Temperature Oscillations in a Metal: Probing Aspects of Fourier Analysis

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In this experiment, we use Fourier analysis to study the propagation of thermal waves in a metal rod. This elegant approach helps us visualize the harmonic components of the temperature oscillations, and subsequently allows us to relate velocity and damping coefficient of the thermal oscillations with thermal diffusivity.

I. INTRODUCTION

Thermal diffusivity is an important material parameter that describes the movement of isothermal surfaces during heat flow [1]. It is related to heat conductivity, specific heat capacity and mass density; hence, its accurate determination allows us to gain significant insight about the thermophysical properties of materials.
In this paper, we use probing aspects of Fourier analysis to study temperature oscillations in a Copper rod. We observe the harmonic content of the temperature oscillations and verify that they are differentially damped. Our analysis helps us separately relate the thermal diffusivity of the metal to two different wave properties, namely the damping coefficient and the velocity. The two properties are experimentally determined, and we arrive at two different results for the thermal diffusivity of the Copper rod used. Finally, we comment on the relative accord or discord between the two values and state their average as the thermal diffusivity of the material in question.

II. EXPERIMENTAL SETUP AND PROCEDURE

Figure 1 shows the schematic for the experimental setup and the circuitry involved.

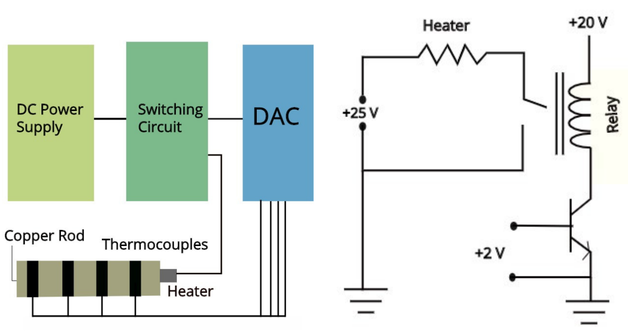

FIG. 1: (a) Schematic of the experimental setup; and (b) Circuitry of the relay switch.

For the purpose of this experiment, we use a Copper rod of length 40 ± 0.05 cm and diameter 3 ± 0.05 cm. The rod has four thermocouples equidistantly embedded into it. The distance between two consecutive thermocouples is 2 ± 0.05 cm. A cartridge heater is inserted toward one end of the rod. It is provided with a square pulse of 20 V at a rate of 0.005 Hz. The pulse frequency is controlled by a simple relay switch, which is in turn controlled using `LabVIEW`. The rod is wrapped in insulation and clamped onto a support.
The system is allowed to run until dynamic equilibrium is attained, which is when temperatures begin to oscillate about a mean value. Data from the four thermocouples is acquired at a rate of 0.01 Hz through out the experiment, and is converted to analog form using a Digital to Analog Converter (DAC) (SCC-68). It is then sent to `LabVIEW`, which saves it in a file.

III. THEORY

To determine the equation that governs the spatial and temporal distribution of temperature in a homogeneous solid, we use the Diffusion equation, which for our one-dimensional case, can be written as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{D}\frac{\partial T}{\partial t}, \tag{1}$$

where T represents temperature oscillations and D represents thermal diffusivity, and is given by

$$D = \frac{\kappa}{\sigma\rho}, \tag{2}$$

where κ represents thermal conductivity, σ represents heat capacity per unit mass and ρ represents mass density.
Conservation of energy implies that energy cannot be destroyed. Hence, if a material is losing energy, the energy must be flowing out of its surface. This is the motivation behind Equation (1), which was obtained by equating the energy flux through a surface with the rate of change of energy density of the volume enclosed by that surface [2]. It must also be noted that Equation (1), although similar to, is not actually the wave equation. This emphasises that thermal oscillations in a substance cannot be characterized as heat waves. This is true since thermal disturbances in media have been observed to not exhibit wave-like phenomena like transfer of energy, reflection and refraction.
In order to see how temperature oscillates through the length of our rod, we seek a steady-state solution to the

differential equation in Equation (1). Since the heating function used in the experiment is periodic, we assume a solution in the form of a Fourier series given by

$$T(x,t) = c_o(x) + \sum_{n=1}^{\infty} \mathbb{R}\left[e^{i(\omega_n - \epsilon_n)}\right], \quad (3)$$

where $T(x,t)$ is the temperature distribution along rod as a function of space and time, c_o are the position dependent Fourier coefficients, $\omega_n = 2n\pi/\tau$ are the Fourier frequencies and ϵ_n are the phase factors. τ is representative of the time period of the thermal oscillation.
Substituting Equation (3) into Equation (1) allows us to get a particular solution, with undetermined coefficients, for our problem. The solution is given by

$$T(x,t) = P_1 x + P_o + \sum_{n}^{\infty} B_n \exp\left(-\sqrt{\frac{\omega_n}{2D}}x\right) \times \exp\left(i(\omega_n t - \sqrt{\frac{\omega_n}{2D}}x - \epsilon_n)\right), \quad (4)$$

where P_o, P_1 and B_n are undetermined coefficients. They can be determined using appropriate boundary conditions and analysis of wave forms [4]. After the substitution of coefficients and further simplification Equation (4) becomes

$$T(x,t) = P_1 x + \langle T(0)\rangle - \frac{4\Delta T}{\pi^2} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^2} \exp\left(-\frac{x}{d_n} \cos\left(\omega_n t - \frac{x}{d_n}\right)\right), \quad (5)$$

where $P_1(x) = \langle (T(\Delta x)\rangle + \langle T(0)\rangle) / \Delta x$, $\langle T(0)\rangle$ is the mean temperature at the location of the first thermocouple (which we have defined as $x = 0$), and the $\exp(-x/d_n)$ term is called the damping coefficient, with $d_n = \sqrt{2D/\omega_n}$ being the damping length.
Equation (5) makes the Fourier composition of thermal oscillations explicit, something which we will experimentally verify later. Evidently, the expression has two components. The oscillatory part represents a periodic function with odd multiples of the pulsing frequency ω. These odd harmonics are spatially damped by the damping coefficients. The damping lengths evidently decrease as n increases. Therefore, at distances sufficiently away from $x = 0$, we may approximate the temperature distribution by the first harmonic only. While the experimental proof for this will come later, we will use this approximation now to derive two expressions for thermal diffusivity.
Assuming a distance significantly far away from $x = 0$, relabelling that point as our new origin and using the first harmonic approximation, we can modify equation (5) so that it becomes

$$T(x,t) = \frac{-4\Delta T}{\pi^2} \exp\left(-\frac{\omega}{2D}\right) \cos\left(\omega\left(t - \frac{x}{\nu}\right)\right), \quad (6)$$

where ν is the phase velocity or velocity of the thermal oscillation. It is given by $\nu = \Delta x/\Delta t$, with Δx being the distance between two thermocouples and Δt being the time lag between two equal phase points. This phase velocity is determined using graphical analysis of the thermocouple data we obtain.
Comparing like terms in Equation (6) and Equation (5), we can write two expressions for thermal diffusivity [5]

$$D_d = \frac{\pi}{\tau d^2} \quad (7)$$

and

$$D_\nu = \frac{\nu^2 \tau}{4\pi}, \quad (8)$$

where we have used D_d to represent thermal diffusivity as a function of damping length and D_ν to represent thermal diffusivity as a function of phase velocity. We shall use experimental data to determine both.

IV. RESULTS AND DISCUSSION

The data acquired from the four thermocouples is loaded and plotted in MATLAB. Figure 2 shows the profiles of temperature oscillations as functions of time, for each thermocouple. It is worth noticing that thermocouples closer to the heater recorded higher mean temperatures. The close up profile of the plots in Figure 3 highlights this further. The figure also makes it evident that the maximum temperature peaks shift toward the right as we move along the rod.

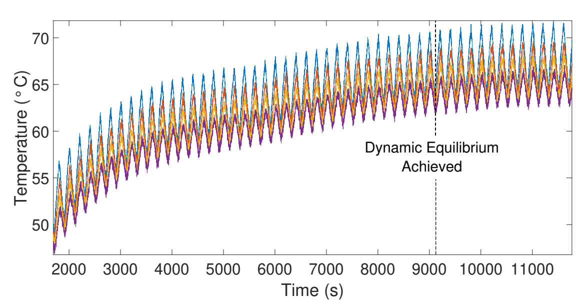

FIG. 2: The curves represent the temperature profiles at different points along the copper rod. Dynamic equilibrium is seen to be reached after approximately 9000 seconds, after which temperature at each point appears to oscillate about a mean value.

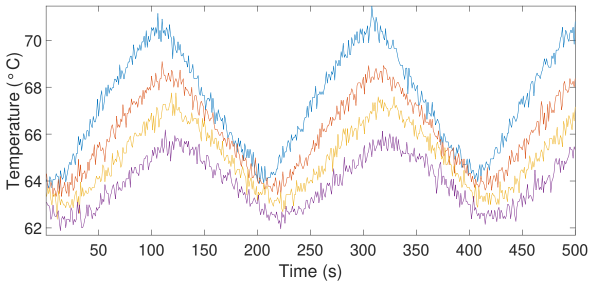

FIG. 3: Close up profiles of the temperature oscillations at different points along the Copper rod. Notice how as we move further along the rod, the curves seem to become more sinusoidal and more shifted toward the right-hand side.

The data for each thermocouple after the point of dynamic equilibrium is extracted from the raw result. We calculate and subtract mean values from each data set and then perform absolute FFT on them. Subtracting the mean removes the peak corresponding to the DC component of the signal. Since half of the data is redundant due to conjugate symmetry, we discard the coefficients corresponding to negative frequencies and plot remaining. Figure 4 shows the result of the Fourier transform. Notice the presence of odd harmonics only. It can also be seen that higher harmonics vanish quickly, thus allowing us to approximate the frequency by the fundamental frequency at distances significantly large from the heater.

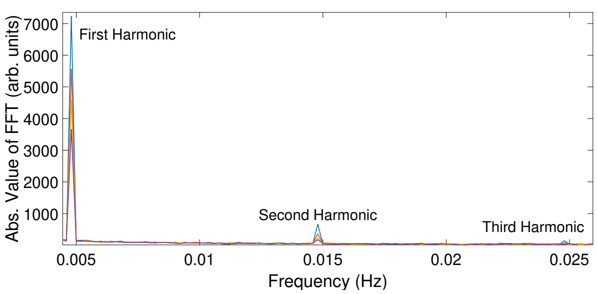

FIG. 4: Fourier transforms of the thermocouple data in the dynamic equilibrium region. The presence of odd harmonics only verifies the claim previously made on purely theoretical grounds. Notice how the higher harmonics vanish quickly.

We use the spectral density curves in Figure 4 at the fundamental frequency to calculate spectral power (logarithm of the area under the curve) at the location of each thermocouple. We ignore the curves at higher frequencies using our first harmonic approximation. We then plot the spectral power against the thermocouple index as shown in Figure 5. The thermocouple index has been used as a measure of distance from the heater.

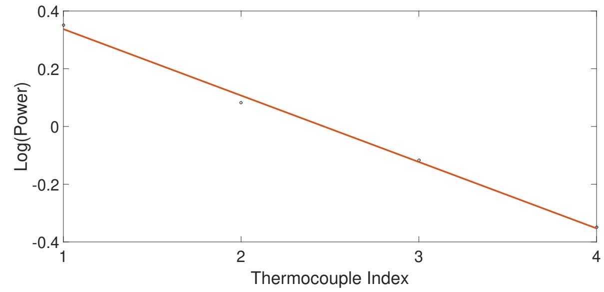

FIG. 5: Spectral power obtained from the curves at fundamental frequency in the Fourier transform.

The data points obtained are fitted using a linear fit. The slope of this line m, is related to the damping length by $m = 1/d$. Since the slope parameter of fit in this case is $m = (-0.23 \pm 0.05)$ cm, we get $d = (4.34 \pm 0.94)$ cm. Substituting this result in Equation (7) with $\tau = 200\,\text{s}$, we obtain $D_d = (8.30 \pm 0.20) \times 10^{-4}\ \text{m}^2\text{s}^{-1}$.
To calculate the phase velocity, we use the time lag between the in-phase points of the first and the last thermocouple. The calculated velocity turns out to be $\nu = 3.0 \pm 0.2\ \text{ms}^{-1}$. We use this result in Equation (8) to determine the velocity dependent thermal diffusivity, which turns out to be $D_\nu = (1.43 \pm 0.21) \times 10^{-4}\ \text{m}^2\text{s}^{-1}$. Under ideal scenarios, the values of diffusivity obtained via both approaches should be identical. Our results are very similar; however, they still have a slight discord. This can be attributed to the fact that while writing the equations, thermal losses to the environment were ignored. These losses influence wave properties differently and may account for different results.
The literature value of thermal diffusivity of Copper under standard conditions is $D = 1.020 \times 10^{-4}\ \text{m}^2\text{s}^{-1}$ [5]. As we can see, D_ν is in good agreement with the published value; however, the calculated value of D_d is slightly off. This can again be catered if we account for heat loss to the surroundings. But since the discord between the values is not highly appreciable, we can use our calculated values to estimate diffusivity. Therefore, $\overline{D} = (4.86 \pm 0.2) \times 10^{-4}\ \text{m}^2\text{s}^{-1}$, gives the thermal diffusivity of the Copper rod with our experimental conditions.

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