

Q.1.

$$\vec{A}(\vec{r}) = \mu_0/4\pi \frac{\vec{p}_1 \times \hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\frac{\hat{r}}{r^2} = -\vec{\nabla} \left(\frac{1}{r} \right)$$

$$= \mu_0/4\pi \vec{\nabla} \times \left(\vec{p}_1 \times \hat{r}/r^2 \right)$$

$$\vec{\nabla} \left(\frac{1}{r} \right) = -\frac{\hat{r}}{r^2}$$

$$= \mu_0/4\pi \left[\vec{p}_1 (\vec{\nabla} \cdot \hat{r}/r^2) - \hat{r}/r^2 (\vec{p}_1 \cdot \vec{\nabla}) \right]$$

$$\vec{\nabla} \cdot \hat{r}/r^2 = 0 \quad \text{unless } r=0. \checkmark$$

$$\vec{\nabla} \cdot \vec{\nabla} \left(\frac{1}{r} \right) = -\vec{\nabla}^2 \left(\frac{1}{r} \right) = 4\pi \delta(r)$$

$$= -\mu_0/4\pi \left((\vec{p}_1 \cdot \vec{\nabla}) \hat{r}/r^2 \right)$$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}}$$

x-component of $(\vec{p}_1 \cdot \vec{\nabla}) \hat{r}/r^2$ will be

$$= \mu_{1x} \frac{\partial}{\partial x} \left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

$$= \mu_{1x} \left[-\frac{3}{2} \cdot \frac{zx^2}{r^5} \hat{i} + \frac{1}{r^3} \hat{i} - \frac{3}{2} \cdot \frac{zxy}{r^5} \hat{j} - \frac{3}{2} \cdot \frac{z^2z}{r^5} \hat{k} \right]$$

$$= \mu_{1x} \left[-\frac{3x}{r^3} \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^2} - \frac{\hat{i}}{3x} \right) \right]$$

$$x\text{-comp.} = -\frac{3\mu_{1x}x}{r^3} \left(\frac{\vec{r}}{r^2} \cdot \hat{i} + \frac{\hat{i}}{3x} \right)$$

Similarly y and z component can be calculated.

$$= -\frac{3\mu_{1y}y}{r^3} \left(\frac{\vec{r}}{r^2} \cdot \hat{j} + \frac{\hat{j}}{3y} \right)$$

$$= -\frac{3\mu_{1z}z}{r^3} \left(\frac{\vec{r}}{r^2} \cdot \hat{k} + \frac{\hat{k}}{3z} \right)$$

Adding these all give

$$= -\frac{3\mu_{1x}x}{r^5} \vec{r} + \frac{3\mu_{1x}x\hat{i}}{3xr^3} - \frac{3\mu_{1y}y}{r^5} \vec{r} + \frac{3\mu_{1y}y\hat{j}}{3yr^3} - \frac{3\mu_{1z}z}{r^5} \vec{r} + \frac{3\mu_{1z}z\hat{k}}{3zr^3}$$

Simplifying this will give us

$$(\mu_1 \cdot \nabla) \frac{\hat{r}}{r^2} = \left[-\frac{3}{r^2} (\mu_1 \cdot \vec{r}) \vec{r} + \vec{\mu}_1 \right] \frac{1}{r^3}$$

$$-(\mu_1 \cdot \nabla) \frac{\hat{r}}{r^2} = \left[\left(\frac{3}{r^2} (\mu_1 \cdot \vec{r}) \vec{r} - \vec{\mu}_1 \right) \right] \frac{1}{r^3}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi r^3} \left[\frac{3}{r^2} (\vec{\mu}_1 \cdot \vec{r}) \vec{r} - \vec{\mu}_1 \right]$$

Energy $E = -\vec{\mu}_2 \cdot \vec{B}$

$$E = \frac{\mu_0}{4\pi r^3} \left[\vec{\mu}_1 \cdot \vec{\mu}_2 - \frac{3}{r^2} (\vec{\mu}_1 \cdot \vec{r})(\vec{\mu}_2 \cdot \vec{r}) \right]$$

$$r_1 = 1\text{\AA}, \quad r_2 = 10\text{\AA}$$

when spins are parallel

$$E = \frac{\mu_0}{4\pi r^3} \left[\mu_1 \mu_2 - 3\mu_1 \mu_2 \right] = -\frac{\mu_0 \mu_1 \mu_2}{2\pi r^3}$$

proton mag. moment $\mu = 1.410 \times 10^{-26} \text{ J/T}$

$$\mu_0 = 1.25 \times 10^{-6} \text{ H} \cdot \text{m}^{-1}$$

=

$$E = -\frac{1.25 \times 10^{-6} \times (1.410 \times 10^{-26})^2}{2\pi (10^{-10})^3} \quad \text{when } r = 1\text{\AA}$$

$$E = \frac{1.25 \times 10^{-6} \times 10^{-52} \times 1.96}{2 \times 3.14 \times 10^{-30}} = -3.9 \times 10^{-27} \text{ J}$$

when $r = 10 \text{ \AA}$

$$E = -3.9 \times 10^{-30} \text{ J}$$

(3 orders of magnitude smaller when r goes up one order of magnitude)

⇒ when μ_1 and μ_2 are antiparallel.

$$E = \frac{\mu_0}{4\pi r^3} [-\mu_1 \mu_2 + 3\mu_1 \mu_2] = \frac{\mu_0 \mu_1 \mu_2}{2\pi r^3}$$

when $r = 1 \text{ \AA}$

$$E = 3.9 \times 10^{-27} \text{ J}$$

when $r = 10 \text{ \AA}$

$$E = 3.9 \times 10^{-30} \text{ J}$$

(c)

lattice constant in Fe is $a = 2.87 \text{ \AA}$ (BCC)

nearest neighbour distance is $a/\sqrt{2}$

$$\text{Dipole energy} \sim \frac{\mu_0 (2.2\mu_B)^2}{4\pi r^3}$$

where

mag. moment of Fe = $2.2\mu_B$

$$\approx 30 \text{ eV}$$

$$\text{exchange constant } J = k_B T_c = 0.09 \text{ eV}$$

$$\text{Ratio of exchange to dipolar energy} = \underline{30000} \quad \checkmark$$

Dipolar couplings cannot explain spin ordering.

Q. 2

Four spin $1/2$ particle

$$\vec{S}_T = \hat{S}_1 + \hat{S}_2 + \hat{S}_3 + \hat{S}_4$$

$$S_T^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 + \hat{S}_4^2 + \hat{S}_1 \cdot \hat{S}_2 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_3 \cdot \hat{S}_4 + \hat{S}_4 \cdot \hat{S}_1 + \hat{S}_1 \cdot \hat{S}_4 + \hat{S}_2 \cdot \hat{S}_4$$

$$\underbrace{(\hat{S}_1 \cdot \hat{S}_2 + \hat{S}_1 \cdot \hat{S}_3 + \hat{S}_1 \cdot \hat{S}_4 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_2 \cdot \hat{S}_4 + \hat{S}_3 \cdot \hat{S}_4)}_{\text{Heisenberg Hamiltonian for two particles}} = \frac{1}{2} [\hat{S}_T^2 - \hat{S}_1^2 - \hat{S}_2^2 - \hat{S}_3^2 - \hat{S}_4^2]$$

Heisenberg Hamiltonian for two particles

$$\hat{H} = -2J \hat{S}_1 \cdot \hat{S}_2$$

$$\hat{H}' = \frac{1}{2} [\hat{S}_T^2 - \hat{S}_1^2 - \hat{S}_2^2 - \hat{S}_3^2 - \hat{S}_4^2]$$

$$\hat{H} = -\frac{2J}{2} [\hat{S}_T^2 - \hat{S}_1^2 - \hat{S}_2^2 - \hat{S}_3^2 - \hat{S}_4^2]$$

$$\hat{H} |S m_s\rangle = -J [\hat{S}_T^2 - \hat{S}_1^2 - \hat{S}_2^2 - \hat{S}_3^2 - \hat{S}_4^2] |S m_s\rangle$$

possible values of $S_T = \cancel{2, 1, 0, -1, -2}$

$$\uparrow \uparrow \uparrow \uparrow \quad S_T = 2$$

$$\uparrow \uparrow \uparrow \downarrow \quad S_T = 1$$

$$\uparrow \uparrow \downarrow \downarrow \quad S_T = 0$$

$$\uparrow \downarrow \downarrow \downarrow \quad S_T = -1$$

$$\downarrow \downarrow \downarrow \downarrow \quad S_T = -2$$

$$\hat{S}_1^2 |s, m_s\rangle = s_1(s_1+1) |s, m_s\rangle = 3/4 \hbar^2 |s, m_s\rangle$$

similarly $\hat{S}_2^2, \hat{S}_3^2, \hat{S}_4^2$ gives eigenvalue of $3/4 \hbar^2$

$$\text{For } S_T = 2, \quad \hat{S}_T^2 |s, m_s\rangle = S(S+1) |s, m_s\rangle$$

$$\Rightarrow \hat{H} |s, m_s\rangle = -J [6\hbar^2 - 4(3/4 \hbar^2)] = -3\hbar^2 J$$

$$\Rightarrow \text{For } S_T = 1$$

$$= -J [2\hbar^2 - 3\hbar^2] = J\hbar^2$$

$$\Rightarrow \text{For } S_T = 0$$

$$= -J [-3\hbar^2] = 3\hbar^2 J$$

$$\Rightarrow \text{For } S_T = -1$$

$$= -J [0 - 3\hbar^2] = 3\hbar^2 J$$

$$\Rightarrow \text{For } S_T = -2$$

$$= -J [3\hbar^2 - 3\hbar^2] = +\hbar^2 J.$$

Q:3

if

$$n_e \leq 2l+1 \quad \text{then}$$

$$s = n/2$$

if $n_e \geq 2l+1$ then

$$s = \frac{1}{2} [2(2l+1) - n]$$

$$s = \frac{1}{2} [(2l+1) - |2l+n-n|]$$

Now if $n \leq 2l+1$.

$$L = l + (l-1) + \dots + (l-n+1)$$

$$L = nl - \frac{n(n-l)}{2} = \frac{s(2l+1-n)}{2}$$

if $n_e \geq 2l+1$

$$L = l + (l-1) + \dots + [l - (n - (2l+1)) + 1]$$

$$= [n - (2l+1)]l - [n - (2l+1)] \left[\frac{n - (2l+1) - 1}{2} \right]$$

$$= \frac{1}{2} [2(2l+1) - n] [n - (2l+1)]$$

$$L = \frac{s(n - (2l+1))}{2}$$

For $n \leq 2l+1$

$$J = |L - S| = |S(2l+1-n) - S|$$
$$= S |2l+1-n-1|$$

$$J = S |2l - n|$$

For $n \geq 2l+1$

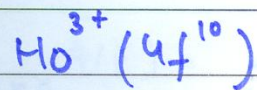
$$J = |L + S|$$

$$= |S[n - (2l+1)] + S|$$

$$= S |n - 2l - 1 + 1|$$

$$J = S |2l + 1|$$

Q.4



$\uparrow\downarrow$	$\uparrow\downarrow$	$\uparrow\downarrow$	\uparrow	\uparrow	\uparrow	\uparrow	
m_l	3	2	1	0	-1	-2	-3

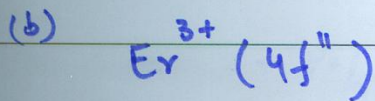
$$S = 2$$

$$\sum m_l = 1 + 2 + 3 = 6 \Rightarrow L = 6$$

$$J = |L + S| \quad \text{more than half filled}$$
$$6 + 2 = 8$$

$$\text{Multiplicity} \quad 2S + 1 = 5$$

$$L_J = {}^S I_8$$



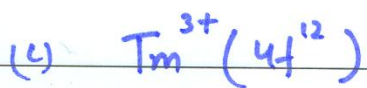
$$S = 3/2$$

$$L = 6$$

$$\text{Multiplicity} \quad 2S + 1 = 2(3/2 + 1) = 4$$

$$J = |L + S| = 6 + 3/2 = 15/2$$

$${}^4 I_{15/2}$$

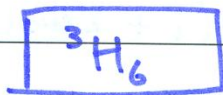


$S = 1$ ✓

$L = 5$ ✓

Multiplicity $2S+1 = 3$ ✓

$J = |L+S| = 6$ ✓



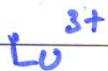
↑	↑	↑	↑	↑	↑
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3 2 1 0 -1 -2 -3

0 1 2 3 4 5

S P D F G H

(d)



$\sum m_s = S = 0$

$\sum m_l = L = 0$

$J = |L+S| = 0$

symbol S



Q.5

There are triplet and a singlet states. Energies are given as

$0, \Delta - g\mu_B B, \Delta, \Delta + g\mu_B B$

partition fn.

$Z = e^{-\beta(0)} + e^{-\beta(\Delta - g\mu_B B)} + e^{-\beta\Delta} + e^{-\beta(\Delta + g\mu_B B)}$

$= 1 + e^{-\beta\Delta} + e^{-\beta\Delta} (e^{\beta g\mu_B B} + e^{-\beta g\mu_B B})$

$$Z = 1 + e^{-\beta \Delta} + e^{-\beta \Delta} 2 \cos(g \mu_B \beta B)$$

μ is given by

$$M = +NK_B T \frac{\partial}{\partial B} (\ln Z)$$

$$= \frac{+NK_B T}{Z} \frac{\partial}{\partial B} \left(1 + e^{-\beta \Delta} (1 + 2 \cos(g \mu_B \beta B)) \right)$$

$$= +K_B T / N \left(0 + 2e^{-\beta \Delta} (-g \mu_B \beta \sin(g \mu_B \beta B)) \right)$$

$$= \frac{-N 2 K_B T \mu_B \beta g \sin(g \mu_B \beta B) e^{-\beta \Delta}}{Z}$$

$$\beta = 1/K_B T$$

$$= \frac{-2 \mu_B g e^{-\beta \Delta} N \sin(g \mu_B \beta B)}{Z}$$

For

$$\text{Magnetization} = \frac{-2 N \mu_B g e^{-\beta \Delta} \sin(g \mu_B \beta B)}{Z}$$

$$\chi = -\mu_0 \frac{\partial M}{\partial B} \quad \text{check } \mu_0$$

$$M = \chi \frac{B}{\mu_0}$$

$$\chi = \mu_0 \frac{M}{B}$$

$$= \frac{2 N \mu_B g e^{-\beta \Delta} (g \mu_B \beta B)}{1 + e^{-\beta \Delta} (1 + 2 (1 - \frac{1}{2} (g \mu_B \beta B)^2 + \dots))}$$

$$\chi = \mu_0 \frac{\partial M}{\partial B}$$

neglecting higher order terms

$$= \frac{2N\mu_0^2 g^2 B \beta}{e^{\beta \Delta} + 1 + 2}$$

$$M \approx = \frac{2Ng^2\mu_0^2 B}{k_B T (e^{\beta \Delta} + 3)}$$

$$\chi = \frac{\delta M}{\delta B}$$

$$\chi = \frac{2Ng^2\mu_0^2}{k_B T (e^{\beta \Delta} + 3)}$$

~~where -~~

~~the~~

