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[12]We define the norm of the matrix A as $\|A\| \equiv \sup_{|c| \neq 0} |Ac|/|c|$, where c ranges over all vectors in the Hilbert space.

[13]A set E of real numbers is said to be relatively dense if there exists a number $L < \infty$ such that any interval on the real axis of length L contains at least one member of E.

[14]This particular result, which has been previously stated by F. Gesztesy and H. Mitter, J. Phys. A 14, L79 (1981), applies to any periodic Hermitian operator.

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[16]We should point out that recurrence in Ψ does not necessarily imply recurrence in E, since the overall envelope of the wave function could reassemble itself with enough small-scale structure so as to produce a large change in energy.

[17]A physical criterion for being away from resonance is to have the frequency of the periodic potential incommensurate with the spectrum of H_0.

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Evidence for Universal Chaotic Behavior of a Driven Nonlinear Oscillator

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A bifurcation diagram for a driven nonlinear semiconductor oscillator is measured directly, showing successive subharmonic bifurcations to $f/32$, onset of chaos, noise band merging, and extensive noise-free windows. The overall diagram closely resembles that computed for the logistic model. Measured values of universal numbers are reported, including effects of added noise.

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Our purpose is to report detailed measurements on a driven nonlinear semiconducting oscillator and to make quantitative comparisons with the predictions of a simple model of period-doubling bifurcation as a route to chaos,[1-3] which stems from earlier work in topology.[4] There is surprising agreement, lending support to the belief and the hope that some nonlinear systems can be approximately understood by a universal model, as has been suggested by some experiments.[5,6] This upsurge of interest in nonlinear behavior has been triggered by the remarkable result that deterministic computer iterations of such a simple nonlinear recursion relation as the logistic equation

$$x_{n+1} = \lambda x_n(1-x_n) \quad (1)$$

yield exceedingly complex pseudorandom or chaotic behavior.[2,3] The results are best summarized by a bifurcation diagram[7-9]: a scatter plot of the iterated value $\{x_n\}$ versus the control parameter λ, which shows that as λ is increased $\{x_n\}$ displays a series of pitchfork bifurcations at λ_n, with period doubling by 2^n, $n=1, 2, \ldots$. These converge geometrically, as $\lambda_c - \lambda_n \propto \delta^{-n}$, to the onset of chaos at λ_c, where $\{x_n\}$ becomes aperiodic; in the chaotic regime, $\lambda > \lambda_c$, noise bands merge and there exist narrow periodic windows in a specific order and pattern.[4] This model is quantified by universal numbers as $n \to \infty$: $\delta = 4.669\ldots$, and the pitchfork scaling parameter $\alpha = 2.502\ldots$, first computed by Feigenbaum. Other universal numbers characterize the spectral power density[10,11] and effects of noise.[8,12]

Our experimental system is a series LRC circuit driven by a controlled oscillator, described by $L\ddot{q} + R\dot{q} + V_c = V_d(t) = V_0 \sin(2\pi f t)$, where V_c is the voltage across a Si varactor diode (type 1N953 supplied by TRW Company), which is the nonlinear element. Under reverse voltage, $V_c = q/C$,

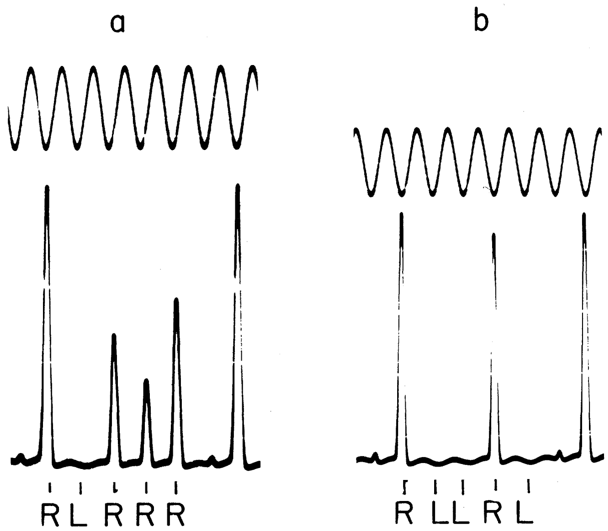

FIG. 1. (a) The varactor voltage $V_c(t)$ and the driving oscillator voltage $V_d(t)$ (upper) for period 6 window at 2.073 V; the pattern is RLRRR, and describes the sequence of visitation of the oscillator to its states according to whether it is to the right or left of zero, following the notation of Ref. 4. (b) Period 6 window at 3.338 V, with different pattern RLLRL.

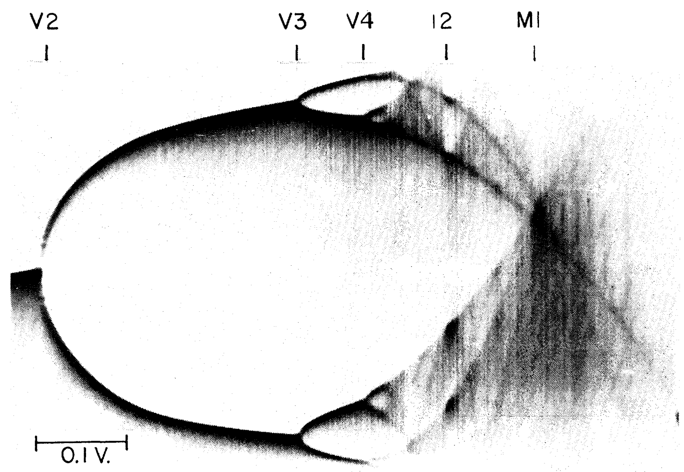

FIG. 3. Expansion of a region of Fig. 2, showing bifurcation thresholds V_2, V_3, and V_4; window of period 12; and band merging M_1.

where $C \simeq C_0/[1+V_c/0.6]^{0.5}$, $C_0 \simeq 300$ pF; under forward voltage the varactor behaves like a normal conducting diode. The coil inductance $L = 10$ mH, the resistance $R = 28\ \Omega$. At low values of V_0, the system behaves like a high-Q resonant circuit at $f_{res} = 93$ kHz; as V_0 is increased, the resonant frequency shifts upward and the Q is lowered. It is not our intention to solve the intractable nonlinear differential equations for this system[13] but rather to do extensive and novel measurements designed to compare its behavior as fully as possible with the simple logistic model. We fix f near f_{res}, vary the driving voltage V_0, and measure the varactor voltage $V_c(t)$. We assume a correspondence between V_0 and λ and between V_c and x of Eq. (1).

A real-time display, e.g., Fig. 1, of $V_c(t)$ and $V_0(t)$ on a dual-beam oscilloscope, with V_0 as a parameter, clearly revealed threshold values V_{0n} for bifurcation; the bifurcation subharmonics $f/2^n$ up to $f/16$; and the pattern of visitation of the oscillator to its stable points. The data shown at two different windows in the chaotic regime, both for period-6 orbits, show different patterns, as expected.[4] During the diode con-

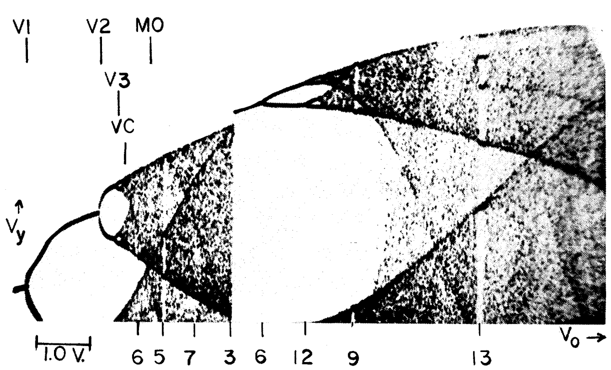

FIG. 2. Bifurcation diagram V_y vs V_0 at $f = 96.85$ kHz, showing thresholds V_1, V_2, and V_3 for periods 2, 4, and 8; threshold for chaos V_c; band merging M_0; and windows of periods 6, 5, 7, 3, 6, 12, 9, and 13. The veiled lines are peaks in the spectral density in the chaotic regime.

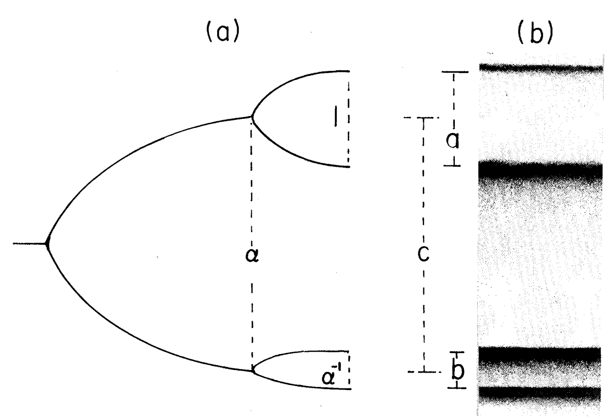

FIG. 4. (a) Schematic of universal metric scaling of pitchfork bifurcation, determined by α (Ref. 2). (b) Data for period 16 between V_4 and V_5, which yield the values $\alpha = a/b = 2.35$ and $\alpha = c/a = 2.61$.

TABLE I. Measured thresholds at 99 kHz.

Period	Threshold V_0 rms volts	Comments
2	0.639	Threshold for periodic bifurcation
4	1.567	
8	1.785	
16	1.836	
32	1.853	
Chaos	1.856	Onset of noise
12	1.901	Window
24	1.902	
6	2.073	Window
12	2.074	
5	2.353	Window
10	2.363	
7	2.693	Window
14	2.696	
3	3.081	Wide Window
6	3.338	
12	3.711	
24	3.821	
9	4.145	Window
18	4.154	

ducting half cycle, V_c is compressed toward the zero line; in the reverse half cycle, V_c has a set of discrete values, which correspond to the upper half of the bifurcation diagram.

To analyze V_c, a window comparator was constructed which selected components between V_y and $V_y + \Delta V$, $\Delta V \simeq 10$ mV. A vertical scan of V_y simultaneously with a slower horizontal scan of V_0 on an oscilloscope yielded Figs. 2 and 3, the first measured bifurcation diagram for a physical system showing subharmonic sequences. It has a striking resemblance to the computed diagram,[7,8] including bifurcation thresholds, onset of chaos, band merging, noise-free windows, and the subtle veiled structure, corresponding to regions of high probability.[8] The diagram allows a direct measurement of the number α; from the expanded region, Fig. 4, the ratio of the pitchfork splittings is directly measured in a series of ten similar measurements:

$$\alpha = 2.41 \pm 0.1. \qquad (2)$$

The diagram shows at least five noise-free windows, which bifurcate within the window: From Fig. 2 and Table I, at $V_0 = 3.081$ V, a noise-free window of period 3 appears, which bifurcates to periods 6, 12, and 24 before onset of chaos again.

The power spectral density of $V_c(t)$ was measured with a spectrum analyzer with 40 dB dynamic range, which showed the expected subharmon-

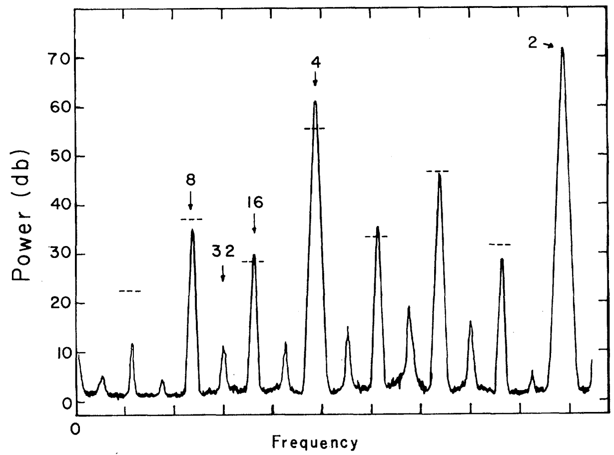

FIG. 5. Power spectral density (dB) vs frequency for $f = 98$ kHz, dynamic range 70 dB, showing subharmonics to $f/32$. The components agree with prediction (dashed bars, Ref. 14) within 2 dB rms deviation, except for the peak at $f/16$.

ics $\frac{1}{2}$; $\frac{1}{4}$, $\frac{3}{4}$; $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, etc., rather symmetrically displayed about $f/2$. The data shown in Fig. 5 were obtained with a more sensitive spectrum analyzer with 85 dB of dynamic range, sensitivity of 300 nV, and range $f = 0$ to 50 kHz $\geqslant f/2$, thus allowing observation of spectral components 95 dB below V_0 at f. Figure 5 shows periodic subharmonics to $f/32$ at V_0 just below the threshold for chaos V_{0c}; the predicted values of the individual spectral components are shown.[14] It is predicted[10] that the average heights of the peaks for a period is $10 \log 20.963 = 13.21$ dB below the previous period; the data are consistent with this, although the region between $f/2$ and f is not available for exact averaging. Spectral analysis showed other noise-free windows (60 dB above noise) at periods 12, 6, 5, 7, and 9, at thresholds listed in Table I; all show bifurcations within the window. The entire V_0 sequence of Table I, identified by period and pattern, is consistent with the universal U sequence of Metropolis, Stein, and Stein[4] (who limit computation to period $\leqslant 11$). From the first four threshold voltages V_{0n} we calculate the convergence rate

$$\delta_1 = \frac{V_{02} - V_{01}}{V_{03} - V_{02}} = 4.257 \pm 0.1;$$
$$\delta_2 = \frac{V_{03} - V_{02}}{V_{04} - V_{03}} = 4.275 \pm 0.1. \qquad (3)$$

We observed the effect on the system of adding a random noise voltage $V_n(t)$ to $V_d(t)$. The bi-

TABLE II. Measured and predicted values for universal numbers.

Number		Measured	Predicted
δ_1	Eq. (3)	4.26 ± 0.1	4.751[a]
δ_2	Eq. (3)	4.28 ± 0.1	4.656[a]
δ_1	Period 3 window	0.69 ± 0.1	0.979[a]
δ_2	Period 3 window	3.38 ± 0.1	4.429[a]
α		2.41 ± 0.1	2.502[b]
ϵ		6.3 ± 0.3	6.55[c]
Average spectral power ratio		11 to 15 dB	13.61 dB[d]

[a]Computed from Eq. (1); cf. asymptotic limit 4.669, Ref. 2.
[b]Ref. 2.
[c]Ref. 12.
[d]Ref. 10.

furcation diagram and the power spectra were observed as $|V_n|$ was increased: periods 16, 8, 4, and 2 were successively obliterated at $V_n = 10$, 62, 400, and 2500 mV_{rms}, respectively, yielding an average value

$$\kappa = 6.3 \tag{4}$$

for the noise voltage factor required to reduce by one the number of observable bifurcations.

To summarize, Table II compares our measured values with predicted values for some universal numbers. There is overall reasonable quantitative agreement between the data and the logistic model. The likely cause for some discrepancy in δ is that the data cannot be taken in the asymptotic limit $n \to \infty$. These are first direct measurements for α and κ. The strong similarity between the predicted and the observed bifurcation diagram gives further support to the utility of simple models as a key to chaotic behavior of nonlinear systems. The measurement of a bifurcation diagram is a powerful method for assessing the degree to which a particular physical system will follow this route, or other routes[14]; it is not yet known how to predict this in advance.

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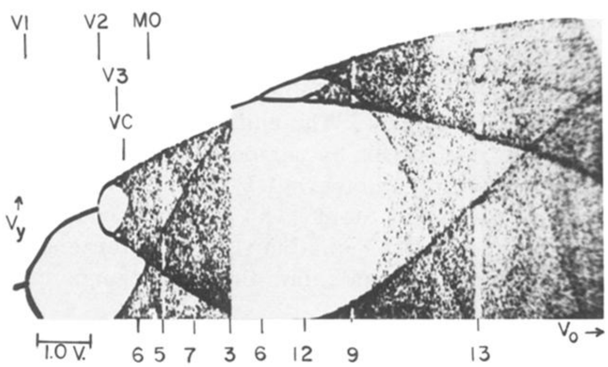

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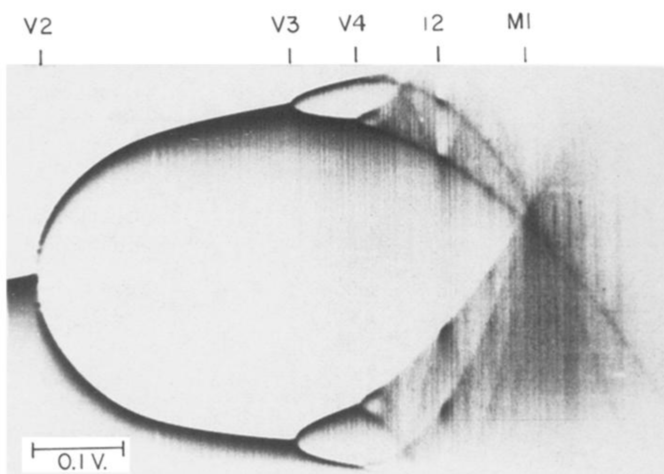

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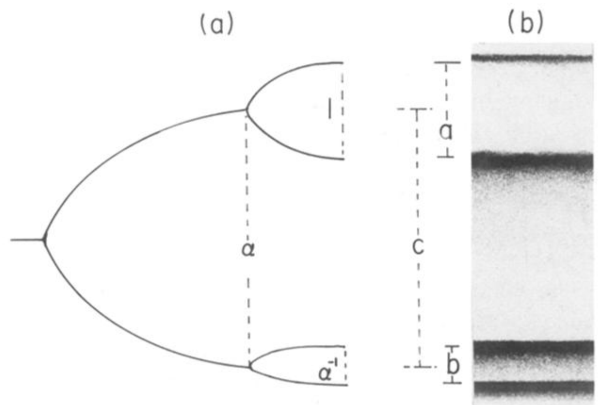

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