✓ Exercises

Q1) $\rho = 7873\ kg\ m^{-3}$

$M_A = 55.847$

$T_c = 1043\ K$

moment / atom $= 2.2\ \mu_B$

$$B_{mf} = \frac{3 k_B T_c}{g_J \mu_B (1+J)}$$

It is given that $g_J (1+J) \approx 2.2$

$$B_{mf} = \frac{3 k_B T_c}{2.2 \mu_B} \approx \frac{3 \times 1.38 \times 10^{-23} \times 1043}{2.2 \times 9.27 \times 10^{-24}} = \boxed{2.12 \times 10^{3}\ T}$$

$\rightarrow M_{sat} = n\ (2.2\ \mu_B)$

$n = ?$

55.847 g has 6.02×10^{23} atoms.

55.847 g has a volume of $\frac{\cancel{58}\ 55.847}{7.873} = 7.09\ cm^3$

$= 7.09 \times 10^{-6}\ m^3$

$\cancel{N} \cancel{=}$ 6.02×10^{23} atoms $\equiv$ 7.09×10^{-6} m^3

$$n = \frac{6.02 \times 10^{23}}{7.09 \times 10^{-6}} = 0.84 \times 10^{29} \; m^{-3} = 8.4 \times 10^{28} \; m^{-3}.$$

$$M_{sat} \approx 8.4 \times 10^{28} \times 2.2 \times 9.27 \times 10^{-24} = 1.71 \times 10^{6} \; A\,m^{-1}$$

$$B_{sat} = \mu_0 M = 2.2 \; T.$$

$$B_{mf} \gg B_{sat}.$$

The molecular field is 3 orders of magnitude larger than what is required to saturate the spins.

HW ✓

Q2

closer to a real antiferromagnet

$$M_+ = m_s \frac{g_J \mu_B J}{k_B T}\left(B + |\alpha| M_+ - |\lambda| M_-\right)\left(\frac{J+1}{3J}\right)$$

$$M_- = m_s \frac{g_J \mu_B J}{k_B T}\left(B + |\alpha| M_- - |\lambda| M_+\right)\left(\frac{J+1}{3J}\right)$$

$$M_+ + M_- = m_s \frac{g_J \mu_B J}{k_B T}\left(\frac{J+1}{3J}\right)\left(2B + |\alpha|\left(M_+ + M_-\right) - |\lambda|\left(M_+ + M_-\right)\right)$$

$$M = \frac{m_s g_J \mu_B J}{k_B T}\left(\frac{J+1}{3J}\right)(2B) + \frac{m_s g_J \mu_B J}{k_B T}\alpha M\left(\frac{J+1}{3J}\right) - |\lambda| \frac{m_s g_J \mu_B J}{k_B T}\left(\frac{J+1}{3J}\right) M$$

$$M\left(1 - \alpha\left(\frac{J+1}{3J}\right)\frac{m_s g_J \mu_B J}{k_B T} + |\lambda| \left(\frac{J+1}{3J}\right)\frac{m_s g_J \mu_B J}{k_B T}\right) = \frac{m_s g_J \mu_B J}{k_B T}(2B)\left(\frac{J+1}{3J}\right)$$

Now $$T_N = \frac{m_s (J+1) g_J \mu_B \lambda}{3 k_B}$$

$$M\left(1 - \alpha \frac{T_N}{T\lambda} + \frac{T_N}{T}\right) = 2B\, T_N / T$$

$$\chi = \frac{2 \mu_0 T_N / T}{1 + \left(\frac{T_N}{T}\left(-\frac{\alpha}{\lambda} + 1\right)\right)} = \frac{2 \mu_0 T_N}{T + (-\alpha/\lambda + 1) T_N}$$

HW4

Q3

(a) $B_J(y) = \tanh y$

$$\tilde{m} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{x - 1/x}{x + 1/x}$$

$$\tilde{m}\left(x + \frac{1}{x}\right) = x - 1/x$$

$$\tilde{m}(x^2 + 1) = x^2 - 1$$

$$x^2(\tilde{m} - 1) + (\tilde{m} + 1) = 0$$

$$x^2 = -\frac{1+\tilde{m}}{\tilde{m} - 1} = \frac{1 + \tilde{m}}{1 - \tilde{m}}.$$

$$\log x = \frac{1}{2}\left(\log(1+\tilde{m}) - \log(1-\tilde{m})\right)$$

where $x = e^y$ $\Rightarrow$ $\log x = \boxed{y = \frac{1}{2}\left(\log(1+\tilde{m}) - \log(1-\tilde{m})\right)}$

Now $y = \frac{g_J \mu_B B}{k_B T} J = \frac{g_J \mu_B \lambda M}{k_B T} J$ (since $B = \lambda M$) - the molecular field

$$\Rightarrow \quad M = \frac{k_B T}{g_J \mu_B \lambda J} y.$$

$$\tilde{m} = \frac{M}{M_s} = \frac{k_B T}{g_J \mu_B \lambda J \quad n g_J \mu_B J} y$$

(as $M_s = n g_J \mu_B J$)

$$\tilde{m} = \frac{k_B T}{n (g_J \mu_B J)^2 \lambda \cancel{\phi}} y \quad — (1)$$

We have $$T_c = \frac{m_s (J+1) g_J \mu_B \lambda}{3 k_B}.$$

~~on~~ Inserting m_s into the expression for T_c:

$$T_c = \frac{n J (g_J \mu_B)^2 (J+1) \lambda}{3 k_B}$$

$$\Rightarrow \frac{n (g_J \mu_B)^2 \lambda}{k_B} = \frac{3 T_c}{J(J+1)}.$$

$$\tilde{m} = \frac{k_B}{n (g_J \mu_B)^2 \lambda} \cdot \frac{T y}{J^2}$$

Insert this into (1):

$$\tilde{m} = \frac{T y}{J^{\cancel{2}} 3 T_c} \cdot \cancel{J}(J+1)$$

$$\frac{3}{\cancel{2}} \cdot \frac{1}{\cancel{3}} \cdot \cancel{2}$$

$$\tilde{m} = \frac{(J+1) y}{3 J} \left(\frac{T}{T_c} \right).$$

Now $J = 1/2 \Rightarrow \tilde{m} = y \frac{T}{T_c}.$

$$y = \tilde{m} \frac{T_c}{T} = \frac{1}{2} \left[\log (1 + \tilde{m}) - \log (1 - \tilde{m}) \right]$$

$$\Rightarrow \boxed{\frac{T}{T_c} = \frac{2\tilde{m}}{\log(1+\tilde{m}) - \log(1-\tilde{m})}}, \text{ as desired.}$$

This expression relates $\tilde{m}(T)$ and T for a given T_c.

(b) For $J = 1$,

$$B_J(y) = \frac{1}{2J}\left\{(2J+1)\coth\left(y\frac{2J+1}{2J}\right) - \coth\left(\frac{y}{2J}\right)\right\}$$

$$B_1(y) = \frac{1}{2}\left(3\coth\left(y\frac{3}{2}\right) - \coth\left(\frac{y}{2}\right)\right)$$

$\coth = \frac{1}{\tan}$

$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$$= \frac{1}{2}\left[3\frac{e^{\frac{3}{2}y} + e^{-\frac{3}{2}y}}{e^{\frac{3}{2}y} - e^{-\frac{3}{2}y}} - \frac{e^{y/2} + e^{-y/2}}{e^{y/2} - e^{-y/2}}\right]$$

$$= \frac{1}{2}\left[\frac{\left(3e^{\frac{3}{2}y} + 3e^{-\frac{3}{2}y}\right)\left(e^{y/2} - e^{-y/2}\right) - \left(e^{y/2} + e^{-y/2}\right)\left(3e^{\frac{3}{2}y} + 3e^{-\frac{3}{2}y}\right)}{\left(e^{\frac{3}{2}y} - e^{-\frac{3}{2}y}\right)\left(e^{y/2} - e^{-y/2}\right)}\right]$$

$$B_1(y) = \frac{2\sinh y}{1 + 2\cosh y}$$

(I used Mathematica)

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Now $x = e^{y}$ (say)

$$\tilde{m} = B_1(y) = \frac{2}{2}\left(e^{y} - e^{-y}\right) \cdot \frac{1}{1 + \frac{2}{2}\left(e^{y} + e^{-y}\right)}$$

$$\tilde{m} = \frac{e^{y} - e^{-y}}{1 + e^{y} + e^{-y}}$$

$$= \frac{x - \frac{1}{x}}{1 + x + \frac{1}{x}}$$

$$\tilde{m} = \frac{x^2 - 1}{x + x^2 + 1}$$

$$\tilde{m}\left(x^2 + x + 1\right) = x^2 - 1$$

$$x^2(\tilde{m} - 1) + \tilde{m}x + (\tilde{m} + 1) = 0$$

$$x = \frac{-\tilde{m} \pm \sqrt{\tilde{m}^2 - 4(\tilde{m} - 1)(\tilde{m} + 1)}}{2(\tilde{m} - 1)}$$

$$= \left(-\tilde{m} \pm \sqrt{\tilde{m}^2 - 4(\tilde{m}^2 - 1)}\right) / 2(\tilde{m} - 1)$$

$$= \left(-\tilde{m} \pm \sqrt{4 - 3\tilde{m}^2}\right) / 2(\tilde{m} - 1)$$

$$= \frac{\cancel{4}\left(\tilde{m} \pm \sqrt{4-3\tilde{m}^2}\right)}{2\left(1-\tilde{m}\right)}$$

Range of $\sqrt{4-3\tilde{m}^2}$ for $\tilde{m} \in [0,1]$ is $[2,1]$, $\therefore$ only the positive root is permissible $(x = e^y > 0)$.

So $$x = e^y = \frac{\tilde{m} + \sqrt{4-3\tilde{m}^2}}{2(1-\tilde{m})}$$

$$y = \log\left(\tilde{m} + \sqrt{4-3\tilde{m}^2}\right) - \log\left(2(1-\tilde{m})\right) \quad -(2)$$

Since $$\tilde{m} = \frac{(J+1)}{3J} y \frac{T}{T_c}$$

for $J = 1$, $$\tilde{m} = \frac{2y}{3}\frac{T}{T_c} \Rightarrow y = \frac{3}{2}\left(\frac{T_c}{T}\right)\tilde{m}$$

Inserting into (2) yields:

$$\frac{3}{2}\left(\frac{T_c}{T}\right)\tilde{m} = \log\left(\tilde{m} + \sqrt{4-3\tilde{m}^2}\right) - \log\left(2(1-\tilde{m})\right)$$

$$\boxed{\frac{T}{T_c} = \frac{\frac{3}{2}\tilde{m}}{\log\left(\tilde{m} + \sqrt{4-3\tilde{m}^2}\right) - \log\left(2(1-\tilde{m})\right)}}$$