## Q.1.

$$\vec{A}(\vec{r}) = \mu_0/4\pi \; \frac{\vec{\mu}_1 \times \hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\frac{\hat{r}}{r^2} = -\vec{\nabla}\left(\frac{1}{r}\right)$$

$$= \mu_0/4\pi \; \vec{\nabla} \times \left(\vec{\mu}_1 \times \hat{r}/r^2\right)$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

$$= \mu_0/4\pi \left[\vec{\mu}_1\left(\vec{\nabla}\cdot \hat{r}/r^2\right) - \hat{r}/r^2\left(\vec{\mu}_1 \cdot \vec{\nabla}\right)\right]$$

$\vec{\nabla}\cdot \hat{r}/r^2 = 0$ unless $r = 0$. ✓

$$-\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla}^2\left(\frac{1}{r}\right) = 4\pi\delta(r)$$

$$= -\mu_0/4\pi \left(\left(\vec{\mu}_1 \cdot \vec{\nabla}\right)\hat{r}/r^2\right)$$

$$\hat{r} = \frac{\vec{r}}{|r|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{1/2}}$$

x-component of $(\vec{\mu}_1 \cdot \vec{\nabla})\hat{r}/r^2$ will be

$$= \mu_{1x} \frac{\partial}{\partial x}\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \mu_{1x}\left[-3/2 \cdot \frac{2x^2\,\hat{i}}{r^5} + 1/r^3\,\hat{i} - 3/2 \cdot \frac{2xy\,\hat{j}}{r^5} - 3/2 \cdot \frac{2xz\,\hat{k}}{r^5}\right]$$

$$= \mu_{1x}\left[-\frac{3x}{r^3}\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^2} - \frac{\hat{i}}{3x}\right)\right]$$

x - comp. $= \dfrac{-3\mu_{1x}x}{r^3}\left(\dfrac{\vec{r}}{r^2} - \dfrac{\hat{i}}{3x}\right)$

similarly y and z component can be calculated.

$$= \frac{-3\mu_{1y}y}{r^3}\left(\vec{r}/r^2 - \frac{\hat{j}}{3y}\right)$$

$$= -3\mu_{1z}z/r^3\left(\vec{r}/r^2 - \frac{\hat{k}}{3z}\right)$$

Adding these all give

$$= -\frac{3\mu_{1x}x\,\vec{r}}{r^5} + \frac{3\mu_{1x}x\,\hat{i}}{3xr^3} - \frac{3\mu_{1}y\,\vec{r}}{r^5} \cancel{3\mu_{1y}}$$

$$+ \frac{3\mu_{1y}y\,\hat{j}}{3yr^3} - \frac{3\mu_{1z}z\,\vec{r}}{r^5} + \frac{3\mu_{1z}z\,\hat{k}}{3zr^3}$$

simplifying this will give us

$$(\mu_1\cdot\nabla)\frac{\hat{r}}{r^2} = \left[-\frac{3}{r^2}(\vec{\mu}_1\cdot\vec{r})\vec{r} + \vec{\mu}_1\right]1/r^3$$

$$-(\mu_1\cdot\nabla)\hat{r}/r^2 = \left[\left(\frac{3}{r^2}(\mu_1\cdot\vec{r})\vec{r} - \vec{\mu}_1\right)\right]1/r^3$$

~~2.45~~ 0.390

$$\Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi r^3}\left[ \frac{3}{r^2}(\vec{\mu}_1\cdot\vec{r})\vec{r} - \vec{\mu}_1 \right]$$

Energy $\quad E = -\vec{\mu}_2 \cdot \vec{B}$

$$\boxed{E = \frac{\mu_0}{4\pi r^3}\left[ \vec{\mu}_1\cdot\vec{\mu}_2 - \frac{3}{r^2}(\vec{\mu}_1\cdot\vec{r})(\vec{\mu}_2\cdot\vec{r}) \right]}$$

$r_1 = 1\,\mathring{A}$ , $r_2 = 10\,\mathring{A}$

when spins are parallel

$$E = \frac{\mu_0}{4\pi r^3}\left[ \mu_1\mu_2 - 3\mu_1\mu_2 \right] = -\frac{\mu_0\mu_1\mu_2}{2\pi r^3}$$

proton mag. moment $\mu = 1.410 \times 10^{-26}\ J/T$

$\mu_0 = 1.25 \times 10^{-6}\ H \cdot m^{-1}$

=

$$E = -\frac{1.25\times10^{-6} \times (1.40\times10^{-26})^2}{2\pi\,(10^{-10})^3} \quad \text{when } r = 1\,\mathring{A}$$

$$\boxed{E = \frac{1.25\times10^{-6}\times10^{-52}\times1.96}{2\times3.14\times10^{-30}} = -3.9\times10^{-27}\ J}$$

when $r = 10 A^\circ$

$$\boxed{E = -3.9 \times 10^{-30} J}$$

(3 orders of magnitude smaller when r goes up one order of magnitude)

$\Rightarrow$ when $\mu_1$ and $\mu_2$ are antiparallel.

$$E = \mu_0/4\pi r^3 \left[ -\mu_1\mu_2 + 3\mu_1\mu_2 \right] = \frac{\mu_0 \mu_1 \mu_2}{2\pi r^3}$$

when $r = 1 A^\circ$

$$E = 3.9 \times 10^{-27} J$$

when $r = 10 \mathring{A}$

$$E = 3.9 \times 10^{-30} J$$

(c)

lattice constant in Fe (BCC) is $a = 2.87 A^\circ$

nearest neighbour distance is $a/\sqrt{2}$

Dipole energy $\sim \frac{\mu_0 2 (2.2\mu_B)^2}{4\pi r^3}$ where mag. moment of Fe $= 2.2\mu_B$

$\cong 30 \mu eV$

exchange constant $J = K_B T_c = 0.09 eV$

Ratio of exchange ~~and~~ to dipolar energy $= \underline{30000}$ ✓

Dipolar couplings cannot explain spin ordering.

Q. 2

Four spin 1/2 particle

$$\vec{S}_T = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 + \hat{\vec{S}}_3 + \hat{\vec{S}}_4$$

$$S_T^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 + \hat{S}_4^2 + \hat{S}_1 \cdot \hat{S}_2 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_3 \cdot \hat{S}_4 + \hat{S}_1 \cdot \hat{S}_3 + \hat{S}_1 \cdot \hat{S}_4 + \hat{S}_2 \cdot \hat{S}_4$$

$$\underbrace{(\hat{S}_1 \cdot \hat{S}_2 + \hat{S}_1 \cdot \hat{S}_3 + \hat{S}_1 \cdot \hat{S}_4 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_2 \cdot \hat{S}_4 + \hat{S}_3 \cdot \hat{S}_4}_{H'}) = \tfrac{1}{2}\left[\hat{S}_T - \hat{S}_1^2 - \hat{S}_2^2 - \hat{S}_3^2 - \hat{S}_4^2\right]$$

Heisenberg Hamiltonian for two particles

$$\hat{H} = -2J\,\hat{S}_1 \cdot \hat{S}_2$$

$$\hat{H}' = \tfrac{1}{2}\left[\hat{S}_T - \hat{S}_1^2 - \hat{S}_2^2 - \hat{S}_3^2 - \hat{S}_4^2\right]$$

$$\hat{H} = -\frac{2J}{2}\left[\hat{S}_T^2 - \hat{S}_1^2 - \hat{S}_2^2 - \hat{S}_3 - \hat{S}_4^2\right]$$

$$\hat{H}|s\,m_s\rangle = -J\left(\hat{S}_T^2 - S_1^2 - S_2^2 - \hat{S}_3^2 - \hat{S}_4^2\right]|s\,m_s\rangle$$

possible values of $S_T$ = 2, 1, ~~3~~ ~~...~~

$\uparrow\uparrow\uparrow\uparrow$ $S_T = 2$

$\uparrow\uparrow\uparrow\downarrow$ $S_T = 1$

$\uparrow\uparrow\downarrow\downarrow$ $S_T = 0$

$\uparrow\downarrow\downarrow\downarrow$ $S_T = -1$

$\downarrow\downarrow\downarrow\downarrow$ $S_T = -2$

$$\hat{S}_1^2 |s_1 m_{s1}\rangle = s_1(s_1+1)|s_1 m_{s1}\rangle = \tfrac{3}{4}\hbar^2 |s_1 m_{s1}\rangle$$

similarly $\hat{S}_2^2$, $\hat{S}_3^2$, $\hat{S}_4^2$ gives eigenvalue of $\tfrac{3}{4}\hbar^2$

For $S_T = 2$ , $S_T^2 |s\ m_s\rangle = s(s+1)|s\ m_s\rangle$

$\Rightarrow$ $\hat{H}|s\ m_s\rangle = -J\left[6\hbar^2 - 4\left(\tfrac{3}{4}\hbar^2\right)\right] = -3\hbar^2 J$

$\Rightarrow$ For $S_T = 1$

$$= -J\left[2\hbar^2 - 3\hbar^2\right] = J\hbar^2$$

$\Rightarrow$ For $S_T = 0$

$$= -J\left[-3\hbar^2\right] = 3\hbar^2 J$$

$\Rightarrow$ For $S_T = -1$

$$= -J\left[0 - 3\hbar^2\right] = 3\hbar^2 J$$

$\Rightarrow$ For $S_T = -2$

$$= -J\left[2\hbar^2 - 3\hbar^2\right] = +\hbar^2 J.$$

Q:3

if

$n_e \leq 2l+1$ then

$s = n/2$

if $n_e \geq 2l+1$ then

$s = 1/2\left[2(2l+1) - n\right]$

$S = 1/2\left[(2l+1) - |2l+n-n|\right]$

Now if $n \leq 2l+1$.

$L = l + (l-1) + \cdots (l-n+1)$

$L = nl - \frac{n(n-l)}{2} = s|2l+1-n|$

if $n_e \geq 2l+1$

$L = l + (l-1) + \cdots\cdots \left[l - (n-(2l+1)) + 1\right)$

$= [n - (2l+1)]l - [n-(2l-1)]\left[\frac{n-(2l+1)-1}{2}\right]$

$= 1/2\left[2(2l+1) - n\right]\left[n-(2l+1)\right]$

$L = s|n-(2l+1)|$

For $n_e \leq 2l+1$

$$J = |L - S| = |S(2l+1-n) - S|$$

$$= S\,|2l + \cancel{1} - n - \cancel{1}|$$

$$\boxed{J = S\,|2l - n|}$$

For $n \geq 2l+1$

$$J = |L + S|$$

$$= |S[n - (2l+1)] + S|$$

$$= S\,|n - 2l - 1 + 1|$$

$$\boxed{J = S\,|2l + 1|}$$

Q.4

$Ho^{3+} (4f^{10})$

| $m_l$ | 3 | 2 | 1 | 0 | -1 | -2 | -3 |
|---|---|---|---|---|---|---|---|
| | ↑↓ | ↑↓ | ↑↓ | ↑ | ↑ | ↑ | ↑ |

$S = 2$

$\Sigma m_l = 1+2+3 = 6 \Rightarrow L = 6$

$J = |L+S|$ more than half filled

$6+2 = 8$

Multiplicity $2S+1 = 5$

$$\boxed{{}^{2S+1}L_J = {}^5I_8}$$

(b) $Er^{3+} (4f^{11})$

$S = 3/2$

$L = 6$

Muliplicity $2S+1 = 2(3/2+1) = 4$

$J = |L+S| = 6 + 3/2 = 15/2$

${}^4I_{15/2}$

(c) $Tm^{3+}(4f^{12})$

$$S = 1$$

$$L = 5$$

Multiplicity $\quad 2S+1 = 3$

$$J = |L+S| = 6$$

$$\boxed{{}^3H_6}$$

| ↑↓ | ↑↓ | ↑↓ | ↑↓ | ↑↓ | ↑ | ↑ |
|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 0 | -1 | -2 | -3 |

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| S | P | D | F | G | H |

(d)

$Lu^{3+}$ $\quad \Sigma m_s = S = 0$

$$\Sigma m_l = L = 0$$

$$J = |L+S| = 0$$

symbol $\quad S$

$$\boxed{{}^1S_0}$$

Q.5

There are triplet and a singlet states. Energies are given as

$$0, \ \Delta - g\mu_B B, \ \Delta, \ \Delta + g\mu_B B$$

partition fn.

$$Z = e^{-\beta(0)} + e^{-(\Delta - g\mu_B B)\beta} + e^{-\beta\Delta} + e^{-(\Delta + g\mu_B B)}$$

$$= 1 + e^{-\beta\Delta} + e^{-\beta\Delta}\left(e^{\beta g\mu_B B} + e^{-\beta g \mu_B B}\right)$$

$$Z = 1 + e^{-\beta\Delta} + e^{-\beta\Delta} 2\cos(\beta g \mu_0 B)$$

$\mu$ is given by

$$M \cancel{\beta} = +NK_BT \frac{\partial}{\partial B}(\ln Z)$$

$$= \frac{+NK_BT}{Z} \frac{\partial}{\partial B}\left(1 + e^{-\beta\Delta}(1 + 2\cos(g\mu_0 B\beta))\right]$$

$$= +K_BT\frac{N}{Z}\left(0 + 2e^{-\beta\Delta}(-\mu_0 g\beta \sin(g\mu_0 B\beta))\right)$$

$$= \frac{-N2K_BT\mu_0\beta g \sin(g\mu_0 B\beta)}{Z} e^{-\beta\Delta}$$

$$\beta = 1/K_BT$$

$$= -\frac{2\mu_0 g e^{-\beta\Delta} N \sin(g\mu_0 B\beta)}{Z}$$

For

$$\text{Magnetization} = -\frac{2N\mu_0 g e^{-\beta\Delta} \sin(g\mu_0 B\beta)}{Z}$$

$$\chi = -\mu_0 \frac{\partial M}{\partial B}$$ check $\mu_0$

$$M = \chi \frac{B}{\mu_0}$$

$$\chi = \mu_0 \frac{M}{B}$$

$$= \frac{2N\mu_0 g e^{-\beta\Delta}(g\mu_0 B\beta)}{1 + e^{-\beta\Delta}(1 + 2(1 - \frac{1}{2}(g\mu_0 B\beta)^2 + \ldots))}$$

$$\chi = \mu_0 \frac{\partial M}{\partial B}$$

neglecting higher order terms

$$= \frac{2N\mu_B^2 g^2 B\beta}{e^{\beta\Delta} + 1 + 2}$$

$$M \cancel{\chi} = \frac{2N g^2 \mu_B^2 B}{k_B T \left( e^{\Delta\beta} + 3 \right)}$$

$$\chi = \partial M / \partial B$$

$$\boxed{\chi = \frac{2N g^2 \mu_B^2}{k_B T \left( e^{\Delta\beta} + 3 \right)}}$$