Chapter 6

Polarization of Light

When the direction of the electric field of light oscillates in a regular, predictable fashion, we say that the light is *polarized*. *Polarization* describes the direction of the oscillating electric field, a distinct concept from dipoles per volume in a material **P** – also called polarization. In this chapter, we develop a formalism for describing polarized light and the effect of devices that modify polarization. If the electric field oscillates in a plane, we say that it is *linearly polarized*. The electric field can also spiral around while a plane wave propagates, and this is called *circular* or *elliptical polarization*. There is a convenient way for keeping track of polarization using a two-dimensional *Jones vector*.

Many devices can affect polarization such as *polarizers* and *wave plates*. Their effects on a light field can be represented by 2×2 *Jones matrices* that operate on the Jones vector representing the light. A Jones matrix can describe, for example, a polarizer oriented at an arbitrary angle or it can characterize the influence of a wave plate, which is a device that introduces a relative phase between two components of the electric field.

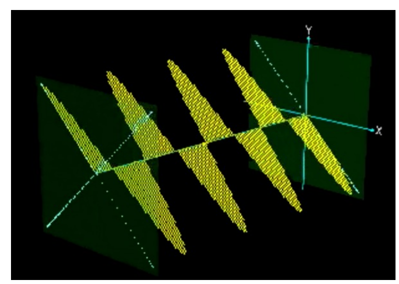


Figure 6.1 Animation showing different polarization states of light.

In this chapter, we will also see how reflection and transmission at a material interface influences field polarization. As we saw previously, s-polarized light can acquire a phase lag or phase advance relative to p-polarized light. This is especially true at metal surfaces, which have complex indices of refraction. The Fresnel coefficients studied in chapters 3 and 4 can be conveniently incorporated into a Jones matrix to keep track of their influence polarization. *Ellipsometry*, outlined in appendix 6.A, is the science of characterizing optical properties of materials through an examination of these effects.

Throughout this chapter, we consider light to have well characterized polarization. However, most common sources of light (e.g. sunlight or a light bulb) have an electric-field direction that varies rapidly and randomly. Such sources are commonly referred to as *unpolarized*. It is common to have a mixture of unpolarized and polarized light, called *partially polarized* light. The Jones vector formalism used in this chapter is inappropriate for describing the unpolarized portions of the light. In appendix 6.B we describe a more general formalism for dealing with light having an arbitrary *degree of polarization*.

6.1 Linear, Circular, and Elliptical Polarization

Consider the plane-wave solution to Maxwell's equations given by

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{6.1}$$

The wave vector $\mathbf{k}$ specifies the direction of propagation. We neglect absorption so that the refractive index is real and $k = n\omega/c = 2\pi n/\lambda_{\text{vac}}$ (see (2.19)–(2.24)). In an isotropic medium we know that $\mathbf{k}$ and $\mathbf{E}_0$ are perpendicular, but even after the direction of $\mathbf{k}$ is specified, we are still free to have $\mathbf{E}_0$ point anywhere in the two dimensions perpendicular to $\mathbf{k}$. If we orient our coordinate system with the z-axis in the direction of $\mathbf{k}$, we can write (6.1) as

$$\mathbf{E}(z,t) = \left(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}\right) e^{i(kz-\omega t)} \tag{6.2}$$

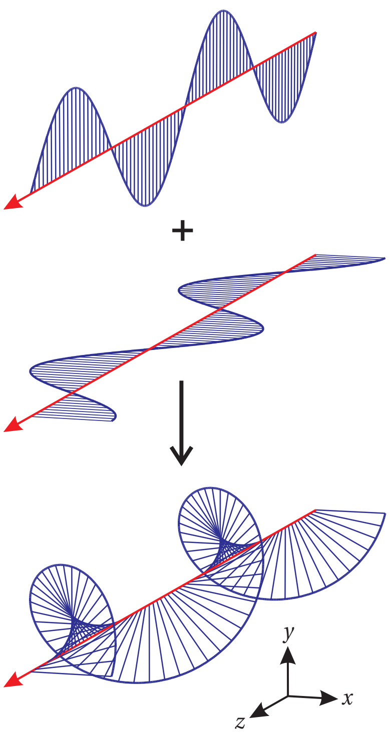


Figure 6.2 The combination of two orthogonally polarized plane waves that are out of phase results in elliptically polarized light. Here we have left circularly polarized light created as specified by (6.3).

As always, only the real part of (6.2) is physically relevant. The complex amplitudes of E_x and E_y keep track of the phase of the oscillating field components. In general the complex phases of E_x and E_y can differ, so that the wave in one of the dimensions lags or leads the wave in the other dimension.

The relationship between E_x and E_y describes the polarization of the light. For example, if E_y is zero, the plane wave is said to be *linearly polarized* along the x-dimension. Linearly polarized light can have any orientation in the x–y plane, and it occurs whenever E_x and E_y have the same complex phase (or a phase differing by π). For our purposes, we will take the x-dimension to be horizontal and the y-dimension to be vertical unless otherwise noted.

As an example, suppose $E_y = iE_x$, where E_x is real. The y-component of the field is then out of phase with the x-component by the factor $i = e^{i\pi/2}$. Taking the real part of the field (6.2) we get

$$\begin{aligned}\mathbf{E}(z,t) &= \text{Re}\left[E_x e^{i(kz-\omega t)}\right]\hat{\mathbf{x}} + \text{Re}\left[e^{i\pi/2}E_x e^{i(kz-\omega t)}\right]\hat{\mathbf{y}} \\ &= E_x\cos(kz-\omega t)\,\hat{\mathbf{x}} + E_x\cos(kz-\omega t+\pi/2)\,\hat{\mathbf{y}} \qquad \text{(left circular)} \\ &= E_x\left[\cos(kz-\omega t)\,\hat{\mathbf{x}} - \sin(kz-\omega t)\,\hat{\mathbf{y}}\right]\end{aligned} \tag{6.3}$$

In this example, the field in the y-dimension lags behind the field in the x-dimension by a quarter cycle. That is, the behavior seen in the x-dimension happens in the y-dimension a quarter cycle later. The field never goes to zero simultaneously in both dimensions. In fact, in this example the strength of the electric field is constant, and it rotates in a circular pattern in the x-y dimensions. For this reason, this type of field is called *circularly polarized.* Figure 6.2 graphically shows the two linear polarized pieces in (6.3) adding to make circularly polarized light.

If we view a circularly polarized light field throughout space at a frozen instant in time (as in Fig. 6.2), the electric field vector spirals as we move along the z-dimension. If the sense of the spiral (with time frozen) matches that of a common wood screw oriented along the z-axis, the polarization is called *right handed.* (It makes no difference whether the screw is flipped end for end.) If instead the field spirals in the opposite sense, then the polarization is called *left handed*. The field shown in Fig. 6.2 is an example of left-handed circularly polarized light.

An equivalent way to view the handedness convention is to imagine the light impinging on a screen as a function of time. The field of a right-handed circularly polarized wave rotates counter clockwise at the screen, when looking along the **k** direction (towards the front side of the screen). The field rotates clockwise for a left-handed circularly polarized wave.

Linearly polarized light can become circularly or, in general, *elliptically* polarized after reflection from a metal surface if the incident light has both s- and p-polarized components. A good experimentalist working with light needs to know this. Reflections from multilayer dielectric mirrors can also exhibit these phase shifts.

6.2 Jones Vectors for Representing Polarization

In 1941, R. Clark Jones introduced a two-dimensional matrix algebra that is useful for keeping track of light polarization and the effects of optical elements that influence polarization.[1] The algebra deals with light having a definite polarization, such as plane waves. It does not apply to un-polarized or partially polarized light (e.g. sunlight). For partially polarized light, a four-dimensional algebra known as Stokes calculus is used (see Appendix 6.B).

In preparation for introducing Jones vectors, we explicitly write the complex phases of the field components in (6.2) as

$$\mathbf{E}(z,t) = \left(|E_x|e^{i\phi_x}\hat{\mathbf{x}} + |E_y|e^{i\phi_y}\hat{\mathbf{y}}\right)e^{i(kz-\omega t)} \tag{6.4}$$

and then factor (6.4) as follows:

$$\mathbf{E}(z,t) = E_{\text{eff}}\left(A\hat{\mathbf{x}} + Be^{i\delta}\hat{\mathbf{y}}\right)e^{i(kz-\omega t)} \tag{6.5}$$

where

$$E_{\text{eff}} \equiv \sqrt{|E_x|^2 + \left|E_y\right|^2}\, e^{i\phi_x} \tag{6.6}$$

$$A \equiv \frac{|E_x|}{\sqrt{|E_x|^2 + \left|E_y\right|^2}} \tag{6.7}$$

$$B \equiv \frac{\left|E_y\right|}{\sqrt{|E_x|^2 + \left|E_y\right|^2}} \tag{6.8}$$

$$\delta \equiv \phi_y - \phi_x \tag{6.9}$$

Please notice that A and B are real non-negative dimensionless numbers that satisfy $A^2 + B^2 = 1$. If E_y is zero, then $B = 0$ and everything is well-defined. On the

R. Clark Jones (1916–2004, American) was born in Toledo Ohio. He was one of six high school seniors to receive a Harvard College National Prize Fellowship. He earned both his undergraduate (summa cum laude 1938) and Ph.D. degrees from Harvard (1941). After working several years at Bell Labs, he spent most of his professional career at Polaroid Corporation in Cambridge MA, until his retirement in 1982. He is well-known for a series of papers on polarization published during the period 1941-1956. He also contributed greatly to the development of infrared detectors. He was an avid train enthusiast, and even wrote papers on railway engineering. See J. Opt. Soc. Am. **63**, 519-522 (1972). Also see SPIE oemagazine, p. 52 (Aug. 2004).

[1] E. Hecht, *Optics*, 3rd ed., Sect. 8.12.2 (Massachusetts: Addison-Wesley, 1998).

other hand, if E_x happens to be zero, then its phase $e^{i\phi_x}$ is indeterminant. In this case we let $E_{\text{eff}} = |E_y|e^{i\phi_y}$, $B = 1$, and $\delta = 0$.

The overall field strength E_{eff} is often unimportant in a discussion of polarization. It represents the strength of an *effective* linearly polarized field that would correspond to the same intensity as (6.4). Specifically, from (2.62) and (6.5) we have

$$I = \langle S \rangle_t = \frac{1}{2} n c \epsilon_0 \mathbf{E} \cdot \mathbf{E}^* = \frac{1}{2} n c \epsilon_0 |E_{\text{eff}}|^2 \tag{6.10}$$

The phase of E_{eff} represents an overall phase shift that one can trivially adjust by physically moving the light source (a laser, say) forward or backward by a fraction of a wavelength.

The portion of (6.5) that is relevant to our discussion of polarization is the vector $A\hat{\mathbf{x}} + Be^{i\delta}\hat{\mathbf{y}}$, referred to as the *Jones vector*. This vector contains the essential information regarding field polarization. Notice that the Jones vector is a kind of unit vector, in that $(A\hat{\mathbf{x}} + Be^{i\delta}\hat{\mathbf{y}}) \cdot (A\hat{\mathbf{x}} + Be^{i\delta}\hat{\mathbf{y}})^* = 1$. (The asterisk represents the complex conjugate.) When writing a Jones vector we dispense with the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ notation and organize the components into a column vector (for later use in matrix algebra) as follows:

$$\begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} \tag{6.11}$$

This vector can describe the polarization state of any plane wave field. Table 6.1 lists some Jones vectors representing various polarization states.

Linearly polarized along x

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Linearly polarized along y

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Linearly polarized at angle α (measured from the x-axis)

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

Right circularly polarized

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Left circularly polarized

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Table 6.1 Jones Vectors for several common polarization states.

6.3 Elliptically Polarized Light

In general, the Jones vector (6.11) represents a polarization state between linear and circular. This 'between' state is known as *elliptically polarized* light. As the wave travels, the field vector makes a spiral motion. If we observe the field vector at a point as the field goes by, the field vector traces out an ellipse oriented perpendicular to the direction of travel (i.e. in the x–y plane). One of the axes of the ellipse occurs at the angle

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2AB\cos\delta}{A^2 - B^2}\right) \tag{6.12}$$

with respect to the x-axis (see P6.8). This angle sometimes corresponds to the minor axis and sometimes to the major axis of the ellipse, depending on the exact values of A, B, and δ. The other axis of the ellipse (major or minor) then occurs at $\alpha \pm \pi/2$ (see Fig. 6.3).

We can deduce whether (6.12) corresponds to the major or minor axis of the ellipse by comparing the strength of the electric field when it spirals through the direction specified by α and when it spirals through $\alpha \pm \pi/2$. The strength of the electric field at α is given by (see P6.8)

$$E_\alpha = |E_{\text{eff}}|\sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha + AB\cos\delta\sin 2\alpha} \qquad (E_{\max} \text{ or } E_{\min}) \tag{6.13}$$

and the strength of the field when it spirals through the orthogonal direction $(\alpha \pm \pi/2)$ is given by

$$E_{\alpha\pm\pi/2} = |E_{\text{eff}}|\sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha - AB\cos\delta\sin 2\alpha} \qquad (E_{\max} \text{ or } E_{\min}) \quad (6.14)$$

After computing (6.13) and (6.14), we decide which represents $E_{\min}$ and which $E_{\max}$ according to

$$E_{\max} \geq E_{\min} \quad (6.15)$$

We could predict in advance which of (6.13) or (6.14) corresponds to the major axis and which corresponds to the minor axis. However, making this prediction is as complicated as simply evaluating (6.13) and (6.14) and determining which is greater.

Elliptically polarized light is often characterized by the *ellipticity*, given by the ratio of the minor axis to the major axis:

$$e \equiv \frac{E_{\min}}{E_{\max}} \quad (6.16)$$

The ellipticity e ranges between zero (corresponding to linearly polarized light) and one (corresponding to circularly polarized light). Finally, the *helicity* or handedness of elliptically polarized light is as follows (see P6.2):

$$0 < \delta < \pi \quad \rightarrow \quad \text{left-handed helicity} \quad (6.17)$$

$$\pi < \delta < 2\pi \quad \rightarrow \quad \text{right-handed helicity} \quad (6.18)$$

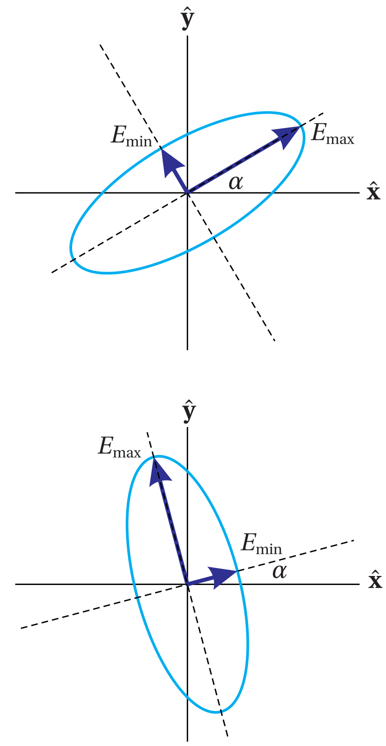


Figure 6.3 The electric field of elliptically polarized light traces an ellipse in the plane perpendicular to its propagation direction. The two plots are for different values of A, B, and δ. The angle α can describe the major axis (top) or the minor axis (bottom), depending on the values of these parameters.

6.4 Linear Polarizers and Jones Matrices

In 1928, Edwin Land invented an inexpensive polarizing device. He did it by stretching a polymer sheet and infusing it with iodine. The stretching caused the polymer chains to align along a common direction, whereupon the sheet was cemented to a substrate. The infusion of iodine caused the individual chains to become conductive, like microscopic wires.

When light impinges upon Land's Polaroid sheet, the component of electric field that is *parallel* to the polymer chains causes a current $\mathbf{J}_{\text{free}}$ to oscillate in that dimension. The resistance to the current quickly dissipates the energy (i.e. the refractive index is complex) and the light is absorbed. The thickness of the Polaroid sheet is chosen sufficiently large to ensure that virtually none of the light with electric field component oscillating along the chains makes it through the device.

The component of electric field that is orthogonal to the polymer chains encounters electrons that are essentially bound to the narrow width of individual polymer molecules. For this polarization component, the wave passes through the material much like it does through typical dielectrics such as glass (i.e. the refractive index is real). Today, there is a wide variety of technologies for making polarizers, many very different from Polaroid.

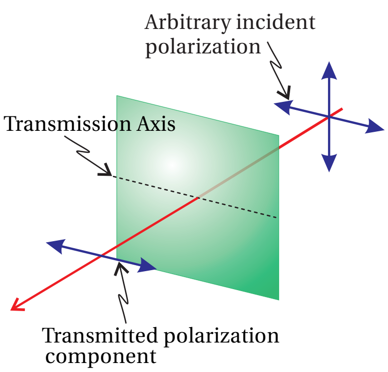


Figure 6.4 Light transmitting through a Polaroid sheet. The conducting polymer chains run vertically in this drawing, and light polarized along the chains is absorbed. Light polarized perpendicular to the polymer chains passes through the polarizer.

A polarizer can be represented as a 2×2 matrix that operates on Jones vectors.[2] The function of a polarizer is to pass only the component of electric field that is oriented along the polarizer transmission axis. If a polarizer is oriented with its transmission axis along the x-dimension, only the x-component of polarization transmits; the y-component is killed. If the polarizer is oriented with its transmission axis along the y-dimension, only the y-component of the field transmits, and the x-component is killed. These two scenarios can be represented with the following Jones matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(polarizer with transmission along x-axis)} \tag{6.19}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(polarizer with transmission along y-axis)} \tag{6.20}$$

These matrices operate on any Jones vector representing the polarization of incident light. The result gives the Jones vector for the light exiting the polarizer.

Example 6.1

Use the Jones matrix (6.19) to calculate the effect of a horizontal polarizer on light that is initially horizontally polarized, vertically polarized, and arbitrarily polarized.

Solution: First we consider a horizontally polarized plane wave traversing a polarizer with its transmission axis oriented also horizontally (x-dimension):

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(horizontal polarizer on horizontally polarized field)}$$

As expected, the polarization state is unaffected by the polarizer. (We have ignored possible attenuation from surface reflections.)

Now consider vertically polarized light traversing the same horizontal polarizer. In this case, we have:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(horizontal polarizer on vertical linear polarization)}$$

As expected, the polarizer extinguishes the light.

Finally, when a horizontally oriented polarizer operates on light with an arbitrary Jones vector (6.11), we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} \qquad \text{(horizontal polarizer on arbitrary polarization)}$$

Only the horizontal component of polarization is transmitted through the polarizer.

[2] E. Hecht, *Optics*, 3rd ed., Sect. 8.12.3 (Massachusetts: Addison-Wesley, 1998).

While you might readily agree that the matrices given in (6.19) and (6.20) can be used to get the right result for light traversing a horizontal or a vertical polarizer, you probably aren't very impressed as of yet. In the next few sections, we will derive Jones matrices for a number of optical elements that can modify polarization: polarizers at arbitrary angle, wave plates at arbitrary angle, and reflection or transmissions at an interface. Table 6.2 shows Jones matrices for each of these devices. Before deriving these specific Jones matrices, however, we take a moment to appreciate why the Jones matrix formulation is useful.

The real power of the formalism becomes clear as we consider situations where light encounters multiple polarization elements in sequence. In these situations, we use a product of Jones matrices to represent the effect of the compound systems. We can represent this situation by

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \mathbf{J}_{\text{system}} \begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} \tag{6.21}$$

where the unprimed Jones vector represents light going into the system and the primed Jones vector represents light emerging from the system.

The matrix $\mathbf{J}_{\text{system}}$ is a Jones matrix formed by a series of polarization devices. If there are N devices in the system, the compound matrix is calculated as

$$\mathbf{J}_{\text{system}} \equiv \mathbf{J}_N \mathbf{J}_{N-1} \cdots \mathbf{J}_2 \mathbf{J}_1 \tag{6.22}$$

where $\mathbf{J}_n$ is the matrix for the n^{th} polarizing optical element encountered in the system. Notice that the matrices operate on the Jones vector in the order that the light encounters the devices. Therefore, the matrix for the first device ($\mathbf{J}_1$) is written on the *right*, and so on until the last device encountered, which is written on the *left*, farthest from the Jones vector.

When part of the light is absorbed by passing through one or more polarizers in a system, the Jones vector of the exiting light does not necessarily remain normalized to magnitude one (see Example 6.1). The factor by which the intensity of the light decreases is given by $\left(A'\hat{\mathbf{x}} + B'\hat{\mathbf{y}}\right) \cdot \left(A'\hat{\mathbf{x}} + B'\hat{\mathbf{y}}\right)^* = \left|A'\right|^2 + \left|B'\right|^2$. The intensity exiting from the system is then

$$I' = \frac{1}{2} n c \epsilon_0 \left|E'_{\text{eff}}\right|^2 \quad \text{where} \quad \left|E'_{\text{eff}}\right|^2 = \left|E_{\text{eff}}\right|^2 \left(\left|A'\right|^2 + \left|B'\right|^2\right) \tag{6.23}$$

Here, E_{eff} is the original effective field before entering the system (see (6.10)), and E'_{eff} is the final effective field.

For the sake of further analysis, if desired, one can renormalize the final Jones vector and write it in standard form as follows:

$$\begin{bmatrix} \tilde{A}' \\ \tilde{B}' e^{i\delta'} \end{bmatrix} = \frac{e^{i\phi_{A'}}}{\sqrt{|A'|^2 + |B'|^2}} \begin{bmatrix} |A'| \\ |B'| e^{i\delta'} \end{bmatrix}$$

This is the Jones vector that is consistent with E'_{eff}. The uninteresting overall phase factor $e^{i\phi_{A'}}$ can be incorporated into E'_{eff}, making $\tilde{A}'$ real and positive. δ' is the phase difference between B' and A'.

Linear polarizer

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Half wave plate

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Quarter wave plate

$$\begin{bmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{bmatrix}$$

Right circular polarizer

$$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Left circular polarizer

$$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

Reflection from an interface

$$\begin{bmatrix} -r_p & 0 \\ 0 & r_s \end{bmatrix}$$

Transmission through an interface

$$\begin{bmatrix} t_p & 0 \\ 0 & t_s \end{bmatrix}$$

Table 6.2 Common Jones Matrices. The angle θ is measured with respect to the x-axis and specifies the transmission axis of a linear polarizer or the fast axis of a wave plate.

6.5 Jones Matrix for a Polarizer

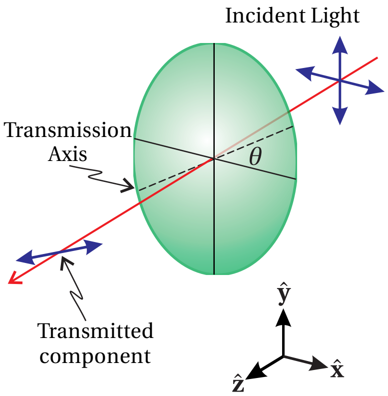


Figure 6.5 Light transmitting through a polarizer oriented with transmission axis at angle θ from x-axis.

In this section we develop a Jones matrix for describing an ideal polarizer with its transmission axis at an arbitrary angle θ from the x-axis. We will do this in a general context so that we can take advantage of the present work when discussing wave plates in the next section. To help keep things on a more conceptual level, we revert back to using electric field components directly. We will make the connection with Jones calculus at the end.

The polarizer acts on a plane wave with arbitrary polarization. The electric field of our plane wave may be written as

$$\mathbf{E}(z,t) = \left(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}\right)e^{i(kz-\omega t)} \tag{6.24}$$

Let the transmission axis of the polarizer be specified by the unit vector $\hat{\mathbf{e}}_1$ and the absorption axis of the polarizer be specified by $\hat{\mathbf{e}}_2$ (orthogonal to the transmission axis). The vector $\hat{\mathbf{e}}_1$ is oriented at an angle θ from the x-axis, as shown in Fig. 6.6. We need to write the electric field components in terms of the new basis specified by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. By inspection of the geometry, the x-y unit vectors are connected to the new coordinate system via:

$$\begin{aligned}\hat{\mathbf{x}} &= \cos\theta\,\hat{\mathbf{e}}_1 - \sin\theta\,\hat{\mathbf{e}}_2\\ \hat{\mathbf{y}} &= \sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2\end{aligned} \tag{6.25}$$

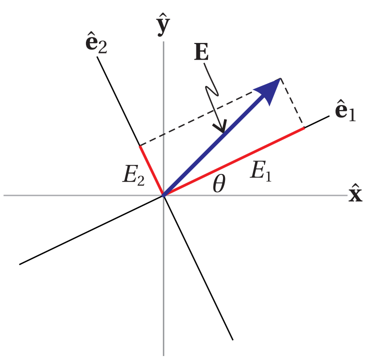


Figure 6.6 Electric field components written in the $\hat{\mathbf{e}}_1$–$\hat{\mathbf{e}}_2$ basis.

Substitution of (6.25) into (6.24) yields for the electric field

$$\mathbf{E}(z,t) = \left(E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2\right)e^{i(kz-\omega t)} \tag{6.26}$$

where

$$\begin{aligned}E_1 &\equiv E_x\cos\theta + E_y\sin\theta\\ E_2 &\equiv -E_x\sin\theta + E_y\cos\theta\end{aligned} \tag{6.27}$$

Now we introduce the effect of the polarizer on the field: E_1 is transmitted unaffected, while E_2 is extinguished. To account for the effect of the device, we multiply E_2 by a parameter ξ. In the case of the polarizer, ξ is zero, but when we consider wave plates we will use other values for ξ. After traversing the polarizer, the field becomes

$$\mathbf{E}_{\text{after}}(z,t) = \left(E_1\hat{\mathbf{e}}_1 + \xi E_2\hat{\mathbf{e}}_2\right)e^{i(kz-\omega t)} \tag{6.28}$$

We now have the field after the polarizer, but it would be nice to rewrite it in terms of the original x–y basis. By inverting (6.25), or again by inspection of Fig. 6.6, we see that

$$\begin{aligned}\hat{\mathbf{e}}_1 &= \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}\\ \hat{\mathbf{e}}_2 &= -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}\end{aligned} \tag{6.29}$$

Substitution of these relationships into (6.28) together with the definitions (6.27) for E_1 and E_2 yields

$$\begin{aligned}\mathbf{E}_{\text{after}}(z,t) &= \left[\left(E_x\cos\theta + E_y\sin\theta\right)\left(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}\right)\right.\\ &\qquad \left.+\xi\left(-E_x\sin\theta + E_y\cos\theta\right)\left(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right)\right]e^{i(kz-\omega t)}\\ &= \left[E_x\left(\cos^2\theta + \xi\sin^2\theta\right) + E_y\left(\sin\theta\cos\theta - \xi\sin\theta\cos\theta\right)\right]\hat{\mathbf{x}}e^{i(kz-\omega t)}\\ &\quad + \left[E_x\left(\sin\theta\cos\theta - \xi\sin\theta\cos\theta\right) + E_y\left(\sin^2\theta + \xi\cos^2\theta\right)\right]\hat{\mathbf{y}}e^{i(kz-\omega t)}\end{aligned} \tag{6.30}$$

Notice that if $\xi = 1$ (i.e. no polarizer), then we get back exactly what we started with (i.e. (6.30) reduces to (6.24)).

To get to the Jones matrix for the polarizer, we note that (6.30) is a linear mixture of E_x and E_y which can be represented with matrix algebra. If we represent the electric field as a two-dimensional column vector with its x component in the top and its y component in the bottom (like a Jones vector), then we can rewrite (6.30) as

$$\mathbf{E}_{\text{after}}(z,t) = \begin{bmatrix}\cos^2\theta + \xi\sin^2\theta & \sin\theta\cos\theta - \xi\sin\theta\cos\theta\\ \sin\theta\cos\theta - \xi\sin\theta\cos\theta & \sin^2\theta + \xi\cos^2\theta\end{bmatrix}\begin{bmatrix}E_x\\ E_y\end{bmatrix}e^{i(kz-\omega t)} \tag{6.31}$$

The matrix here is a properly normalized Jones matrix, even though we did not bother factoring out E_{eff} to make a properly normalized Jones vector, as specified in (6.5). We can now write down the Jones matrix for a polarizer by inserting $\xi = 0$ into the matrix:

$$\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix} \quad \text{(polarizer with transmission axis at angle } \theta\text{)} \tag{6.32}$$

Notice that when $\theta = 0$ this matrix reduces to that of a horizontal polarizer (6.19), and when $\theta = \pi/2$, it reduces to that of a vertical polarizer (6.20).

6.6 Jones Matrix for Wave Plates

We next consider *wave plates* (or *retarders*), which are usually made from birefringent crystals. The index of refraction in the crystal depends on the orientation of the electric field polarization. A wave plate has the appearance of a thin window through which the light passes. The crystal is cut such that the wave plate has a *fast* and a *slow axis*, which are 90° apart in the plane of the window. If the light is polarized along the fast axis, it experiences index n_{fast}. The orthogonal polarization component experiences higher index n_{slow}.

When a plane wave passes through a wave plate, the component of the electric field oriented along the fast axis travels faster than its orthogonal counterpart, which introduces a relative phase between the two polarization components. As light passes through a wave plate of thickness d, the phase difference that accumulates between the fast and the slow polarization components is

$$k_{\text{slow}}d - k_{\text{fast}}d = \frac{2\pi d}{\lambda_{\text{vac}}}\left(n_{\text{slow}} - n_{\text{fast}}\right) \tag{6.33}$$

Edwin H. Land (1909–1991, American) was born in Bridgeport Connecticut. He began college at Harvard University, but dropped out to work on his idea of making an inexpensive polarizer. He had access to scientific literature at New York Public Library. He gained access to laboratory equipment by sneaking into Columbia University after hours. In a major breakthrough. Land invented what later would be called polaroid film. He resumed his studies at Harvard, but never graduated. This was in spite of the efforts of his wife who would extract answers from him and write up his homework. A few years later, Land and a financial backer formed Polaroid Corporation, which had tremendous success and growth thanks to Land's continued innovations over the years, including his development of an instant camera. Land would often work on a problem for days without going home or changing his clothes. He sometimes needed to be reminded to eat. (Wikipedia)

By adjusting the thickness of the wave plate, one can introduce any desired phase difference.

The most common types of wave plates are the *quarter-wave plate* and the *half-wave plate*. The quarter-wave plate introduces a phase difference of

$$k_{\text{slow}} d - k_{\text{fast}} d = \pi/2 + 2\pi m \qquad \text{(quarter-wave plate)} \quad (6.34)$$

between the two polarization components, where m is an integer. This means that the polarization component along the slow axis is delayed spatially by a quarter wavelength (or five quarters, etc.).

The half-wave plate introduces a phase difference of

$$k_{\text{slow}} d - k_{\text{fast}} d = \pi + 2\pi m \qquad \text{(half-wave plate)} \quad (6.35)$$

where m is an integer. This means that the polarization component along the slow axis is delayed spatially by a half wavelength (or three halves, etc.). When $m = 0$ in either (6.34) or (6.35), the wave plate is said to be *zero order*.

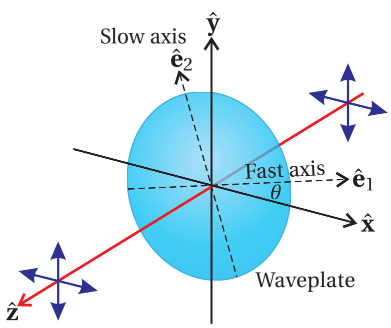


Figure 6.7 Wave plate interacting with a plane wave.

The derivation of the Jones matrix for the two wave plates is essentially the same as the derivation for the polarizer in the previous section. Let $\hat{\mathbf{e}}_1$ correspond to the fast axis, and let $\hat{\mathbf{e}}_2$ correspond to the slow axis, as illustrated in Fig. 6.7. We proceed as before. However, instead of setting ξ equal to zero in (6.31), we must choose values for ξ appropriate for each wave plate. Since nothing is absorbed, ξ should have a magnitude equal to one. The important feature is the phase of ξ. As seen in (6.33), the field component along the slow axis accumulates excess phase relative to the component along the fast axis, and we let ξ account for this. In the case of the quarter-wave plate, the appropriate factor from (6.34) is

$$\xi = e^{i\pi/2} = i \qquad \text{(quarter-wave plate)} \quad (6.36)$$

This describes a *relative* phase delay for the light emerging with polarization along the slow axis. Substituting (6.36) into (6.30) yields the Jones matrix for a quarter wave plate:

$$\begin{bmatrix} \cos^2\theta + i\sin^2\theta & \sin\theta\cos\theta - i\sin\theta\cos\theta \\ \sin\theta\cos\theta - i\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{bmatrix} \qquad \text{(quarter-wave plate)} \quad (6.37)$$

For the half-wave plate, the appropriate factor applied to the slow axis is

$$\xi = e^{i\pi} = -1 \qquad \text{(half-wave plate)} \quad (6.38)$$

and the Jones matrix becomes:

$$\begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \qquad \text{(half-wave plate)} \quad (6.39)$$

Remember that θ refers to the angle that the fast axis makes with respect to the x-axis.

Before moving on, consider the following two examples that illustrate how wave plates are often used:

Example 6.2

Calculate the Jones matrix for a quarter-wave plate at $\theta = 45°$, and determine its effect on horizontally polarized light.

Solution: At $\theta = 45°$, the Jones matrix for the quarter-wave plate (6.37) reduces to

$$\frac{e^{i\pi/4}}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \qquad \text{(quarter-wave plate, fast axis at } \theta = 45° \text{)} \tag{6.40}$$

The overall phase factor $e^{i\pi/4}$ in front is unimportant since it can be adjusted arbitrarily by moving the light source forwards or backwards through a fraction of a wavelength.

Now we calculate the effect of the quarter-wave plate (oriented at $\theta = 45°$) operating on horizontally polarized light:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{6.41}$$

The linearly polarized light becomes right-circularly polarized (see Table 6.1)

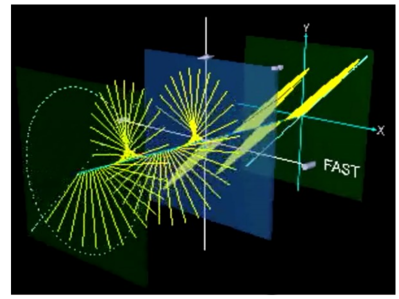


Figure 6.8 Animation showing effects of polarizers and wave plates on polarized light.

The previous example shows that a quarter-wave plate (properly oriented) can change linearly polarized light into circularly polarized light. A quarter wave plate can perform the reverse operation as well. On the other hand, as seen in the next example, a half-wave plate can rotate the polarization angle of linearly polarized light by varying degrees while preserving the linear polarization.

Example 6.3

Calculate the effect of a half wave plate at an arbitrary θ on horizontally polarized light.

Solution: Multiplying by the half-wave matrix (6.39), we obtain

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix} \tag{6.42}$$

The resulting Jones vector describes linearly polarized light at an angle $\alpha = 2\theta$ from the x-axis (see Table 6.1).

6.7 Polarization Effects of Reflection and Transmission

When light encounters a material interface, the amount of reflected and transmitted light depends on the polarization. The Fresnel coefficients (3.20)–(3.23) dictate how much of each polarization is reflected and how much is transmitted. In addition, the Fresnel coefficients keep track of phases intrinsic in the reflection phenomenon. This is true also for reflections from multilayer coatings with effective Fresnel coefficients (4.59), (4.60), (4.64) and (4.65).

To the extent that the s and p components of the field behave differently, the overall polarization state is altered. For example, a linearly-polarized field upon reflection can become elliptically polarized (see L 6.9). Even when a wave reflects at normal incidence so that the s and p components are indistinguishable, right-circular polarized light becomes left-circular polarized. This is the same effect that causes a right-handed person to appear left-handed when viewed in a mirror.

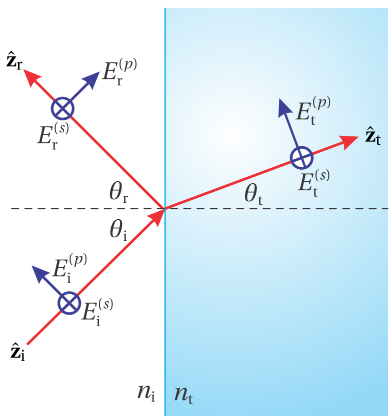


Figure 6.9 Incident, reflected and transmitted plane waves, each propagating along the z-axis of its own reference frame.

We can use Jones calculus to keep track of how reflection and transmission influences polarization. However, before proceeding, we emphasize that in this context we do not strictly adhere to a single coordinate system as we did in chapter 3, for example in Fig. 3.1. Instead, we consider each plane wave, whether incident, reflected or transmitted, to propagate in the z-direction of its own frame, regardless of the relative angles between the incident and reflected wave. This loose manner of defining coordinate systems, depicted in Fig. 6.9, has a great advantage. The x and y dimensions in each individual frame is aligned parallel to their respective s and p field component. We will adopt the convention that p-polarized light in all cases is associated with the x-dimension (horizontal, say). The s-polarized component then lies along the y-dimension (vertical). These conventions are different from those used in chapter 3 but will do us no harm.

We are now in a position to see why there is a handedness inversion upon reflection from a mirror. Notice in Fig. 6.9 that for the incident light, the s-component of the field crossed into the p-component of the field yields a vector pointing along the beam's propagation direction. However, for the reflected light, the s-component crossed into the p-component points opposite to that beam's propagation direction.

The Jones matrix corresponding to reflection from a surface is

$$\begin{bmatrix} -r_p & 0 \\ 0 & r_s \end{bmatrix} \quad \text{(Jones matrix for reflection)} \tag{6.43}$$

By convention, we place the minus sign on the coefficient r_p to take care of handedness inversion. We could put the minus sign on r_s instead; the important point is that the two polarizations acquire a relative phase differential of π when the propagation direction flips.[3]

The Fresnel coefficients specify the ratios of the exiting fields to the incident ones. When (6.43) operates on an arbitrary Jones vector such as (6.11), $-r_p$ multiplies the horizontal component of the field, and r_s multiplies the vertical component of the field. Especially in the case of reflection from an absorbing surface such as a metal, the phases of the two polarization components can vary markedly (see P6.13). Thus, linearly polarized light containing both s- and p-components in general becomes elliptically polarized when reflected from such a surface. When light undergoes total internal reflection, again the phases of the s-

[3] The minus sign is needed for our specific convention of field directions, as drawn in Fig. 6.9. In our convention, r_s and r_p are identical at normal incidence.

and p-components differ markedly, which can cause linearly polarized light to become elliptically polarized (see P6.14).

Transmission through a material interface can also influence the polarization of the field, although typically to a lesser degree. However, there is no handedness inversion, since the light continues on in a forward sense. The Jones matrix for transmission is

$$\begin{bmatrix} t_p & 0 \\ 0 & t_s \end{bmatrix} \quad \text{(Jones matrix for transmission)} \tag{6.44}$$

If a beam of light encounters a series of mirrors, the final polarization is determined by multiplying the sequence of appropriate Jones matrices (6.43) onto the initial polarization. This procedure is straightforward if the normals to all of the mirrors lie in a single plane (say parallel to the surface of an optical bench). However, if the beam path deviates from this plane (due to vertical tilt on the mirrors), then we must reorient our coordinate system before each mirror to have a new 'horizontal' (p-polarized dimension) and the new 'vertical' (s-polarized dimension). Earlier in this chapter we performed a rotation of a coordinate system through an angle θ, described in (6.27), which is also useful here. The rotation can be accomplished by multiplying the following matrix onto the incident Jones vector:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{(rotation of coordinates through an angle } \theta\text{)} \tag{6.45}$$

This is understood as a rotation about the z-axis. The angle of rotation θ is chosen such that the rotated x-axis lies in the plane of incidence for the mirror. When such a reorientation of coordinates is necessary, the two orthogonal field components in the initial coordinate system are stirred together to form the field components in the new system. This does not change the intrinsic characteristics of the polarization, just its representation.

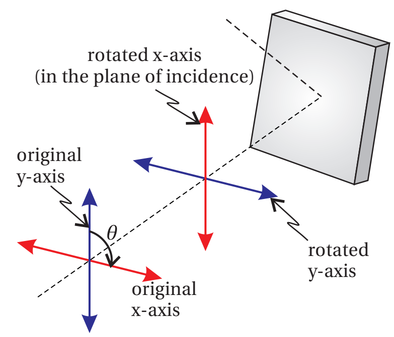


Figure 6.10 If the plane of incidence does not coincide for successive elements in an optical system, a rotation matrix must be applied to rotate the x-axis to the plane of incidence before computing the effect of each element.

Appendix 6.A Ellipsometry

Measuring the polarization of light reflected from a surface can yield information regarding the optical properties of that surface. As done in L 6.9, it is possible to characterize the polarization of a beam of light using a quarter-wave plate and a polarizer. However, we often want to measure reflections at a range of frequencies, and this would require a different quarter-wave plate thickness d for each wavelength used (see (6.34)). Therefore, many commercial *ellipsometers* do not try to extract the helicity of the light, but only the ellipticity. In this case only polarizers are needed, which can be made to work over a wide range of wavelengths. If, in addition, a variety of incident angles are measured, it is possible to extract detailed information about the optical constants n and κ and the thicknesses of possibly many layers of materials influencing the reflection.

Commercial ellipsometers[4] typically employ two polarizers, one before and

[4] See Spectroscopic Ellipsometry Tutorial at J. A. Woollam Co.

one after the sample, where s and p-polarized reflections take place. The first polarizer ensures that linearly polarized light arrives at the test surface (polarized at angle α to give both s and p-components). The Jones matrix for the test surface reflection is given by (6.43), and the Jones matrix for the analyzing polarizer oriented at angle θ is given by (6.32). The Jones vector for the light arriving at the detector is then

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} -r_p & 0 \\ 0 & r_s \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} -r_p\cos\alpha\cos^2\theta + r_s\sin\alpha\sin\theta\cos\theta \\ -r_p\cos\alpha\sin\theta\cos\theta + r_s\sin\alpha\sin^2\theta \end{bmatrix} \quad (6.46)$$

and the intensity arriving to the detector is

$$\begin{aligned} I &\propto \left|-r_p\cos\alpha\cos^2\theta + r_s\sin\alpha\cos\theta\sin\theta\right|^2 + \left|-r_p\cos\alpha\cos\theta\sin\theta + r_s\sin\alpha\sin^2\theta\right|^2 \\ &= \left|r_p\right|^2\cos^2\alpha\cos^2\theta + |r_s|^2\sin^2\alpha\sin^2\theta - \frac{\left(r_p r_s^* + r_s r_p^*\right)}{4}\sin 2\alpha\sin 2\theta \end{aligned} \quad (6.47)$$

For ellipsometry measurements, it is customary to express the ratio of Fresnel coefficients as

$$r_p/r_s \equiv \tan\Psi e^{i\Delta} \quad (6.48)$$

In this case, the intensity may be shown to be proportional to (see problem P6.15)

$$I \propto 1 - \eta\sin 2\theta + \xi\cos 2\theta \quad (6.49)$$

where

$$\eta \equiv 2\frac{\tan\Psi\cos\Delta\tan\alpha}{\tan^2\Psi + \tan^2\alpha} \quad \text{and} \quad \xi \equiv \frac{\tan^2\Psi - \tan^2\alpha}{\tan^2\Psi + \tan^2\alpha} \quad (6.50)$$

In commercial ellipsometers, the angle θ of the analyzing polarizer often rotates at a high speed, and the time dependence of the light reaching a detector is analyzed. From this type of measurement, the coefficients η and ξ can be extracted with high precision. Then equations (6.50) can be inverted (see problem P6.15) to reveal

$$\tan\Psi = \sqrt{\frac{1+\xi}{1-\xi}}\,|\tan\alpha| \quad \text{and} \quad \cos\Delta = \frac{\eta}{\sqrt{1-\xi^2}}\,\text{sign}(\alpha) \quad (6.51)$$

From a series of these types of measurements, it is possible to extract the values of n and κ for materials from the expressions for r_s and r_p (with the aid of a computer!). With a sufficiently large number of unique measurements, it is possible even to characterize multilayer coatings involving layers with varying thicknesses and indices.

Appendix 6.B Partially Polarized Light

We outline here an approach for dealing with partially polarized light, which is a mixture of *polarized* and *unpolarized* light. Most natural light such as sunshine is unpolarized. The transverse electric field direction in natural light varies rapidly (and quasi randomly). Such variations imply the superposition of multiple frequencies as opposed to the single frequency assumed in the formulation of Jones calculus earlier in this chapter. Unpolarized light can become partially polarized when it, for example, reflects from a surface at oblique incidence, since *s* and *p* components of the polarization might reflect with differing strength.

Stokes vectors are used to keep track of the partial polarization (and attenuation) of a light beam as the light progresses through an optical system.[5] In contrast, Jones vectors are designed for pure polarization states. We can consider any light beam as an intensity sum of completely unpolarized light and perfectly polarized light:

$$I = I_{\text{pol}} + I_{\text{un}} \tag{6.52}$$

It is assumed that both types of light propagate in the same direction.

The main characteristic of unpolarized light is that it cannot be extinguished by a single polarizer (even in combination with a wave plate). Moreover, the transmission of unpolarized light through an ideal polarizer is always 50%. On the other hand, polarized light (be it linearly, circularly, or elliptically polarized) can always be represented by a Jones vector, and it is always possible to extinguish it using a quarter wave plate and a single polarizer.

We may introduce the *degree of polarization* as the fraction of the intensity that is in a definite polarization state:

$$\xi\text{pol} \equiv \frac{I_{\text{pol}}}{I_{\text{pol}} + I_{\text{un}}} \tag{6.53}$$

The degree of polarization takes on values between zero and one. Thus, if the light is completely unpolarized (such that $I_{\text{pol}} = 0$), the degree of polarization is zero, and if the beam is fully polarized (such that $I_{\text{un}} = 0$), the degree of polarization is one.

A Stokes vector, which characterizes a partially polarized beam, is written as

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}$$

The parameter

$$S_0 \equiv \frac{I}{I_{\text{in}}} = \frac{I_{\text{pol}} + I_{\text{un}}}{I_{\text{in}}} \tag{6.54}$$

is a comparison of the beam's intensity (or power) to a benchmark or 'input' intensity, I_{in}, measured before the beam enters the optical system under consideration. I represents the intensity at the point of investigation, where one wishes to characterize the beam. Thus, the value $S_0 = 1$ represents the input intensity,

Sir George Gabriel Stokes (1819–1903, Irish) was born in Skreen, Ireland. He entered Cambridge University at age 18 and graduated four years later with the distinction of senior wrangler. In 1849, he became a professor of mathematics at Cambridge where he later worked with James Clerk Maxwell and Lord Kelvin to form the Cambridge School of Mathematical Physics. Stokes was a powerful mathematician as well as good experimentalist, often testing his theoretical solutions in the laboratory. In addition to his contributions to optics, Stokes made important contributions to fluid dynamics (e.g. the Navier-Stokes equations) and to mathematical physics; Stokes' theorem is employed several places in this in this book.(Wikipedia)

[5] E. Hecht, *Optics*, 3rd ed., Sect. 8.12.1 (Massachusetts: Addison-Wesley, 1998).

and S_0 can drop to values less than one, to account for attenuation of light by polarizers in the system. (S_0 could increase in the atypical case of amplification.)

The next parameter, S_1, describes how much the light looks either horizontally or vertically polarized, and it is defined as

$$S_1 \equiv \frac{2I_{\text{hor}}}{I_{\text{in}}} - S_0 \tag{6.55}$$

Here, I_{hor} represents the amount of light detected if an ideal linear polarizer is placed with its axis aligned horizontally directly in front of the detector (inserted where the light is characterized). S_1 ranges between negative one and one, taking on its extremes when the light is linearly polarized either horizontally or vertically, respectively. If the light has been attenuated, it may still be perfectly horizontally polarized even if S_1 has a magnitude less than one. (Alternatively, one might examine S_1/S_0, which is guaranteed to range between negative one and one.)

The parameter S_2 describes how much the light looks linearly polarized along the diagonals. It is given by

$$S_2 \equiv \frac{2I_{45^\circ}}{I_{\text{in}}} - S_0 \tag{6.56}$$

Similar to the previous case, I_{45° represents the amount of light detected if an ideal linear polarizer is placed with its axis at 45° directly in front of the detector (inserted where the light is characterized). As before, S_2 ranges between negative one and one, taking on extremes when the light is linearly polarized either at 45° or 135°.

Finally, S_3 characterizes the extent to which the beam is either right or left circularly polarized:

$$S_3 \equiv \frac{2I_{\text{r-cir}}}{I_{\text{in}}} - S_0 \tag{6.57}$$

Here, $I_{\text{r-cir}}$ represents the amount of light detected if an ideal right-circular polarizer is placed directly in front of the detector. A right-circular polarizer is one that passes right-handed polarized light, but blocks left handed polarized light. One way to construct such a polarizer is a quarter wave plate, followed by a linear polarizer with the transmission axis aligned 45° from the wave-plate fast axis, followed by another quarter wave plate at −45° from the polarizer (see P6.12).[6] Again, this parameter ranges between negative one and one, taking on the extremes for right and left circular polarization, respectively.

Importantly, if any of the parameters S_1, S_2, or S_3 take on their extreme values (i.e. a magnitude equal to S_0), the other two parameters necessarily equal zero. As an example, if a beam is linearly polarized in the horizontal direction with $I = I_{\text{in}}$, then we have $I_{\text{hor}} = I_{\text{in}}$, $I_{45^\circ} = I_{\text{in}}/2$, and $I_{\text{r-cir}} = I_{\text{in}}/2$. This yields $S_0 = 1$, $S_1 = 1$, $S_2 = 0$, and $S_3 = 0$. As a second example, suppose that the light has been attenuated to $I = I_{\text{in}}/3$ but is purely left circularly polarized. Then we have

[6] The final quarter wave plate is to put the light back into the original circular state – not needed to measure the Stokes parameter.

$I_{\text{hor}} = I_{\text{in}}/6$, $I_{45°} = I_{\text{in}}/6$, and $I_{\text{r-cir}} = 0$. Whereas the Stokes parameters are $S_0 = 1/3$, $S_1 = 0$, $S_2 = 0$, and $S_3 = -1/3$.

Another interesting case is completely unpolarized light, which transmits 50% through all of the polarizers discussed above. In this case, $I_{\text{hor}} = I_{45°} = I_{\text{r-cir}} = I/2$ and $S_1 = S_2 = S_3 = 0$.

Example 6.4

Find the Stokes parameters for perfectly polarized light, represented by an arbitrary Jones vector $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ where A and B are complex.[7] Depending on the values A and B, the polarization can follow any ellipse.

Solution: The input intensity of this polarized beam is $I_{\text{in}} = I_{\text{pol}} = |A|^2 + |B|^2$, according to Eq. (6.23), where we absorb the factor $\frac{1}{2}\epsilon_0 c|E_{\text{eff}}|^2$ into $|A|^2$ and $|B|^2$ for convenience. The Jones vector for the light that passes through a horizontal polarizer is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}$$

which gives a measured intensity of $I_{\text{hor}} = |A|^2$. Similarly, the Jones vector when the beam is passed through a polarizer oriented at 45° is

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{A+B}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

leading to an intensity of

$$I_{45°} = \frac{|A+B|^2}{2} = \frac{|A|^2 + |B|^2 + A^*B + AB^*}{2}$$

Finally, the Jones vector for light passing through a right-circular polarizer (see P6.12) is

$$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{A+iB}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

giving an intensity of

$$I_{\text{r-cir}} = \frac{|A+iB|^2}{2} = \frac{|A|^2 + |B|^2 + i\left(A^*B - AB^*\right)}{2}$$

Thus, the Stokes parameters become

$$S_0 = \frac{|A|^2 + |B|^2}{I_{\text{in}}} = 1$$

$$S_1 = \frac{2|A|^2}{I_{\text{in}}} - \frac{|A|^2 + |B|^2}{I_{\text{in}}} = \frac{|A|^2 - |B|^2}{I_{\text{in}}}$$

$$S_2 = \frac{|A|^2 + |B|^2 + A^*B + AB^*}{I_{\text{in}}} - \frac{|A|^2 + |B|^2}{I_{\text{in}}} = \frac{A^*B + AB^*}{I_{\text{in}}}$$

$$S_3 = \frac{|A|^2 + |B|^2 + i\left(A^*B - AB^*\right)}{I_{\text{in}}} - \frac{|A|^2 + |B|^2}{I_{\text{in}}} = i\frac{\left(A^*B - AB^*\right)}{I_{\text{in}}}$$

[7] We will find it easier in this appendix to write $\begin{bmatrix} A \\ B \end{bmatrix}$ instead of $\begin{bmatrix} |A| \\ |B|e^{i\delta} \end{bmatrix}$, where δ is the phase difference between B and A.

Note that the unpolarized portion of the light does not contribute to S_1, S_2, or S_3. Half of the unpolarized light survives any of the test filters, which cancels neatly with the unpolarized portion of $S_0 = \frac{I_{\text{pol}}+I_{\text{un}}}{I_{\text{in}}}$ in Eqs. (6.55)–(6.57).

With the aid of the results in Example 6.4, a completely general form of the Stokes vector may then be written as

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \frac{1}{I_{\text{in}}} \begin{bmatrix} I_{\text{pol}} + I_{\text{un}} \\ |A|^2 - |B|^2 \\ A^* B + AB^* \\ i\left(A^* B - AB^*\right) \end{bmatrix} \tag{6.58}$$

where the Jones vector for the polarized portion of the light is

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

and the intensity of the polarized portion of the light is

$$I_{\text{pol}} = |A|^2 + |B|^2 \tag{6.59}$$

Again, we have hidden the factor $\frac{1}{2}\epsilon_0 c\,|E_{\text{eff}}|^2$ for the polarized portion of the light inside $|A|^2$ and $|B|^2$.

We would like to express the degree of polarization in terms of the Stokes parameters. We first note that the quantity $\sqrt{S_1^2 + S_2^2 + S_3^2}$ can be expressed as

$$\begin{aligned} \sqrt{S_1^2 + S_2^2 + S_3^2} &= \sqrt{\left(\frac{|A|^2 - |B|^2}{I_{\text{in}}}\right)^2 + \left(\frac{\left(A^* B + AB^*\right)}{I_{\text{in}}}\right)^2 + \left(\frac{i\left(A^* B - AB^*\right)}{I_{\text{in}}}\right)^2} \\ &= \frac{|A|^2 + |B|^2}{I_{\text{in}}} \\ &= \frac{I_{\text{pol}}}{I_{\text{in}}} \end{aligned} \tag{6.60}$$

Substituting (6.54) and (6.60) into the expression for the degree of polarization (6.53) yields

$$\xi\text{pol} \equiv \frac{1}{S_0}\sqrt{S_1^2 + S_2^2 + S_3^2} \tag{6.61}$$

If the light is polarized such that it perfectly transmits through or is perfectly extinguished by one of the three test polarizers associated with S_1, S_2, or S_3, then the degree of polarization will be unity. Obviously, it is possible to have pure polarization states that are not aligned with the axes of any one of these test polarizers. In this situation, the degree of polarization is still one, although the values S_1, S_2, and S_3 may all three contribute to (6.61).

Finally, it is possible to represent polarizing devices as matrices that operate on the Stokes vectors in much the same way that Jones matrices operate on Jones vectors. Since Stokes vectors are four-dimensional, the matrices used are four-by-four. These are known as *Mueller matrices*.[8]

[8] E. Hecht, *Optics*, 3rd ed., Sect. 8.12.3 (Massachusetts: Addison-Wesley, 1998).

Derivation: Mueller Matrix for a Linear Polarizer

We know that 50% of the unpolarized light transmits through a polarizer, ending up as polarized light with Jones vector

$$\begin{bmatrix} A_1' \\ B_1' \end{bmatrix} = \sqrt{\frac{I_{\text{un}}}{2}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

(see table 6.1). As usual, let θ give the angle of the transmission axis relative to the horizontal. The Jones matrix (6.23) acts on the polarized portion of the light as follows

$$\begin{bmatrix} A_2' \\ B_2' \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = [A\cos\theta + B\sin\theta] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

One might be tempted to add $\begin{bmatrix} A_1' \\ B_1' \end{bmatrix}$ and $\begin{bmatrix} A_2' \\ B_2' \end{bmatrix}$, but this would be wrong, since the two beams are not coherent. As mentioned previously, unpolarized light necessarily contains multiple frequencies, and so the fields from the polarized and unpolarized beams destructively interfere as often as they constructively interfere. In this case, we simply add intensities rather than fields. That is, we have

$$\begin{aligned} \left|A'\right|^2 = \left|A_1'\right|^2 + \left|A_2'\right|^2 &= \left[\frac{I_{\text{un}}}{2} + |A\cos\theta + B\sin\theta|^2\right]\cos^2\theta \\ &= \left[\frac{I_{\text{un}}}{2} + |A|^2\cos^2\theta + |B|^2\sin^2\theta + \left(A^*B + AB^*\right)\sin\theta\cos\theta\right]\cos^2\theta \\ &= I_{\text{in}}\left[\frac{S_0}{2} + \frac{\cos 2\theta}{2}S_1 + \frac{\sin 2\theta}{2}S_2\right]\cos^2\theta \end{aligned}$$

Similarly,

$$\left|B'\right|^2 = \left|B_1'\right|^2 + \left|B_2'\right|^2 = I_{\text{in}}\left[\frac{S_0}{2} + \frac{\cos 2\theta}{2}S_1 + \frac{\sin 2\theta}{2}S_2\right]\sin^2\theta$$

Since the light has gone through a linear polarizer, we are guaranteed that A' and B' have the same phase. Therefore, $A'^*B' = A'B'^* = |A'||B'|$. In view of (6.58), these results lead to

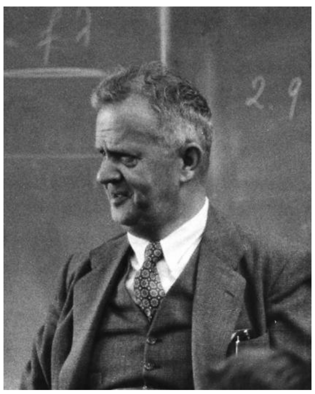
$$\begin{aligned} S_0' &= \frac{\left|A'\right|^2 + \left|B'\right|^2}{I_{\text{in}}} = \frac{S_0}{2} + \frac{\cos 2\theta}{2}S_1 + \frac{\sin 2\theta}{2}S_2 \\ S_1' &= \frac{\left|A'\right|^2 - \left|B'\right|^2}{I_{\text{in}}} = \left[\frac{S_0}{2} + \frac{\cos 2\theta}{2}S_1 + \frac{\sin 2\theta}{2}S_2\right]\left(\cos^2\theta - \sin^2\theta\right) \\ &= \frac{\cos 2\theta}{2}S_0 + \frac{\cos^2 2\theta}{2}S_1 + \frac{\sin 4\theta}{4}S_2 \\ S_2' &= \frac{\left|A'\right|\left|B'\right| + \left|A'\right|\left|B'\right|}{I_{\text{in}}} = 2\left[\frac{S_0}{2} + \frac{\cos 2\theta}{2}S_1 + \frac{\sin 2\theta}{2}S_2\right]\cos\theta\sin\theta \\ &= \frac{\sin 2\theta}{2}S_0 + \frac{\sin 4\theta}{4}S_1 + \frac{\sin^2 2\theta}{2}S_2 \\ S_3' &= i\frac{\left|A'\right|\left|B'\right| - \left|A'\right|\left|B'\right|}{I_{\text{in}}} = 0 \end{aligned}$$

These transformations expressed in matrix format become

$$\begin{bmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \frac{1}{2}\sin 4\theta & 0 \\ \sin 2\theta & \frac{1}{2}\sin 4\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}$$

which reveals the Mueller matrix for a linear polarizer.

The Mueller matrix for a half wave plate is worked out below. The Mueller matrix for a quarter wave plate is deferred to problem P6.16



Hans Mueller (1900-1965, Swiss) was a shepherd until his late teens. As a physics professor at MIT, he built on the work of Stokes and in 1943 formulated a matrix method for manipulating Stokes vectors. He was an engaging lecturer into the 1950s and was known for his exciting demonstrations. He was a student of Arnold Sommerfeld, and did seminal work on ferroelectricity (he is reported to have coined the term). See Laszlo Tisza,"Adventures of a Theoretical Physicist, Part II: America," Phys. Perspect. **11**, 120-168 (2009).

Derivation: Mueller Matrix for a Half Wave Plate

We know that all of the light transmits through the wave plate. This immediately gives

$$S_0' = S_0$$

The wave plate does nothing to unpolarized light. On the other hand, the polarized portion of the light is influenced by the wave plate as follows (see (6.39)):

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A\cos 2\theta + B\sin 2\theta \\ A\sin 2\theta - B\cos 2\theta \end{bmatrix}$$

As usual, θ is the angle of the fast axis relative to the horizontal. (As expected, $\left|A'\right|^2 + \left|B'\right|^2 = |A|^2 + |B|^2$; the intensity of the light is unaltered.) Using (6.58) we get

$$\begin{aligned}
S_1' &= \frac{\left|A'\right|^2 - \left|B'\right|^2}{I_{\text{in}}} = \frac{|A\cos 2\theta + B\sin 2\theta|^2 - |A\sin 2\theta - B\cos 2\theta|^2}{I_{in}} \\
&= \frac{\left(|A|^2 - |B|^2\right)\cos 4\theta + (A^*B + AB^*)\sin 4\theta}{I_{in}} = S_1\cos 4\theta + S_2\sin 4\theta \\
S_1' &= \frac{\left|A'\right|^2 - \left|B'\right|^2}{I_{\text{in}}} = \frac{|A\cos 2\theta + B\sin 2\theta|^2 - |A\sin 2\theta - B\cos 2\theta|^2}{I_{in}} \\
&= \frac{\left(|A|^2 - |B|^2\right)\cos 4\theta + (A^*B + AB^*)\sin 4\theta}{I_{in}} = S_1\cos 4\theta + S_2\sin 4\theta \\
S_2' &= \frac{A'^*B' + A'B'^*}{I_{in}} \\
&= \frac{(A^*\cos 2\theta + B^*\sin 2\theta)(A\sin 2\theta - B\cos\theta)}{I_{in}} \\
&+ \frac{(A\cos 2\theta + B\sin 2\theta)(A^*\sin 2\theta - B^*\cos\theta)}{I_{in}} \\
&= \frac{|A|^2 - |B|^2}{I_{in}}\sin 4\theta - \frac{AB^* + A^*B}{I_{in}}\cos 4\theta = S_1\sin 4\theta - S_2\cos 4\theta \\
S_3' &= i\,\frac{A'^*B' - A'B'^*}{I_{in}} \\
&= i\,\frac{(A^*\cos 2\theta + B^*\sin 2\theta)(A\sin 2\theta - B\cos\theta)}{I_{in}}
\end{aligned}$$

$$
-i\,\frac{(A\cos 2\theta + B\sin 2\theta)\,(A^{*}\sin 2\theta - B^{*}\cos\theta)}{I_{in}}
$$

$$
= -i\,\frac{A^{*}B - AB^{*}}{I_{in}} = -S_{3}
$$

These transformations expressed in matrix format become

$$
\begin{bmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\theta & \sin 4\theta & 0 \\ 0 & \sin 4\theta & -\cos 4\theta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}
$$

which reveals the Mueller matrix for a half wave plate.

Exercises

Exercises for 6.2 Jones Vectors for Representing Polarization

P6.1 Show that $\left(A\hat{\mathbf{x}}+Be^{i\delta}\hat{\mathbf{y}}\right)\cdot\left(A\hat{\mathbf{x}}+Be^{i\delta}\hat{\mathbf{y}}\right)^{*}=1$, as defined in connection with (6.5).

P6.2 Prove that if $0<\delta<\pi$, the helicity is left-handed, and if $\pi<\delta<2\pi$ the helicity is right-handed.

HINT: Write the relevant real field associated with (6.5)

$$\mathbf{E}(z,t)=|E_{\text{eff}}|\left[\hat{\mathbf{x}}A\cos\left(kz-\omega t+\phi\right)+\hat{\mathbf{y}}B\cos\left(kz-\omega t+\phi+\delta\right)\right]$$

where ϕ is the phase of E_{eff}. Freeze time at, say, $t=\phi/\omega$. Determine the field at $z=0$ and at $z=\lambda/4$ (a quarter cycle), say. If $\mathbf{E}(0,t)\times\mathbf{E}(\lambda/4,t)$ points in the direction of $\mathbf{k}$, then the helicity matches that of a wood screw.

L6.3 Determine how much right-handed circularly polarized light ($\lambda_{\text{vac}}=$ 633 nm) is delayed (or advanced if ϕ is negative) with respect to left-handed circularly polarized light as it goes through approximately 3 cm of Karo syrup (the neck of the bottle). This phenomenon is called *optical activity*. Because of a definite-handedness to the molecules in the syrup, right- and left-handed polarized light experience slightly different refractive indices. (video)

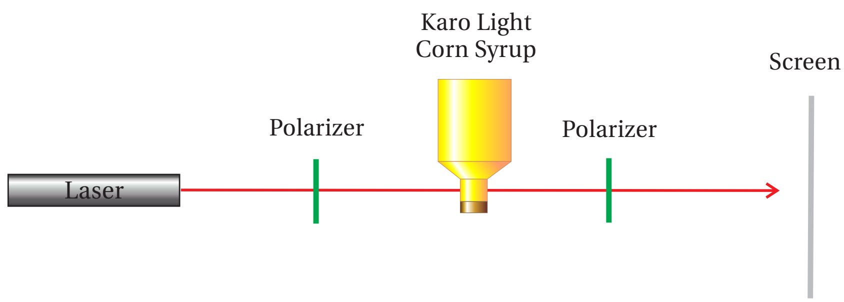


Figure 6.11 Lab schematic for L 6.3

HINT: Linearly polarized light contains equal amounts of right and left circularly polarized light. Consider

$$\frac{1}{2}\begin{bmatrix}1\\ i\end{bmatrix}+\frac{e^{i\phi}}{2}\begin{bmatrix}1\\ -i\end{bmatrix}$$

where ϕ is the phase delay of the right circular polarization. Show that this can be written as

$$e^{i\delta}\begin{bmatrix}\cos\phi/2\\ \sin\phi/2\end{bmatrix}$$

The overall phase δ is unimportant. Compare this with

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

where α is the angle of linearly polarized light (see table 6.1).

Exercises for 6.3 Elliptically Polarized Light

P6.4 Consider the Jones vector

$$\begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix}$$

For the following cases, what is the orientation of the major axis, and what is the ellipticity of the light? Case I: $A = B = 1/\sqrt{2}$; $\delta = 0$ Case II: $A = B = 1/\sqrt{2}$; $\delta = \pi/2$; Case III: $A = B = 1/\sqrt{2}$; $\delta = \pi/4$.

Exercises for 6.4 Linear Polarizers and Jones Matrices

P6.5 (a) Suppose that linearly polarized light is oriented at an angle α with respect to the horizontal or x-axis (see table 6.1). What fraction of the original *intensity* gets through a vertically oriented polarizer?

(b) If the original light is right-circularly polarized, what fraction of the original *intensity* gets through the same polarizer?

Exercises for 6.5 Jones Matrix for a Polarizer

P6.6 Horizontally polarized light ($\alpha = 0$) is sent through two polarizers, the first oriented at $\theta_1 = 45°$ and the second at $\theta_2 = 90°$. What fraction of the original intensity emerges? What is the fraction if the ordering of the polarizers is reversed?

P6.7 (a) Suppose that linearly polarized light is oriented at an angle α with respect to the horizontal or x-axis. What fraction of the original intensity emerges from a polarizer oriented with its transmission at angle θ from the x-axis?

Answer: $\cos^2(\theta - \alpha)$; compare with P6.5.

(b) If the original light is right circularly polarized, what fraction of the original intensity emerges from the same polarizer?

P6.8 Derive (6.12), (6.13), and (6.14).

HINT: Analyze the Jones vector just as you would analyze light in the laboratory. Put a polarizer in the beam and observe the intensity of the light as a function of polarizer angle. Compute the intensity via (6.23).

Then find the polarizer angle (call it α) that gives a maximum (or a minimum) of intensity. The angle then corresponds to an axis of the ellipse inscribing the E-field as it spirals. When taking the arctangent, remember that it is defined only over half of the unit circle. You can add π for another valid result (which gives a second ellipse axis).

Exercises for 6.6 Jones Matrix for Wave Plates

L6.9 Create a source of unknown elliptical polarization by reflecting a linearly polarized laser beam (with both s and p-components) from a metal mirror with a large incident angle (i.e. $\theta_i \geq 80°$). Use a quarter-wave plate and a polarizer to determine the Jones vector of the reflected beam. Find the ellipticity, the helicity (right or left handed), and the orientation of the major axis. (video)

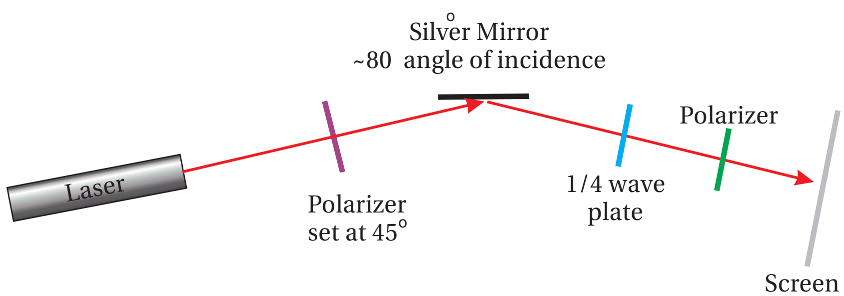


Figure 6.12 Lab schematic for L 6.9

HINT: A polarizer alone can reveal the direction of the major and minor axes and the ellipticity, but it does not reveal the helicity. Use a quarter-wave plate (oriented at a special angle θ) to convert the unknown elliptically polarized light into linearly polarized light. A subsequent polarizer can then extinguish the light, from which you can determine the Jones vector of the light coming through the wave plate. This must equal the original (unknown) Jones vector (6.11) operated on by the wave plate (6.37). As you solve the matrix equation, it is helpful to note that the inverse of (6.37) is its own complex conjugate.

P6.10 What is the minimum thickness (called zero-order thickness) of a quartz plate made to operate as a quarter-wave plate for $\lambda_{vac} = 500$ nm? The indices of refraction are $n_{fast} = 1.54424$ and $n_{slow} = 1.55335$.

P6.11 Light that is linearly polarized along $\alpha = 30°$ traverses a quarter wave plate with fast axis at $\theta_1 = 60°$. The light then goes through a polarizer with transmission axis at $\theta_2 = 90°$ followed by a half wave plate with fast axis at $\theta_3 = 30°$.

(a) What is the Jones vector of the light emerging from the final element?

(b) What fraction of the original intensity transmits through the system?

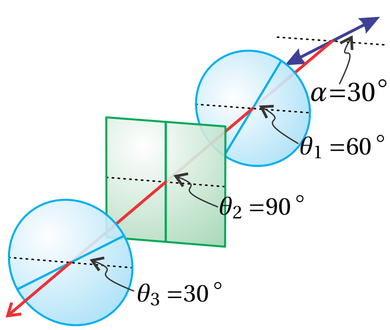


Figure 6.13 Arrangement for P6.11

P6.12 A right-circular polarizer can be constructed using a quarter wave plate with fast axis at 45°, followed by a linear polarizer oriented vertically, and finally a quarter wave plate with fast axis at −45°.

(a) Calculate the Jones matrix for this system.

Answer: $\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

(b) Check that the device leaves right-circularly polarized light unaltered while killing left-circularly polarized light.

Exercises for 6.7 Polarization Effects of Reflection and Transmission

P6.13 Light is linearly polarized at $\alpha = 45°$ with a Jones vector according to table 6.1. The light is reflected from a vertical silver mirror with angle of incidence $\theta_\text{i} = 80°$, as described in (P3.15). Find the Jones vector representation for the polarization of the reflected light.

NOTE: The answer may be somewhat different than the result measured in L 6.9. For one thing, we have not considered that a silver mirror inevitably has a thin oxide layer or, more often, a special protective coating applied.

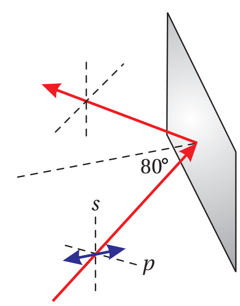


Figure 6.14 Geometry for P6.13

P6.14 Calculate the angle θ to cut the glass in a Fresnel rhomb such that after the two internal reflections there is a phase difference of $\pi/2$ between the two polarization states. The rhomb then acts as a quarter wave plate.

HINT: You need to find the phase difference between (3.42) and (3.43). Set the difference equal to $\pi/4$ for each bounce. The equation you get does not have a clean analytic solution, but you can plot it to find a numerical solution.

Answer: There are two angles that work: $\theta \cong 50°$ and $\theta \cong 53°$.

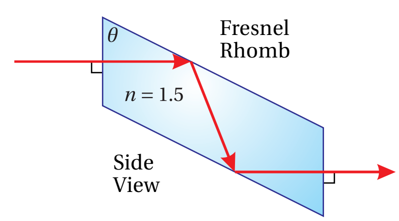


Figure 6.15 Fresnel Rhomb geometry for P6.14

Exercises for 6.A Ellipsometry

P6.15 Derive (6.49) and (6.51), often used for ellipsometry measurements.

HINT: Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, first show

$$I \propto 1 - \frac{\left(r_p r_s^* + r_s r_p^*\right) \Big/ |r_s|^2 \tan\alpha}{|r_p|^2 \Big/ |r_s|^2 + \tan^2\alpha} \sin 2\theta + \frac{|r_p|^2 \Big/ |r_s|^2 - \tan^2\alpha}{|r_p|^2 \Big/ |r_s|^2 + \tan^2\alpha} \cos 2\theta$$

Exercises for 6.B Partially Polarized Light

P6.16 Derive the Mueller matrix for a quarter wave plate.

Answer:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta & \frac{1}{2}\sin 4\theta & -\sin 2\theta \\ 0 & \frac{1}{2}\sin 4\theta & \sin^2 2\theta & \cos 2\theta \\ 0 & \sin 2\theta & -\cos 2\theta & 0 \end{bmatrix}$$