HW 1: Mathematical Preliminaries

In this sheet vectors are typefaced as bold while you are required to identify vectors by overhanging arrows in your handwritten solutions. An irrotational vector $\mathbf{A}$ is defined by $\nabla \times \mathbf{A} = 0$ and a solenoidal vector is defined as $\nabla \cdot \mathbf{A} = 0$. The symbols ϕ, u, v represent scalar functions.

1. Prove the following relations or statements.

   (a) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$.

   (b) If $\mathbf{A}$ is irrotational, $\mathbf{A} \times \mathbf{r}$ is also irrotational.

   (c) $\mathbf{A}$ and $\mathbf{B}$ are constant vectors. Show that $\nabla(\mathbf{A} \cdot \mathbf{B} \times \mathbf{r}) = \mathbf{A} \times \mathbf{B}$.

   (d) $\nabla \times (\phi \nabla \phi) = 0$.

   (e) $\nabla u \times \nabla v$ is solenoidal.

2. Using Gauss's theorem, prove the following:

   (a) $\oint_{\partial V} d\mathbf{S} = 0$.

   (b) $\frac{1}{3} \oint_{\partial V} \mathbf{r} \cdot d\mathbf{S} = V$ where V is the volume enclosed by the surface.

3. If $\mathbf{B} = \nabla \times \mathbf{A}$, show that $\oint \mathbf{B} \cdot d\mathbf{S} = 0$.

4. Given a vector $\mathbf{t} = -y\hat{\mathbf{e}}_x + x\hat{\mathbf{e}}_y$, show with the help of Stokes' theorem that the line integral of $\mathbf{t}$ around a continuous closed curve in the xy plane staisfies

   $$\frac{1}{2} \oint \mathbf{t} \cdot d\mathbf{l} = \frac{1}{2} \oint (xdy - ydx) = A$$

   where A is the area enclosed by the curve.

5. Evaluate $\oint \mathbf{r} \times d\mathbf{r}$ using Stokes' theorem. The line integral is assumed to be computed on a curve in the xy plane.

6. Prove that

   $$\oint u \nabla v \cdot d\mathbf{l} = \int \nabla u \times \nabla v \cdot d\mathbf{S}.$$