The solution for greater values of t may be written down in a similar manner. It is noted that at

$$t=T \qquad x=-a+\frac{2A}{\omega^2} \tag{2.16}$$

$$t=2T \qquad x=a-\frac{4A}{\omega^2} \qquad \text{etc.} \tag{2.17}$$

That is, each successive swing is $2A/\omega^2$ shorter than the preceding one. The solution should be continued until one of the swings lies inside the dead region $x=\pm A/\omega^2$. In such a case the motion stops.

3 THE FREE OSCILLATIONS OF A PENDULUM

As another simple example of a nonlinear dynamical system, let us discuss the motion of the simple pendulum of Fig. 3.1. Figure 3.1 represents a particle of mass m suspended from a fixed point O by a massless inextensible rod of length s, free to oscillate in the plane of the paper under the influence of gravity. If the pendulum is displaced by an angle θ from its position of equilibrium as shown in the figure, then the potential energy of the system V is the work done against gravity to lift the mass of the pendulum a distance h, given by

$$h=s(1-\cos\theta) \tag{3.1}$$

Hence the potential energy is

$$V=mgs(1-\cos\theta) \tag{3.2}$$

The kinetic energy of motion of the pendulum is given by

$$T=\tfrac{1}{2}mv^2 \tag{3.3}$$

where v is the linear velocity of the pendulum. In terms of the angle θ we have

$$v=s\frac{d\theta}{dt}=s\dot{\theta} \tag{3.4}$$

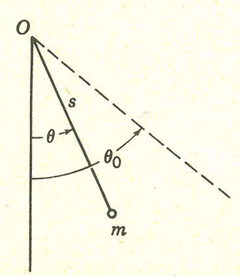

Fig. 3.1

Hence

$$T=\tfrac{1}{2}ms^2\dot{\theta}^2 \tag{3.5}$$

Now, since there is no loss of energy from the system, we must have

$$T+V=C \tag{3.6}$$

where C is a constant representing the total energy. Hence, substituting (3.2) and (3.5) into (3.6), we have

$$mgs(1-\cos\theta)+\tfrac{1}{2}ms^2\dot{\theta}^2=C \tag{3.7}$$

To determine the constant C, let us assume that $\theta=\theta_0$ is the maximum amplitude of swing so that

$$\dot{\theta}=0 \qquad \text{when } \theta=\theta_0 \tag{3.8}$$

Substituting this condition into (3.7), we have

$$C=mgs(1-\cos\theta_0) \tag{3.9}$$

Hence we have

$$g(1-\cos\theta)+\frac{s}{2}\dot{\theta}^2=g(1-\cos\theta_0) \tag{3.10}$$

or

$$\dot{\theta}^2=\frac{2g}{s}(\cos\theta-\cos\theta_0) \tag{3.11}$$

To obtain the equation of motion, we differentiate (3.11) with respect to time and obtain

$$2\dot{\theta}\ddot{\theta}=-\frac{2g}{s}\sin\theta\dot{\theta} \tag{3.12}$$

or

$$\frac{d^2\theta}{dt^2}+\frac{g}{s}\sin\theta=0 \tag{3.13}$$

This is the equation of motion, and Eq. (3.11) is its first integral.

Equation (3.13) is a nonlinear differential equation because of the presence of the trigonometric function $\sin\theta$. In the theory of *small* oscillations of a pendulum we expand $\sin\theta$ into a Maclaurin series of the form

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots \tag{3.14}$$

If θ is always small, the approximation

$$\sin\theta \doteq \theta \tag{3.15}$$

is a very good one, and in this case (3.13) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{s}\theta = 0 \tag{3.16}$$

This is a linear equation with constant coefficients whose general solution is of the form

$$\theta = A \sin(\omega t + \phi) \tag{3.17}$$

where

$$\omega = \sqrt{\frac{g}{s}} \tag{3.18}$$

and A and ϕ are arbitrary constants that depend on the initial conditions of the motion. The period of the motion P_0 is given by

$$P_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{g}{s}} \tag{3.19}$$

We thus see that the period is independent of the amplitude of oscillation in this case. By measuring s and P_0 the acceleration g due to gravity may be determined by (3.19) to a high degree of accuracy.

To determine the period of oscillation for large amplitudes, we return to Eq. (3.11) and write it in the form

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{s}}\sqrt{\cos\theta - \cos\theta_0} \tag{3.20}$$

or

$$dt = \sqrt{\frac{s}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{3.21}$$

Now the period P_{θ_0} is twice the time taken by the pendulum to swing from $\theta = -\theta_0$ to $\theta = \theta_0$. Therefore

$$P_{\theta_0} = 2\sqrt{\frac{s}{2g}}\int_{-\theta_0}^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{3.22}$$

If we use the trigonometric identities

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \qquad \cos\theta_0 = 1 - 2\sin^2\frac{\theta_0}{2} \tag{3.23}$$

then we may write (3.22) in the form

$$P_{\theta_0} = \sqrt{\frac{s}{g}}\int_{-\theta_0}^{+\theta_0}\frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} \tag{3.24}$$

Let us now introduce the two new variables k and ϕ by the equations

$$k = \sin\frac{\theta_0}{2} \tag{3.25}$$

and

$$\sin\frac{\theta}{2} = k\sin\phi \tag{3.26}$$

Hence

$$d\theta = \frac{2k\cos\phi\, d\phi}{\cos(\theta/2)} = \frac{2k\cos\phi\, d\phi}{\sqrt{1 - k^2\sin^2\phi}} \tag{3.27}$$

In terms of these two new variables the integral of (3.24) becomes

$$P_{\theta_0} = 4\sqrt{\frac{s}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \tag{3.28}$$

Now the integral

$$K(k) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \tag{3.29}$$

is called the complete elliptic integral of the first kind and is tabulated.[1] We therefore may write

$$P_{\theta_0} = 4\sqrt{\frac{s}{g}}K(k) \tag{3.30}$$

Equation (3.25) determines k in terms of the maximum angle of swing θ_0. From the tables of the function $K(k)$ and Eq. (3.30), we may determine the period P_{θ_0}. We thus find that, for $\theta_0 = 60°$, we have

$$P_{60°} = 1.07P_0 \tag{3.31}$$

and, for $\theta_0 = 2°$, we have

$$P_{2°} = 1.000076P_0 \tag{3.32}$$

where P_0 is given by (3.19).

We thus see that the period of a pendulum depends on its amplitude of oscillation.

4 RESTORING FORCE A GENERAL FUNCTION OF THE DISPLACEMENT

Let us consider the motion of a particle of mass m. Let the mass be subjected to a restoring force $F(x)$ of an elastic nature tending to restore the mass m to

[1] See, for example, B. O. Peirce, "A Short Table of Integrals," p. 121, Ginn & Company, Boston, 1929.