquad (6.1\text{-}10)$$

Given $\mathbf{J}$, we can determine the total intensity of the wave, $I = (|A_x|^2 + |A_y|^2)/2\eta$, and use the ratio $\mathrm{R} = \mathrm{a}_y/\mathrm{a}_x = |A_y|/|A_x|$ and the phase difference $\varphi = \varphi_y - \varphi_x = \arg\{A_y\} - \arg\{A_x\}$ to determine the orientation and shape of the polarization ellipse, as well as the Poincaré sphere and the Stokes parameters.

The Jones vectors for some special polarization states are provided in Table 6.1-1. The intensity in each case has been normalized so that $|A_x|^2 + |A_y|^2 = 1$ and the phase of the x component is taken to be $\varphi_x = 0$.

Table 6.1-1 Jones vectors of linearly polarized (LP) and right- and left-circularly polarized (RCP, LCP) light.

LP in x direction	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$		LP at angle θ	$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$	
RCP	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix}$		LCP	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix}$	

Orthogonal Polarizations

Two polarization states represented by the Jones vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ are said to be orthogonal if the inner product between $\mathbf{J}_1$ and $\mathbf{J}_2$ is zero. The inner product is defined by

$$(\mathbf{J}_1, \mathbf{J}_2) = A_{1x}A_{2x}^* + A_{1y}A_{2y}^*, \qquad (6.1\text{-}11)$$

where A_{1x} and A_{1y} are the elements of $\mathbf{J}_1$ and A_{2x} and A_{2y} are the elements of $\mathbf{J}_2$. An example of orthogonal Jones vectors are the linearly polarized waves in the x and

y directions, or any other pair of orthogonal directions. Another example is provided by right and left circularly polarized waves.

Expansion of Arbitrary Polarization as a Superposition of Two Orthogonal Polarizations

An arbitrary Jones vector $\mathbf{J}$ can always be analyzed as a weighted superposition of two orthogonal Jones vectors, say $\mathbf{J}_1$ and $\mathbf{J}_2$, that form the expansion basis; thus $\mathbf{J} = \alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2$. If $\mathbf{J}_1$ and $\mathbf{J}_2$ are normalized such that $(\mathbf{J}_1, \mathbf{J}_1) = (\mathbf{J}_2, \mathbf{J}_2) = 1$, the expansion coefficients are the inner products $\alpha_1 = (\mathbf{J}, \mathbf{J}_1)$ and $\alpha_2 = (\mathbf{J}, \mathbf{J}_2)$.

EXAMPLE 6.1-1. ***Expansions in Linearly Polarized and Circularly Polarized Bases.*** Using the x and y linearly polarized vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ as an expansion basis, the expansion coefficients for a Jones vector of components A_x and A_y with $|A_x|^2 + |A_y|^2 = 1$ are, by definition, $\alpha_1 = A_x$ and $\alpha_2 = A_y$. The same polarization state may be expanded in other bases.

- In a basis of linearly polarized vectors at angles 45° and 135°, i.e., $\mathbf{J}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{J}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$, the expansion coefficients α_1 and α_2 are:

$$A_{45} = \frac{1}{\sqrt{2}}(A_x + A_y), \quad A_{135} = \frac{1}{\sqrt{2}}(A_y - A_x). \tag{6.1-12}$$

- Similarly, if the right and left circularly polarized waves $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$ are used as an expansion basis, the coefficients α_1 and α_2 are:

$$A_R = \frac{1}{\sqrt{2}}(A_x - jA_y), \quad A_L = \frac{1}{\sqrt{2}}(A_x + jA_y). \tag{6.1-13}$$

For example, a linearly polarized wave with a plane of polarization that makes an angle θ with the x axis (i.e., $A_x = \cos\theta$ and $A_y = \sin\theta$) is equivalent to a superposition of right and left circularly polarized waves with coefficients $\frac{1}{\sqrt{2}}e^{-j\theta}$ and $\frac{1}{\sqrt{2}}e^{j\theta}$, respectively. A linearly polarized wave therefore equals a weighted sum of right and left circularly polarized waves.

EXERCISE 6.1-1

Measurement of the Stokes Parameters. Show that the Stokes parameters defined in (6.1-9) for light with Jones vector components A_x and A_y are given by

$$\mathrm{S}_0 = |A_x|^2 + |A_y|^2 \tag{6.1-14a}$$

$$\mathrm{S}_1 = |A_x|^2 - |A_y|^2 \tag{6.1-14b}$$

$$\mathrm{S}_2 = |A_{45}|^2 - |A_{135}|^2 \tag{6.1-14c}$$

$$\mathrm{S}_3 = |A_R|^2 - |A_L|^2, \tag{6.1-14d}$$

where A_{45} and A_{135} are the coefficients of expansion in a basis of linearly polarized vectors at angles 45° and 135° as in (6.1-12), and A_R and A_L are the coefficients of expansion in a basis of the right and left circularly polarized waves set forth in (6.1-13). Suggest a method of measuring the Stokes parameters of light with arbitrary polarization.

Matrix Representation of Polarization Devices

Consider the transmission of a plane wave of arbitrary polarization through an optical system that maintains the plane-wave nature of the wave, but alters its polarization, as illustrated schematically in Fig. 6.1-6. The system is assumed to be linear, so that the principle of superposition of optical fields is obeyed. Two examples of such systems are the reflection of light from a planar boundary between two media, and the transmission of light through a plate with anisotropic optical properties.

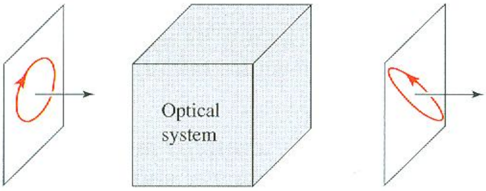


Figure 6.1-6 An optical system that alters the polarization of a plane wave.

The complex envelopes of the two electric-field components of the input (incident) wave, A_{1x} and A_{1y}, and those of the output (transmitted or reflected) wave, A_{2x} and A_{2y}, are in general related by the weighted superpositions

$$\begin{aligned} A_{2x} &= T_{11}A_{1x} + T_{12}A_{1y} \\ A_{2y} &= T_{21}A_{1x} + T_{22}A_{1y}, \end{aligned} \tag{6.1-15}$$

where T_{11}, T_{12}, T_{21}, and T_{22} are constants describing the device. Equations (6.1-15) are general relations that all linear optical polarization devices must satisfy.

The linear relations in (6.1-15) may conveniently be written in matrix notation by defining a 2×2 matrix $\mathbf{T}$ with elements T_{11}, T_{12}, T_{21}, and T_{22} so that

$$\begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix}. \tag{6.1-16}$$

If the input and output waves are described by the Jones vectors $\mathbf{J}_1$ and $\mathbf{J}_2$, respectively, then (6.1-16) may be written in the compact matrix form

$$\mathbf{J}_2 = \mathbf{T}\mathbf{J}_1. \tag{6.1-17}$$

The matrix $\mathbf{T}$, called the **Jones matrix**, describes the optical system, whereas the vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ describe the input and output waves.

The structure of the Jones matrix $\mathbf{T}$ of a given optical system determines its effect on the polarization state and intensity of the wave. The following is a compilation of the Jones matrices of some systems with simple characteristics. Physical devices that have such characteristics will be discussed subsequently in this chapter.

Linear polarizers. The system represented by the Jones matrix

$$\boxed{\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \tag{6.1-18}$$

Linear Polarizer
Along x Direction

transforms a wave of components (A_{1x}, A_{1y}) into a wave of components $(A_{1x}, 0)$ by eliminating the y component, thereby yielding a wave polarized along the x direction,

as illustrated in Fig. 6.1-7. The system is a **linear polarizer** with its transmission axis pointing in the x direction.

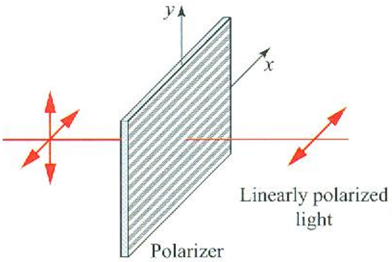


Figure 6.1-7 The linear polarizer. The lines in the polarizer represent the field direction that is permitted to pass.

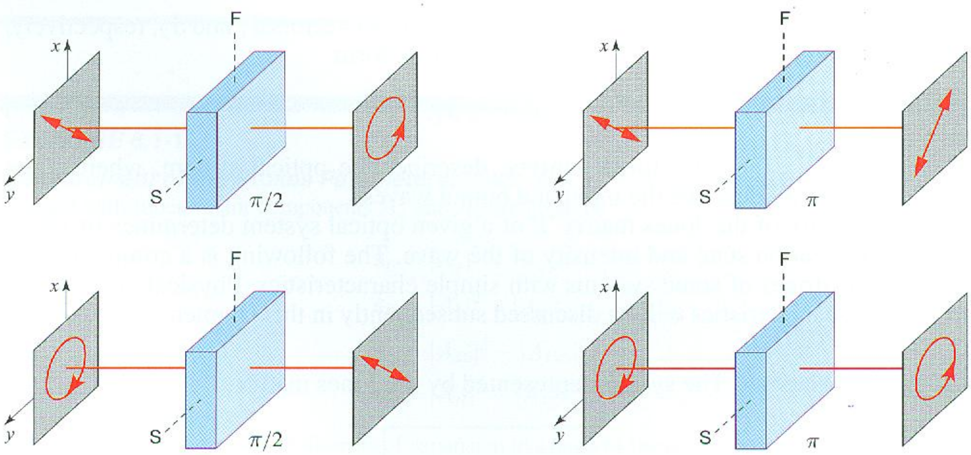
Wave retarders. The system represented by the matrix

$$\boxed{\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\Gamma} \end{bmatrix}} \tag{6.1-19}$$

Wave-Retarder
(Fast Axis Along x Direction)

transforms a wave with field components (A_{1x}, A_{1y}) into another with components $(A_{1x}, e^{-j\Gamma}A_{1y})$, thereby delaying the y component by a phase Γ while leaving the x component unchanged. It is therefore called a **wave retarder**. The x and y axes are called the fast and slow axes of the retarder, respectively.

The simple application of matrix algebra permits the results illustrated in Fig. 6.1-8 to be understood:



(*a*) Quarter-wave retarder (*b*) Half-wave retarder

Figure 6.1-8 Operations of quarter-wave ($\pi/2$) and half-wave (π) retarders on several particular states of polarization are shown in (*a*) and (*b*), respectively. F and S represent the fast and slow axes of the retarder, respectively.

- When $\Gamma = \pi/2$, the retarder (called a **quarter-wave retarder**) converts the linearly polarized wave $\begin{bmatrix}1\\1\end{bmatrix}$ into the left circularly polarized wave $\begin{bmatrix}1\\-j\end{bmatrix}$, and converts the right circularly polarized wave $\begin{bmatrix}1\\j\end{bmatrix}$ into the linearly polarized wave $\begin{bmatrix}1\\1\end{bmatrix}$.
- When $\Gamma = \pi$, the retarder (called a **half-wave retarder**) converts the linearly polarized wave $\begin{bmatrix}1\\1\end{bmatrix}$ into the linearly polarized wave $\begin{bmatrix}1\\-1\end{bmatrix}$, thereby rotating the plane of polarization by 90°. The half-wave retarder converts the right circularly polarized wave $\begin{bmatrix}1\\j\end{bmatrix}$ into the left circularly polarized wave $\begin{bmatrix}1\\-j\end{bmatrix}$.

Polarization rotators. While a wave retarder can transform a wave with one form of polarization into another, a **polarization rotator** always maintains the linear polarization of a wave but rotates the plane of polarization by a particular angle. The Jones matrix

$$\boxed{\mathbf{T} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}} \qquad (6.1\text{-}20)$$

Polarization Rotator

represents a device that converts a linearly polarized wave $\begin{bmatrix}\cos\theta_1\\ \sin\theta_1\end{bmatrix}$ into another linearly polarized wave $\begin{bmatrix}\cos\theta_2\\ \sin\theta_2\end{bmatrix}$, where $\theta_2 = \theta_1 + \theta$. It therefore rotates the plane of polarization of a linearly polarized wave by an angle θ.

Cascaded Polarization Devices

The action of cascaded optical systems on polarized light may be conveniently determined by using conventional matrix multiplication formulas. A system characterized by the Jones matrix $\mathbf{T}_1$ followed by another characterized by $\mathbf{T}_2$ are equivalent to a single system characterized by the product matrix $\mathbf{T} = \mathbf{T}_2\mathbf{T}_1$. The matrix of the system through which light is first transmitted must stand to the right in the matrix product since it is the first to affect the input Jones vector.

EXERCISE 6.1-2

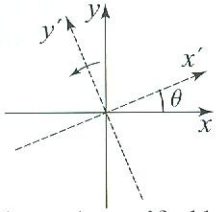
Cascaded Wave Retarders. Show that two cascaded quarter-wave retarders with parallel fast axes are equivalent to a half-wave retarder. What is the result if the fast axes are orthogonal?

Coordinate Transformation

The elements of the Jones vectors and Jones matrices are dependent on the choice of the coordinate system. However, if these elements are known in one coordinate system, they can be determined in another coordinate system by using matrix methods. If $\mathbf{J}$ is the Jones vector in the x–y coordinate system, then in a new coordinate system x'–y', with the x' direction making an angle θ with the x direction, the Jones vector $\mathbf{J}'$ is given by

$$\mathbf{J}' = \mathbf{R}(\theta)\,\mathbf{J}, \qquad (6.1\text{-}21)$$

where $\mathbf{R}(\theta)$ is the matrix



$$\boxed{\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.} \tag{6.1-22}$$

Coordinate Transformation

This can be verified by relating the components of the electric field in the two coordinate systems.

The Jones matrix $\mathbf{T}$, which represents an optical system, is similarly transformed into $\mathbf{T}'$, in accordance with the matrix relations

$$\mathbf{T}' = \mathbf{R}(\theta)\,\mathbf{T}\,\mathbf{R}(-\theta) \tag{6.1-23}$$

$$\mathbf{T} = \mathbf{R}(-\theta)\,\mathbf{T}'\,\mathbf{R}(\theta), \tag{6.1-24}$$

where $\mathbf{R}(-\theta)$ is given by (6.1-22) with $-\theta$ replacing θ. The matrix $\mathbf{R}(-\theta)$ is the inverse of $\mathbf{R}(\theta)$, so that $\mathbf{R}(-\theta)\,\mathbf{R}(\theta)$ is a unit matrix. Equation (6.1-23) can be obtained by using the relation $\mathbf{J}_2 = \mathbf{T}\mathbf{J}_1$ and the transformation $\mathbf{J}_2' = \mathbf{R}(\theta)\,\mathbf{J}_2 = \mathbf{R}(\theta)\,\mathbf{T}\,\mathbf{J}_1$. Since $\mathbf{J}_1 = \mathbf{R}(-\theta)\,\mathbf{J}_1'$, we have $\mathbf{J}_2' = \mathbf{R}(\theta)\,\mathbf{T}\,\mathbf{R}(-\theta)\,\mathbf{J}_1'$; since $\mathbf{J}_2' = \mathbf{T}'\,\mathbf{J}_1'$, (6.1-23) follows.

EXERCISE 6.1-3

Jones Matrix of a Polarizer. Show that the Jones matrix of a linear polarizer with a transmission axis making an angle θ with the x axis is

$$\boxed{\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}.} \tag{6.1-25}$$

Linear Polarizer at Angle θ

Derive (6.1-25) using (6.1-18), (6.1-22), and (6.1-24).

Normal Modes

The normal modes of a polarization system are the states of polarization that are not changed when the wave is transmitted through the system (see Appendix C). These states have Jones vectors satisfying

$$\mathbf{T}\mathbf{J} = \mu\mathbf{J}, \tag{6.1-26}$$

where μ is constant. The normal modes are therefore the eigenvectors of the Jones matrix $\mathbf{T}$, and the values of μ are the corresponding eigenvalues. Since the matrix $\mathbf{T}$ is of size 2×2 there are only two independent normal modes, $\mathbf{T}\mathbf{J}_1 = \mu_1\mathbf{J}_1$ and $\mathbf{T}\mathbf{J}_2 = \mu_2\mathbf{J}_2$. If the matrix $\mathbf{T}$ is a Hermitian, i.e., if $T_{12} = T_{21}^*$, the normal modes are orthogonal: $(\mathbf{J}_1, \mathbf{J}_2) = 0$. The normal modes are usually used as an expansion basis, so that an arbitrary input wave $\mathbf{J}$ may be expanded as a superposition of normal modes: $\mathbf{J} = \alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2$. The response of the system may then be easily evaluated since $\mathbf{T}\mathbf{J} = \mathbf{T}(\alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2) = \alpha_1\mathbf{T}\mathbf{J}_1 + \alpha_2\mathbf{T}\mathbf{J}_2 = \alpha_1\mu_1\mathbf{J}_1 + \alpha_2\mu_2\mathbf{J}_2$ (see Appendix C).