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Electrodynamics I
Homework 4

$\boxed{1.}$ Uniformly charged ring of radius R, and negligible thickness, & total charge Q.

i. Monopole Charge:

$$Q' = \int \rho(r')\, d^3r' \qquad Q = \lambda(2\pi R)$$

$$Q' = \int \left(\frac{Q}{2\pi R}\right)(R\, d\phi)$$

$$Q' = \frac{Q}{2\pi}\int d\phi = \frac{Q}{2\pi}(2\pi) = Q$$

$$Q' = Q \quad \text{or} \quad \boxed{Q = \lambda(2\pi R)}$$

ii. Dipole Moment:

$$\vec{p} = \int d^3r\, \rho(r)\, \vec{r} \qquad \vec{r} = R(\cos\phi\, \hat{e}_x + \sin\phi\, \hat{e}_y)$$

$$\vec{p} = \int \lambda R\, d\phi\, \vec{r}$$

$$\vec{p}_x = \lambda R^2 \int_0^{2\pi} \cos\phi\, d\phi$$

$$\vec{p}_x = \lambda R^2 \left[+\sin\phi\right]_0^{2\pi} = 0$$

$$\vec{p}_y = \lambda R^2 \int \sin\phi\, d\phi$$

$$\vec{p}_y = \lambda R^2 \left[-\cos\phi\right]_0^{2\pi} = 0$$

$$\vec{p}_z = 0$$

Dipole moment of a uniformly charged ring is 0.

iii Quadrupole Moment:

$$dq = \rho(\theta) R\, d\theta \qquad (r_1, r_2, r_3) = (x, y, z)$$

$$Q_{ij} = \int_0^{2\pi} d\theta\, \rho(\theta) R\, (3 r_i r_j - r^2 \delta_{ij})$$

$(x, y) = (R\cos\theta, R\sin\theta)$, $z = 0$ (since ring is in the xy plane where $z = 0$)

Since, $z=0$, this means that $Q_{xz} = Q_{yz} = Q_{zx} = Q_{zy} = 0$

$Q_{zz} = 0$ because:

$$Q_{zz} = -\int_0^{2\pi} d\theta\, \rho(\theta) R^2 = 0$$

$$Q_{xx} = \int_0^{2\pi} d\theta\, \rho(\theta) R^3 (3\cos^2\theta - 1)$$

$$= 2R^3 \int_0^{\pi/2} d\theta\, \rho(\theta)(3\cos(2\theta)+2) + 2R^3 \int_{\pi/2}^{\pi} d\theta\, \rho(\theta)(3\cos(2\theta)+2)$$

$$= 2R^3 \int_0^{\pi/2} d\theta\, \rho(\theta)[3\cos(2\theta)+2] + 2R^3 \lambda \int_0^{\pi/2} \rho(\theta+\pi/2)\, d\theta\, [3\cos(2\theta+\pi)+2]$$

$$= 6R^3 \int_0^{\pi/2} d\theta\, [\rho(\theta) - \rho(\theta)][3\cos(2\theta+2]$$

$$= 0$$

Similarly $Q_{yy} = 0$

$$Q_{xy} = Q_{yx} = \int_0^{2\pi} d\theta\, \rho(\theta) R^3 (3\cos\theta\sin\theta)$$

$$= \frac{3R^3}{2} \int_0^{2\pi} d\theta\, \rho(\theta) \sin 2\theta$$

$$= \frac{3R^3}{2}\lambda \int_0^{2\pi} d\theta\, (\sin 2\theta)$$

$$= \frac{3R^3}{2}\lambda(4) = 6R^3\lambda$$

$$Q_{ij} = \begin{pmatrix} 0 & 6R^3\lambda & 0 \\ 6R^3\lambda & 0 & 0 \\ 0 & 0 & \end{pmatrix}$$

$$\frac{\lambda\pi R^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rho(\vec{r}) = \lambda\, \delta(z)\, \delta(r-R)$

$Q_{xz} = Q_{yz} = Q_{zx} = Q_{zy} = Q_{zz} = 0$

$Q_{xy} = Q_{yx} = 0$ (show yourself)

$Q_{xx} = \frac{1}{2}\int d^3r\, \rho(\vec{r})\, x x$

(since $\sin\theta = 1$)

$$= \frac{\lambda}{2}\int_{\phi=0}^{2\pi} \sin\phi \sin\phi\, d\phi \int_0^{\infty} R^2\, \delta(r-R)\, dr$$

$$= \frac{\lambda}{2} R^2 \int_{\phi=0}^{2\pi} d\phi\, \cos^2\phi$$

$$= \frac{\lambda R^2}{4}\int^{2\pi} (1+\cos 2\phi)\, d\phi$$

$$= \frac{\lambda R^2}{4} \cdot 2\pi$$

$$= \frac{\lambda\pi R^2}{2}$$

2. a). Show that $\nabla_j \nabla_i \frac{1}{r} = \frac{3r_i r_j - \delta_{ij} r^2}{r^5}$

✓ $\nabla_i\left(\frac{1}{r}\right) = \left(\nabla \frac{1}{r}\right)_i = -\left(\frac{\hat{r}}{r^2}\right)_i = -\left(\frac{\vec{r}}{r^3}\right)_i = \frac{-r_i}{r^3}$

$$-\nabla_j\left(\frac{r_i}{r^3}\right) = -\left[\frac{1}{r^3}\underbrace{\nabla_j r_i}_{\delta_{ij}} + r_i \nabla_j \frac{1}{r^3}\right] \quad \text{—— (a)}$$

✓

$$\nabla_j \frac{1}{r^3} = \left(\nabla \frac{1}{r^3}\right) = -3\left(\frac{\hat{r}}{r^4}\right)_j = -3\left(\frac{\vec{r}}{r^5}\right) = \frac{-3r_j}{r^5}$$

Eq (a) now becomes:

$$\nabla_j \nabla_i \frac{1}{r} = \frac{-\delta_{ij}}{r^3} + \frac{3r_i r_j}{r^5}$$

$$\boxed{\nabla_j \nabla_i \frac{1}{r} = \frac{3r_i r_j - \delta_{ij} r^2}{r^5}}$$ ✓

b). $$\phi(r) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho_Q(r')}{|\vec{r}-\vec{r}'|}$$ ✓

this sign is inverted

$$= \frac{-1}{4\pi\varepsilon_0} \int d^3r' \frac{Q_{ij} \nabla_i' \nabla_j' \delta(r'-r_0)}{|\vec{r}-\vec{r}'|}$$ ✓

$$= (+)\frac{1}{4\pi\varepsilon_0} \int d^3r' \delta(r'-r_0)\, Q_{ij} \nabla_i' \nabla_j' \frac{1}{|r-r'|}$$ ✓ (Using Integration by parts)

$$= (+)\frac{1}{4\pi\varepsilon_0} Q_{ij} \cancel{\nabla_i \nabla_j} \int d^3r' \delta(r'-r_0)\, \nabla_i' \nabla_j' \frac{1}{|r-r'|}$$ (Using sampling theorem $r' = r_0$)

$$\boxed{\phi(r) = -\frac{1}{4\pi\varepsilon_0} Q_{ij} \nabla_i \nabla_j \frac{1}{|r-r_0|}}$$ ✓

c). $$F = \int d^3r\, \rho(r) E(r)$$ ✓

$$= Q_{ij} \int d^3r\, E(r) \nabla_i \nabla_j \delta(r-r_0)$$ ✓

$$\boxed{F = Q_{ij} \nabla_i \nabla_j E(r_0)}$$ ✓

d). $$U_E = \int d^3r\, \phi(r) \rho(r) = Q_{ij} \int d^3r\, \phi(r) \nabla_i \nabla_j \delta(r-r_0)$$ ← Apply integration by parts

$$= Q_{ij} \int d^3r\, \delta(\vec{r}-\vec{r}_0) \nabla_i \nabla_j \phi(r)$$ ← Using: $E(r) = -\nabla \phi(\vec{r})$

$$= -Q_{ij} \int d^3r\, \delta(r-r_0) \nabla_i E_j(r)$$

Label your vectors properly.

$$\boxed{U_E = -Q_{ij} \nabla_i E_j(\vec{r}_0)}$$ ✓

(−)

$\boxed{3.}$ $N = \int d^3r' \, r' \times E(r') \rho_D(r')$

$$N = \int d^3r' \, \delta(r'-r)(p\cdot\nabla')(r' \times E)$$

$$N_i = \varepsilon_{ijk} \int d^3r' \, \delta(r'-r) \, p_m \nabla_l (r_j E_k)$$

$$N_i = \varepsilon_{ijk} \int d^3r' \, \delta(r'-r) \, p_m \{ E_k \nabla_l r_j + r_j \nabla_l E_i \}$$

$$= \varepsilon_{ijk} \int d^3r' \, \delta(r'-r) \, p_m \{ \delta_{lj} E_k + r_j \nabla_l E_i \}$$

$$N_i = \varepsilon_{ijk} \, p_m \{ \delta_{lj} E_k + r_j \nabla_i E_m \}$$

Using $F = (p\cdot\nabla) E(r)$, we get:

$$N_i = \varepsilon_{ijk} p_m \delta_{lj} E_k + r_j \varepsilon_{ijk} p_m \nabla_i E_m$$

$$N_i = (p \times E)_i + (r \times F)_i$$

$$\boxed{N = (p \times E) + (r \times F)}$$

$\boxed{4.}$ a). $\rho_D(r) = -p\cdot\nabla \, \delta(r - r_0)$

$$\phi(r) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\rho_D(r')}{|r-r'|}$$

$$= \frac{-1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\vec{p}\cdot\nabla\delta(r-r_0)}{|r-r'|}$$

$$= \frac{-\vec{P}}{4\pi\varepsilon_0} \cdot \int d^3r' \, \underbrace{\frac{1}{|r-r'|}}_{u} \, \underbrace{\vec{\nabla}\delta(\vec{r}-\vec{r}_0)}_{v} \qquad \text{——— } (1)$$

We will use integration by parts for the above:

$$\int_{x\to-\infty}^{x\to+\infty} dx \left[\frac{d}{dx}(\delta(x))\right] f(x)$$

$$= \int \underbrace{dx\, f(x)}_{u}\ \underbrace{\frac{d}{dx}\delta(x)}_{v'}$$

$$= f(\cancel{x})^{\nearrow 0}\,\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{+\infty} dx\, \delta(x)\, \frac{df}{dx}$$

potentials at ∞ is 0 (for bound functions).

$$= \left| -\int dx\, \delta(x)\, \frac{df}{dx} \right|$$

We will use the above result on 1

$$\phi(r) = \frac{\vec{p}}{4\pi\varepsilon_0} \cdot \int d^3r'\, \delta(r'-r_0)\, \nabla' \frac{1}{|\vec{r}-r'|}$$ ✓

Now use the sampling theorem, $r' = r_0$ ✓

$$\phi(r) = \frac{\vec{p}}{4\pi\varepsilon_0} \cdot \nabla \frac{1}{|\vec{r}-\vec{r}_0|}.$$ ✓

$$\boxed{\phi(r) = \frac{p\cdot(\vec{r}-\vec{r}_0)}{4\pi\varepsilon_0 |r-r_0|^3}}$$

For $\vec{r}_0 \equiv$ origin $\phi(\vec{x}) = \frac{\vec{p}\cdot\vec{x}}{4\pi\varepsilon_0|\vec{x}|^3}$.

Be rigorous about vector notation.

For $r \neq 0$

$$E = -\vec{\nabla}\phi(r)$$

$$E_j = \frac{-1}{4\pi\varepsilon_0} p_i \nabla_j \left(\frac{\vec{r}_i}{r^3}\right)$$ unclear $r = |r - r_0|$

Solving the term in bracket:

$$\nabla_j\left(\frac{r_i}{r^3}\right) = \frac{1}{r^3}\nabla_j r_i + r_i \nabla_j \frac{1}{r^3}$$

$$= \frac{1}{r^3}\delta_{ij} + r_i \nabla_j \frac{1}{r^3} \qquad \text{—— (2)}$$

$$\nabla_j \frac{1}{r^3} = \left(\nabla \frac{1}{r^3}\right)_j = -3\left(\frac{\hat{r}}{r^4}\right)_j = -3\left(\frac{\vec{r}}{r^5}\right)_j = -\frac{3r_j}{r^5}$$

The eq ② now becomes:

Show

$$\nabla_j\left(\frac{r_i}{r^3}\right) = \frac{1}{r^3}\left(\delta_{ij} - 3\frac{r_i r_j}{r^2}\right) \qquad \vec{E} = -\vec{\nabla}\frac{1}{4\pi\varepsilon_0}\left(\frac{\vec{p}\cdot\hat{r}}{r^2}\right)$$

So, $\vec{E}_j$ now becomes:

$$= -\vec{\nabla}\frac{1}{4\pi\varepsilon_0}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

$$E_j = -\frac{1}{4\pi\varepsilon_0} p_i \frac{1}{r^3}\left(\delta_{ij} - 3\frac{r_i r_j}{r^2}\right)$$

unclear

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3r_i r_j}{r^2}p_i - p_i\delta_{ij}\right)$$

$$E_j = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3r_i r_j}{r^2}p_i - p_j\right)$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\hat{r}(\hat{r}\cdot p) - p\right)}$$

b). The work done by us to bring p_2 into position (with p_1 already present) is exactly the interaction energy with the electric field of p_1 playing the role of E_{ext}.

$$V_E = -p_2 \cdot E_1(r_2)$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{p_1\cdot p_2}{|r_2-r_1|^3} - \frac{3p_1\cdot(r_2-r_1)\,p_2\cdot(r_2-r_1)}{|r_2-r_1|^5}\right\} ✓$$

$r = |r_2 - r_1|$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{p_1\cdot p_2}{r^3} - \frac{3(p_1\cdot\hat{r})(p_2\cdot\hat{r})}{r^3}\right\}$$

$$\boxed{V_E = \frac{1}{4\pi\varepsilon_0 r^3}\left(p_1\cdot p_2 - 3(p_1\cdot\hat{r})(p_2\cdot\hat{r})\right)}$$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

c). We choose a polar coordinate system with $p_1 \parallel \hat{z}$. The field produced by p_1 in this system is:

$$\vec{E}_1 = \frac{p_1}{4\pi\varepsilon_0 r^3}(2\cos\theta_1 \hat{r} + \sin\theta_1 \hat{\theta})$$

At equilibrium, the potential energy $V_E = -p_2 \cdot E_1$ is a minimum.

$$p_2 = p_2 \cos\theta_2 \hat{r} + p_2 \sin\theta_2 \hat{\theta}$$

✓

$$V_E = -p_2 \cdot E_1$$
$$= \frac{1}{4\pi\varepsilon_0}\left[-p_1 p_2 2\cos\theta_1 \cos\theta_2 - p_1 p_2 \sin\theta_1 \sin\theta_2\right]$$

To find minimum V_E, we take its derivative & set it equal to 0.

$$0 = \frac{\partial V_E}{\partial \theta_2} = -\frac{p_1 p_2}{4\pi\varepsilon_0}\left[-2\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2\right]$$

$$2\cos\theta_1 \sin\theta_2 = \sin\theta_1 \cos\theta_2$$
$$\tan\theta_1 = 2\tan\theta_2$$

If $\theta_1 = \frac{\pi}{4}$, then $\theta_2 = \tan^{-1}(0.5)$
$= 0.463$ radians

5. $\rho(r) = \dfrac{R}{r^2}(R-2r)\sin\theta$

Let us first calculate the monopole charge:

$$Q = \int d^3r\, \rho(r)$$
$$= \int \frac{R}{r^2}(R-2r)\sin\theta\,(r^2\sin\theta\, d\theta\, d\phi\, dr)$$
$$= \int (R^2 - 2Rr)\sin^2\theta\, d\theta\, d\phi\, dr$$
$$= \int (R^2 - 2Rr)\,dr \int \sin^2\theta\, d\theta \int d\phi$$
$$= \left[\cancel{R^2 r - \frac{2Rr^2}{2}}\right]_0^R \left[\frac{\sin 2\theta}{4} - \frac{1}{2}\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$Q = 0$$

✓

Now, let's calculate the dipole moment:

$$\vec{p} = \int d^3r\, \rho(r)\,\vec{r} \qquad \vec{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$= \int (R^2 - 2Rr)\sin^2\theta\, d\theta\, d\phi\, dr\, \vec{r}$$

$$\vec{p}_x = \int (R^2 - 2Rr)\sin^2\theta\, d\theta\, d\phi\, dr\, r\sin\theta\cos\phi$$

$$\vec{p}_x = \int R^2 r - 2Rr^2\, dr \int \sin\theta\frac{\cos 2\theta}{2} - \frac{\sin\theta}{2}\, d\theta \int \cos\phi\, d\phi$$

$$\vec{p}_x = \left[\frac{R^2r^2}{2} - \frac{2Rr^3}{3}\right]\left[\frac{1}{3}\cos^3\theta - \cos\theta\right]\left[\sin\phi\right]$$

$$\boxed{\vec{p}_x = 0}$$

✓

$$\vec{p}_y = \left[\frac{R^2r^2}{2} - \frac{2Rr^3}{3}\right]\left[\frac{1}{3}\cos^3\theta - \cos\theta\right]_0^\pi \left[-\cos\phi\right]_0^{2\pi}$$

$$\boxed{\vec{p}_y = 0}$$

$$\vec{p}_z = \left[\frac{R^2r^2}{2} - \frac{2Rr^3}{3}\right]\left[\frac{\sin^3\theta}{3}\right]_0^\pi = \left[\frac{R^4}{2} - \frac{2R^4}{3}\right]\left[0\right]$$

$$\boxed{\vec{p}_z = 0}$$ — Lowest order term – ✓ the multipole expansion is

$$Q_{xx} = Q_{yy} = \frac{1}{2}\int d^3r \; \rho(\vec{r})\, x^2$$

$$= \frac{1}{2}\int d^3r \; \frac{R}{\cancel{r^2}}(R-2r)\sin\theta \; \cancel{r^2}\sin^2\theta\cos^2\phi$$

$$= \frac{1}{2}\int d\theta\, d\phi\, dr \left(r^2 \sin^4\theta\right) R(R-2r)\cos^2\phi$$

$$= \frac{1}{2}\int_{\phi=0}^{2\pi} d\phi \; \cos^2\phi \int_{\theta=0}^{\pi} d\theta \; \sin^4\theta \int_{r=0}^{R} R\, r^2 (R-2r)\, dr$$

$$= -\pi^2 R^5/32$$

Similarly $Q_{zz} = -\pi^2 R^5/48$.

$$\overleftrightarrow{Q} = \frac{-\pi^2 R^5}{32}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \quad 1/2 \end{pmatrix}$$

One can show that other terms are zero.