Assignment 6: Various aspects of quantum magnetism

1. The Hamiltonian for a spin system is given by:

$$\hat{H} = -2\sum_{i<j} J\hat{S}_i \cdot \hat{S}_j - 2\sum_{i<j} K\hat{S}_{i,z}\hat{S}_{j,z}. \tag{1}$$

   Assume all the pairwise Heisenberg coupling constants J and the Ising coupling constants K are the same. Only next neighbor interactions are operative.

   (a) Use the second quantization formalism to derive the dispersion relation for magnons.

   (b) What is the dispersion relation for a one-dimensional spin chain in the (i) general case, (ii) when $K = 0$ and (iii) when $J = 0$?

   (c) Derive the dispersion relation for spins arranged on a two-dimensional square lattice.

2. Consider a spin-1/2 paramagnetic system with isolated spins placed inside a magnetic field. Using the partition function, derive and plot the internal energy, heat capacity and entropy of the system.

3. The dispersion relation for spin waves is given by $\omega = \alpha k^n$. How does the heat capacity vary as a function of temperature T for a d-dimensional system, where $d = 1, 2, 3$?

4. Consider an Ising chain in one dimension. The Hamiltonian is

$$\hat{H} = -2J\sum_{i<j} \hat{S}_{i,z}\hat{S}_{i+1,z}. \tag{2}$$

   (a) Define the operator $\zeta_i = \hat{S}_{i,z}\hat{S}_{i+1,z}$. What are its eigenvalues? (Assume $\hbar = 1$).

   (b) Find the partition function of the system.

   (c) What is the probability that inside a pair of spins, both spins are parallel? Sketch this probability as a function of temperature.

   (d) Find the probability that M *consecutive* spins are parallel. How does this probability change with M?