The solution for greater values of t may be written down in a similar manner. It is noted that at

$$t=T \qquad x=-a+\frac{2A}{\omega^2} \tag{2.16}$$

$$t=2T \qquad x=a-\frac{4A}{\omega^2} \qquad \text{etc.} \tag{2.17}$$

That is, each successive swing is $2A/\omega^2$ shorter than the preceding one. The solution should be continued until one of the swings lies inside the dead region $x=\pm A/\omega^2$. In such a case the motion stops.

3 THE FREE OSCILLATIONS OF A PENDULUM

As another simple example of a nonlinear dynamical system, let us discuss the motion of the simple pendulum of Fig. 3.1. Figure 3.1 represents a particle of mass m suspended from a fixed point O by a massless inextensible rod of length s, free to oscillate in the plane of the paper under the influence of gravity. If the pendulum is displaced by an angle θ from its position of equilibrium as shown in the figure, then the potential energy of the system V is the work done against gravity to lift the mass of the pendulum a distance h, given by

$$h=s(1-\cos\theta) \tag{3.1}$$

Hence the potential energy is

$$V=mgs(1-\cos\theta) \tag{3.2}$$

The kinetic energy of motion of the pendulum is given by

$$T=\tfrac{1}{2}mv^2 \tag{3.3}$$

where v is the linear velocity of the pendulum. In terms of the angle θ we have

$$v=s\frac{d\theta}{dt}=s\dot{\theta} \tag{3.4}$$

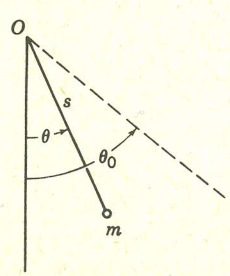

Fig. 3.1

Hence

$$T=\tfrac{1}{2}ms^2\dot{\theta}^2 \tag{3.5}$$

Now, since there is no loss of energy from the system, we must have

$$T+V=C \tag{3.6}$$

where C is a constant representing the total energy. Hence, substituting (3.2) and (3.5) into (3.6), we have

$$mgs(1-\cos\theta)+\tfrac{1}{2}ms^2\dot{\theta}^2=C \tag{3.7}$$

To determine the constant C, let us assume that $\theta=\theta_0$ is the maximum amplitude of swing so that

$$\dot{\theta}=0 \qquad \text{when } \theta=\theta_0 \tag{3.8}$$

Substituting this condition into (3.7), we have

$$C=mgs(1-\cos\theta_0) \tag{3.9}$$

Hence we have

$$g(1-\cos\theta)+\frac{s}{2}\dot{\theta}^2=g(1-\cos\theta_0) \tag{3.10}$$

or

$$\dot{\theta}^2=\frac{2g}{s}(\cos\theta-\cos\theta_0) \tag{3.11}$$

To obtain the equation of motion, we differentiate (3.11) with respect to time and obtain

$$2\dot{\theta}\ddot{\theta}=-\frac{2g}{s}\sin\theta\dot{\theta} \tag{3.12}$$

or

$$\frac{d^2\theta}{dt^2}+\frac{g}{s}\sin\theta=0 \tag{3.13}$$

This is the equation of motion, and Eq. (3.11) is its first integral.

Equation (3.13) is a nonlinear differential equation because of the presence of the trigonometric function $\sin\theta$. In the theory of *small* oscillations of a pendulum we expand $\sin\theta$ into a Maclaurin series of the form

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots \tag{3.14}$$

If θ is always small, the approximation

$$\sin\theta\doteq\theta \tag{3.15}$$