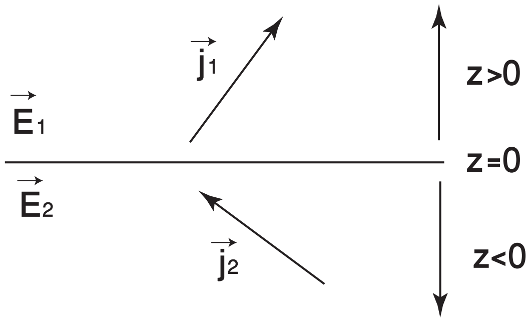
Solution HW 7: Miscellaneous topics in electrostatics and steady currents

1. **Answer:**



Let

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_1(\mathbf{r})\Theta(z) + \mathbf{E}_2(\mathbf{r})\ \Theta(-z) \\
\nabla \times \mathbf{E} &= \big(\nabla \times \mathbf{E}_1(\mathbf{r})\big)\Theta(z) + \nabla\Theta(z) \times \mathbf{E}_1(\mathbf{r}) \\
&+ \big(\nabla \times \mathbf{E}_2(\mathbf{r})\big)\Theta(-z) + \nabla\Theta(-z) \times \mathbf{E}_2(\mathbf{r})
\end{aligned}$$

using $\nabla \times \mathbf{E}_i(\mathbf{r}) = 0$, (on either side)

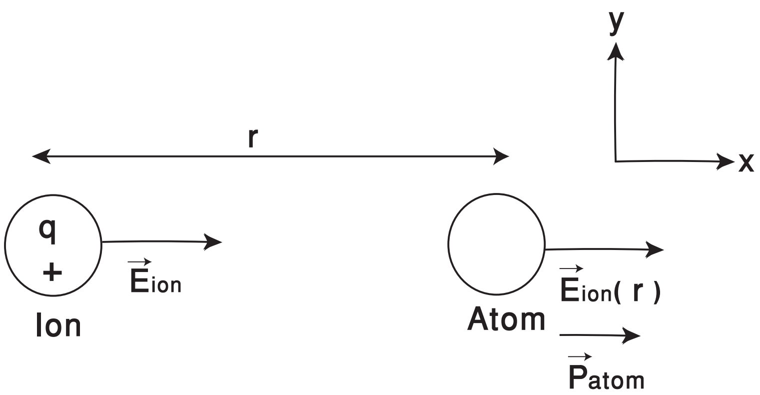
$$\begin{aligned}
0 &= \mathbf{E}_1(z=0) \times \hat{\mathbf{e}}_z - \mathbf{E}_2(z=0) \times \hat{\mathbf{e}}_z \\
0 &= \left(\mathbf{E}_1\Big|_S - \mathbf{E}_2\Big|_S\right) \times \hat{\mathbf{n}}
\end{aligned}$$

$$\boxed{\left(\mathbf{E}_1\Big|_S - \mathbf{E}_2\Big|_S\right) \times \hat{\mathbf{n}} = 0}$$

is the desired boundary condition.

2. **Answer:**

(a)



$$\begin{aligned} E_{\text{ion}} &= \frac{q}{4\pi\varepsilon_0 r^2} \\ \mathbf{E}_{\text{ion}}(\mathbf{r}) &= \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{e}}_x. \end{aligned}$$

$\mathbf{E}_{\text{ion}}(\mathbf{r})$ induces a dipole in the atom, $\mathbf{P}_{\text{atom}} = \alpha\mathbf{E}_{\text{ion}}(\mathbf{r})$

$$\mathbf{P}_{\text{atom}} = \alpha\frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{e}}_x$$

We now like to find the electric field produced by the polarized atom at the location of the ion. $\big($Use $\mathbf{E}(r,\Theta) = \dfrac{P}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\mathbf{e}}_r + \sin\theta\hat{\mathbf{e}}_\theta)$ with $\theta = 0.\big)$

$$\begin{aligned} \mathbf{E}_{\text{atom}}(\text{location of ion}) &= \frac{1}{2\pi\varepsilon_0}\frac{\mathbf{P}_{\text{atom}}\cdot\hat{\mathbf{r}}}{r^3}\hat{\mathbf{e}}_x \\ &= \frac{1}{2\pi\varepsilon_0}\left(\frac{\alpha q}{4\pi\varepsilon_0 r^2}\right)\frac{1}{r^3}\hat{\mathbf{e}}_x. \\ \mathbf{E}_{\text{ion}} &= q\mathbf{E}_{\text{atom}} = \frac{2\alpha q}{(4\pi\varepsilon_0)^2 r^5}\hat{\mathbf{e}}_x. \end{aligned}$$

The force varies as $\dfrac{1}{r^5}$.

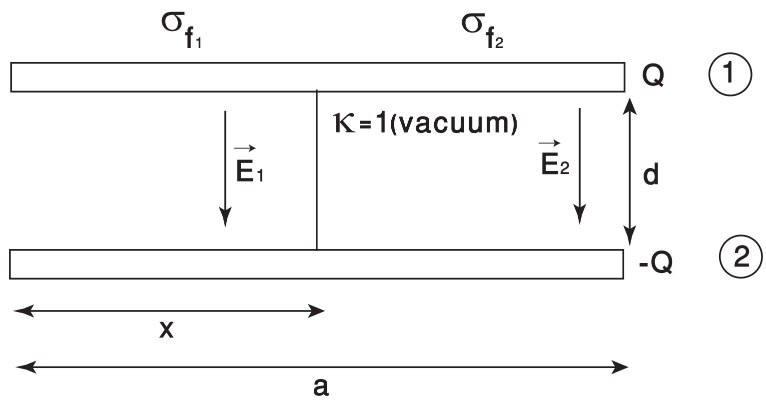
(b)

$$\begin{aligned} u(r) &= -\int_\infty^r \frac{-2\alpha q}{(4\pi\varepsilon_0)^2 r'^5}dr' \\ &= \frac{2\alpha q}{(4\pi\varepsilon_0)^2}\int_\infty^r r'^{(-5)}dr' \\ &= \frac{2\alpha q}{(4\pi\varepsilon_0)^2}\frac{r'^{(-4)}}{-4}\Bigg|_\infty^r \\ &= \frac{-\alpha q}{2(4\pi\varepsilon_0)^2}\left(\frac{1}{r^4}\right). \end{aligned}$$

The potential is $\dfrac{-\alpha q}{2(4\pi\varepsilon_0)^2 r^4}$ with respect to when the atom and ion are spaced infinitely apart.

3. **Answer:**

(a)



$$\begin{aligned} D_2 &= \varepsilon_0 E_2 = \sigma_{f_2} \quad \text{(by Gauss's law)} \\ D_1 &= \varepsilon_0 E_1 = \sigma_{f_1} \end{aligned}$$

Since $\phi_1 - \phi_2$ is the same across the two regions:

$$\begin{aligned} E_1 = E_2 &= \frac{\Delta\phi}{d} \\ \Rightarrow \quad \frac{\sigma_{f_1}}{\varepsilon} &= \frac{\sigma_{f_2}}{\varepsilon_0} \\ \Rightarrow \quad \sigma_{f_1} &= \frac{\varepsilon}{\varepsilon_0}\sigma_{f_2} = \kappa\sigma_{f_2} \end{aligned}$$

We require,

$$\begin{aligned} \sigma_{f_1} xb + \sigma_{f_2}(a-x)b &= Q \\ \sigma_{f_2}(\kappa xb + ab - xb) &= Q \end{aligned}$$

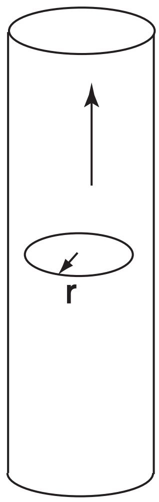
$$\sigma_{f_2} = \frac{Q}{b(x(\kappa-1)+a)} \quad , \quad \sigma_{f_1} = \frac{\kappa}{b(x(\kappa-1)+a)}.$$

(b) We have two capacitors in parallel,

$$
\begin{aligned}
C_1 = \varepsilon_0 \kappa \frac{xb}{d} \quad &, \quad C_2 = \varepsilon_0 (a-x) \frac{b}{d} \\
C = C_1 + C_2 &= \varepsilon_0 \frac{b}{d}\big(\kappa x + (a-x)\big) \\
u = \frac{Q^2}{2C} &= \frac{Q^2}{2\varepsilon_0 (b/d)}\big(\kappa x + (a-x)\big) \\
F = -\frac{du}{dx} &= \frac{-Q^2}{2\varepsilon_0 (b/d)} \frac{d}{dx}\big(\kappa x + (a-x)\big)^{-1} \\
&= \frac{-Q^2}{2\varepsilon_0 (b/d)}(-1)\frac{1}{\big(\kappa x + (a-x)\big)^2}\kappa \\
&= \frac{Q^2 \kappa}{2\varepsilon_0 (b/d)} \frac{1}{\big(\kappa x + (a-x)\big)^2}.
\end{aligned}
$$

4. **Answer:**

(a)



$$
\begin{aligned}
\mathbf{j} &= j(s)\hat{\mathbf{e}}_z \\
\mathbf{j} &= qn\mathbf{v} \\
\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= q\left(\mathbf{E} + \frac{1}{qn}\mathbf{j} \times \mathbf{B}\right) \\
\mathbf{E} &= -\frac{1}{qn}\mathbf{j}(s) \times \mathbf{B}
\end{aligned}
$$

From Ampere's law:

$$\begin{aligned} B_\phi(s)\ 2\pi r &= \mu_0 2\pi \int_0^r ds\ s\ j(s) \\ \Rightarrow \quad B_\phi(s) &= \frac{1}{r}\int_0^r ds\ s\ j(s) \\ &= \frac{\mu_0}{r}\int_0^r ds\ s\ j(s)\ \hat{\mathbf{e}}_\phi \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{F} &= q\left(\mathbf{E} + \frac{\mathbf{j}(s)}{nq} \times \mathbf{B}(s)\right) \\ \mathbf{E} &= -\frac{1}{nq}\left(\mathbf{j}(s) \times \mathbf{B}(s)\right) \\ &= \frac{1}{nq}\left(\mathbf{j}(s) \times \mathbf{B}(s)\right)\hat{\mathbf{e}}_r \end{aligned}$$

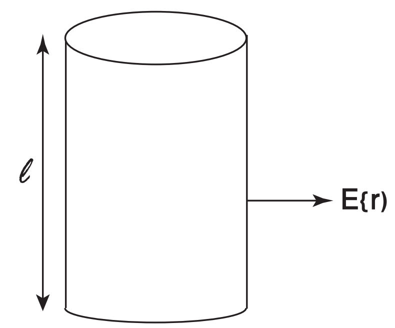
There must exist a radial electric field.

$$\boxed{E(r) = \frac{1}{nq} j(s) \frac{\mu_0}{r} \int_0^r ds\ s\ j(s)}.$$

(c)

$$\rho = \rho_1 + \rho_2$$

where $\rho_1 = \dfrac{j}{v}$ and ρ_2 is other charge, ρ_2 is uniform, while $\rho_1 = \rho_1(s)$ varies as the distance from the axis of the cylinder.



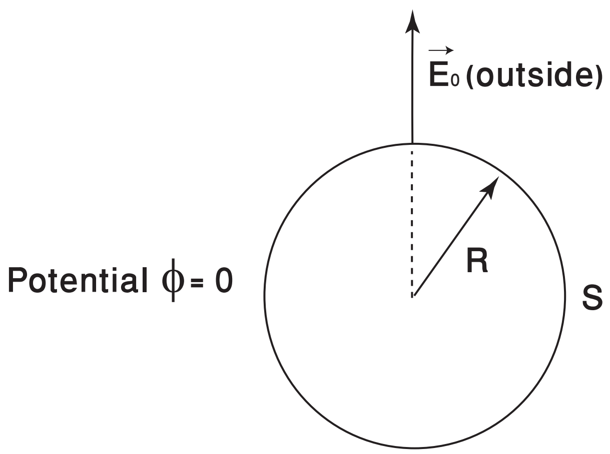
$$
\begin{aligned}
E(r)\ 2\pi r\ell &= \frac{1}{\varepsilon_0}\ell 2\pi \int\limits_0^r ds\ s\ (\rho_1^{(s)} + \rho_2) \\
E(r) &= \frac{1}{\varepsilon_0 r}\left[\int ds\ s\ \rho_1(s) + \frac{\rho_2 r^2}{2}\right] \\
&= \frac{1}{\varepsilon_0 r}\left[\frac{1}{v}\int\limits_0^r ds\ s\ j_1(s) + \frac{\rho_2 r^2}{2}\right] \\
\frac{v\mu_0}{r}\int\limits_0^r ds\ s\ j(s) &= \frac{1}{\varepsilon_0 r v}\int\limits_0^r ds\ s\ j(s) + \frac{1}{\varepsilon_0 r}\int\limits_0^r ds\ s\ \rho_2 \\
\left(\frac{v\mu_0}{r} - \frac{1}{\varepsilon_0 r v}\right)\int\limits_0^r ds\ s\ j(s) &= \frac{\rho_2}{\varepsilon_0 r}\int\limits_0^r ds\ s
\end{aligned}
$$

Now $j(s) = \rho_1(s)v$

$$
\begin{aligned}
\left(\frac{v^2\mu_0}{r} - \frac{1}{\varepsilon_0 r}\right)\int\limits_0^r ds\ s\ \rho_1(s) &= \frac{\rho_2}{\varepsilon_0 r}\int\limits_0^r ds\ s \\
\int\limits_0^r ds\ s\ \rho_1(s) &= \frac{\rho_2}{\varepsilon_0 r\left(\frac{v^2\mu_0}{r} - \frac{1}{\varepsilon_0 r}\right)}\int\limits_0^r ds\ s \\
\int\limits_0^r ds\ s\ \rho_1(s) &= \frac{\rho_2}{\frac{v^2}{c^2} - 1}\int\limits_0^r ds\ s \\
\Rightarrow\quad \rho_1^{(s)} &= -\frac{\rho_2}{1 - v^2/c^2}.
\end{aligned}
$$

Positive charges are squeezed in a little bit.

5. **Answer:**



$$\begin{aligned}\phi = 0 \;&=\; \frac{1}{4\pi\varepsilon_0}\int \frac{ds\ \sigma_p(\mathbf{r}_s)}{R} \\ &=\; \frac{1}{4\pi\varepsilon_0}\frac{R^2}{R}\int\int \sin\theta\ d\theta\ d\phi\ \sigma_p(\theta,\phi)\end{aligned}$$

σ_p cannot depend on ϕ.

$$\begin{aligned}\therefore\quad 0 \;&=\; \frac{1}{4\pi\varepsilon_0}\,2\pi\int\limits_{\theta=0}^{\pi}\sin\theta\ \sigma_p(\theta)\ d\theta \\ \Rightarrow\quad \sigma_p(\theta) \;&=\; \theta_0\cos\theta.\end{aligned}$$