Solution HW 1: Mathematical Preliminaries

1. **Answer:**

   (a)

$$
\begin{aligned}
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \\
\text{LHS} &= \partial_i (\mathbf{a} \times \mathbf{b})_i \\
&= \partial_i \left(\epsilon_{ijk}\, a_j\, b_k\right) \\
&= \epsilon_{ijk}\, \partial_i \left(a_j\, b_k\right) \\
&= \epsilon_{ijk} \left(\partial_i\, a_j\right) b_k + \epsilon_{ijk}\, a_j \left(\partial_i\, b_k\right) \\
&= \left(\epsilon_{ijk}\, \partial_i\, a_j\right) b_k + a_j (\epsilon_{ijk}\, \partial_i\, b_k) \\
&= \left(\epsilon_{kij}\, \partial_i\, a_j\right) b_k + a_j (\epsilon_{jki}\, \partial_i\, b_k) \\
&= \left(\epsilon_{kij}\, \partial_i\, a_j\right) b_k - a_j (\epsilon_{jki}\, \partial_k\, b_i) \\
&= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) = \text{RHS}.
\end{aligned}
$$

   (b) We are given that $\mathbf{A}$ is irrotational, thus curl of $\mathbf{A}$ is zero i.e., $\nabla \times \mathbf{A} = \mathbf{0}$. Let's prove that $\mathbf{A} \times \mathbf{r}$ is also irrotational.

$$
\begin{aligned}
(\mathbf{A} \times \mathbf{r})_i &= \epsilon_{ijk}\, A_j\, r_k = 0 \\
\big(\nabla \times (\mathbf{A} \times \mathbf{r})\big)_i &= \epsilon_{ijk}\, \partial_j\, (A \times r)_k \\
&= \epsilon_{ijk}\, \partial_j\, (\epsilon_{k\ell m}\, A_\ell\, r_m) \\
&= \epsilon_{ijk}\, \epsilon_{k\ell m}\, \partial_j\, (A_\ell\, r_m) \\
&= \epsilon_{kij}\, \epsilon_{k\ell m}\, \partial_j\, (A_\ell\, r_m),
\end{aligned}
$$

   Use the identity $\epsilon_{kij}\epsilon_{k\ell m} = \delta_{i\ell}\, \delta_{jm} - \delta_{im}\, \delta_{j\ell}$;

$$
\begin{aligned}
\Rightarrow \big(\nabla \times (\mathbf{A} \times \mathbf{r})\big)_i &= (\delta_{i\ell}\, \delta_{jm} - \delta_{im}\, \delta_{j\ell})\big(\partial_j\, (A_\ell\, r_m)\big) \\
&= \delta_{i\ell}\, \delta_{jm}\big(\partial_j\, (A_\ell\, r_m)\big) - \delta_{im}\, \delta_{j\ell}\big(\partial_j\, (A_\ell\, r_m)\big) \\
&= \partial_j\, (A_i\, r_j) - \partial_j\, (A_j\, r_i) \\
&= \partial_j\, (A_i\, r_j - A_j\, r_i) \\
&= A_i\, \delta_{ij} - A_j\, \delta_{ij} = 0
\end{aligned}
$$

   $\therefore \big(\nabla \times (\mathbf{A} \times \mathbf{r})\big) = 0 \qquad \Rightarrow \; (\mathbf{A} \times \mathbf{r})$ is irrotational.

(c) We are given that $\mathbf{A}$ and $\mathbf{B}$ are constant. We want to prove that $\nabla(\mathbf{A}\cdot\mathbf{B}\times\mathbf{r}) = \mathbf{A}\times\mathbf{B}$.

$$\begin{aligned}
\textbf{LHS} &= \nabla(A\cdot\mathbf{B}\times\mathbf{r}) \\
&= \partial_i\,(A\cdot\mathbf{B}\times\mathbf{r})\,\hat{e}_i \\
&= \partial_i\,(A_j\cdot(\mathbf{B}\times\mathbf{r})_j)\,\hat{e}_i \\
&= \partial_i\,(A_j\,\epsilon_{jk\ell}\,B_k\,r_\ell)\,\hat{e}_i \\
&= \epsilon_{jk\ell}\,A_j\,B_k\,(\partial_i\,r_\ell)\,\hat{e}_i \\
&= \epsilon_{jk\ell}\,A_j\,B_k\,\delta_{i\ell})\,\hat{e}_i \\
&= \epsilon_{jki}\,A_j\,B_k\,\hat{e}_i \\
&= \epsilon_{ijk}\,A_j\,B_k\,\hat{e}_i \\
&= \mathbf{A}\times\mathbf{B} = \text{RHS}.
\end{aligned}$$

(d) We want to prove that $\nabla\times(\phi\nabla\phi) = \mathbf{0}$.

$$\begin{aligned}
(\nabla\times f\mathbf{V})_i &= \epsilon_{ijk}\,\partial_j\,(f\,V_k) \\
&= \epsilon_{ijk}\,\partial_j\,f\,V_k + \epsilon_{ijk}\,f\,\partial_j\,V_k \\
&= \nabla f\times\mathbf{V} + f\nabla\times\mathbf{V} \\
\therefore\ \nabla\times(\phi\nabla\phi) &= \nabla\phi\times\nabla\phi + \phi\nabla\times\nabla\phi \\
&= \phi\nabla\times\nabla\phi \\
(\nabla\times\nabla\phi)_i &= \epsilon_{ijk}\,\partial_j\,(\nabla\phi)_k = \epsilon_{ijk}\,\partial_j\,\partial_k\,\phi = 0.
\end{aligned}$$

(e) We want to prove that $(\nabla u\times\nabla v)$ is solenoidal. In order to prove the above function is solenoidal we should prove that the divergence of the given function is zero at all points, i.e., $\nabla\cdot(\nabla u\times\nabla v) = 0$.

$$\begin{aligned}
\nabla\cdot(\nabla u\times\nabla v) &= \partial_i(\nabla u\times\nabla v)_i \\
&= \partial_i\big(\epsilon_{ijk}\,(\nabla u)_j\,(\nabla v)_k\big) \\
&= \partial_i\big(\epsilon_{ijk}\,\partial_j u\,\partial_k v\big) \\
&= \epsilon_{ijk}\,\partial_i\,(\partial_j u\,\partial_k v) \\
&= \epsilon_{ijk}\,(\partial_i\,\partial_j u)\,(\partial_k v) + \epsilon_{ijk}\,(\partial_j u)\,(\partial_i\,\partial_k v) \\
&= 0.
\end{aligned}$$

2. **Answer:**

   (a) From Gauss's theorem, we showed in class that

   $$\int \nabla f \, dV = \oint f \, d\mathbf{S}$$

   put $f = 1$, the result follows.

   (b)

   $$\begin{aligned} \oint_{\partial V} \mathbf{r} \cdot d\mathbf{S} &= \int_V (\nabla \cdot \mathbf{r}) \, dV = 3V \\ \text{since} \quad \nabla \cdot \mathbf{r} &= \partial_i n_i = 3. \\ \text{Hence } \frac{1}{3} \oint_{\partial V} \mathbf{r} \cdot d\mathbf{S} &= V. \end{aligned}$$

3. **Answer:**

   $$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{S} &= \int_V \nabla \cdot \mathbf{B} \, dV \\ &= \int \nabla \cdot (\nabla \times \mathbf{A}) \, dV = 0 \qquad \text{Since div(curl)=0.} \end{aligned}$$

4. We are given a vector $\mathbf{t} = -y\hat{e}_x + x\hat{e}_y$.

   $$\begin{aligned} \oint_{\partial S} \mathbf{t} \cdot d\ell &= \int_S (\nabla \times \mathbf{t}) \cdot d\mathbf{S} \\ \nabla \times \mathbf{t} &= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} \\ &= \hat{e}_x(0) - \hat{e}_y(0) + \hat{e}_z(1+1) = 2\hat{e}_z. \\ d\mathbf{S} &= dx \; dy \; \hat{e}_z \\ \oint_{\partial S} \mathbf{t} \cdot d\ell &= 2\hat{e}_z \cdot \hat{e}_z \iint dx \, dy = 2A \end{aligned}$$

   which is the desired result.

5. We want to evaluate $\oint_{\partial S} \mathbf{r} \times d\mathbf{r}$. Starting with Stoke's theorem

   $$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\ell.$$

Let $\mathbf{F} = \mathbf{A} \times \mathbf{C}$, where $\mathbf{C}$ is a constant vector.

$$\begin{aligned}
\textbf{LHS} &= \int \nabla \times (\mathbf{A} \times \mathbf{C}) \cdot d\mathbf{S} \\
&= \int (\mathbf{C} \cdot \nabla)\ \mathbf{A} \cdot d\mathbf{S} - \mathbf{C} \cdot \int (\nabla \cdot \mathbf{A}) \cdot d\mathbf{S} \\
\textbf{RHS} &= \oint \mathbf{A} \times \mathbf{C} \cdot d\ell = \mathbf{C} \cdot \int d\ell \times \mathbf{A} \\
\text{where} \qquad \mathbf{C} \cdot \int d\ell \times \mathbf{A} &= \int (\mathbf{C} \cdot \nabla)\mathbf{A} \cdot d\mathbf{S} - \mathbf{C} \cdot \int (\nabla \cdot \mathbf{A}) \cdot d\mathbf{S}
\end{aligned}$$

Simplify the above expression;

$$\begin{aligned}
\int d\mathbf{S} \cdot (\mathbf{C} \cdot \nabla)\mathbf{A} &= \int d\mathbf{S} \cdot \left(C_x \frac{\partial}{\partial x} + C_y \frac{\partial}{\partial y} + C_z \frac{\partial}{\partial z} \right) \left(A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z \right) \\
&= \int d\mathbf{S} \cdot \left(C_x \frac{\partial}{\partial x}\ \mathbf{A} + C_y \frac{\partial}{\partial y}\ \mathbf{A} + C_z \frac{\partial}{\partial z}\ \mathbf{A} \right) \\
&= \int dS_x \left(C_x \frac{\partial}{\partial x}\ A_x + C_y \frac{\partial}{\partial y}\ A_x + C_z \frac{\partial}{\partial z}\ A_x \right) \\
&+ \int dS_y \left(C_x \frac{\partial}{\partial x}\ A_y + C_y \frac{\partial}{\partial y}\ A_y + C_z \frac{\partial}{\partial z}\ A_y \right) \\
&+ \int dS_z \left(C_x \frac{\partial}{\partial x}\ A_z + C_y \frac{\partial}{\partial y}\ A_z + C_z \frac{\partial}{\partial z}\ A_z \right) \\
&= C_x \left(\int dS_x\ \frac{\partial}{\partial x}\ A_x + \int dS_y\ \frac{\partial}{\partial x}\ A_y + \int dS_z\ \frac{\partial}{\partial x}\ A_z \right) \\
&+ C_y \left(\int dS_x\ \frac{\partial}{\partial y}\ A_x + \int dS_y\ \frac{\partial}{\partial y}\ A_y + \int dS_z\ \frac{\partial}{\partial y}\ A_z \right) \\
&+ C_z \left(\int dS_x\ \frac{\partial}{\partial z}\ A_x + \int dS_y\ \frac{\partial}{\partial z}\ A_y + \int dS_z\ \frac{\partial}{\partial z}\ A_z \right) \\
\nabla A_x &= \frac{\partial}{\partial x}\ A_x \hat{e}_x + \frac{\partial}{\partial y}\ A_x \hat{e}_y + \frac{\partial}{\partial z}\ A_x \hat{e}_z \\
\nabla A_y &= \frac{\partial}{\partial x}\ A_y \hat{e}_x + \frac{\partial}{\partial y}\ A_y \hat{e}_y + \frac{\partial}{\partial z}\ A_y \hat{e}_z \\
\nabla A_z &= \frac{\partial}{\partial x}\ A_z \hat{e}_x + \frac{\partial}{\partial y}\ A_z \hat{e}_y + \frac{\partial}{\partial z}\ A_z \hat{e}_z \\
\mathbf{C} \cdot \int dS_k \nabla A_k &= C_k \int dS_k \frac{\partial}{\partial k} A_k \\
\therefore \oint d\ell \times \mathbf{A} &= \int dS_k \nabla A_k - \int (\nabla \cdot \mathbf{A}) d\mathbf{S} \\
\text{Let} \qquad \mathbf{A} &= \mathbf{r}, \quad d\ell = d\mathbf{r} \\
\nabla \cdot \mathbf{A} &= \nabla \cdot \mathbf{r} = 3 \\
\nabla A_k &= \nabla r_k = \partial_i r_k \hat{e}_i = \delta_{ik} \hat{e}_i = \hat{e}_k
\end{aligned}$$

$d\mathbf{S} = a\hat{e}_k$, where a is area bounded by ∂S.

$$
\begin{aligned}
\int dS_k \nabla A_k &= \int dS_k \ \hat{e}_k = a\hat{e}_k \\
\therefore \oint d\mathbf{r} \times \mathbf{r} &= -2a\hat{e}_k \\
\Rightarrow \oint\limits_{\partial S} \mathbf{r} \times d\mathbf{r} &= 2a\hat{e}_k = 2\int\limits_S d\mathbf{S}.
\end{aligned}
$$

6.

$$
\begin{aligned}
\textbf{LHS} &= \oint u\nabla v \cdot d\ell \\
&= \int \nabla \times u\nabla v \cdot d\mathbf{S} \qquad \text{from Stoke's law}
\end{aligned}
$$

Now curl of a product $(\nabla \times u\nabla v)$ is,

$$
\begin{aligned}
\nabla \times u\nabla v &= \epsilon_{ijk}\, \partial_j \,(u\partial_k\, v) \\
&= \epsilon_{ijk}\,(\partial_j\, u)(\partial_k\, v) + \epsilon_{ijk}\, u\, \partial_j\, \partial_k\, v) \\
&= \epsilon_{ijk}\,(\partial_j\, u)(\partial_k\, v) + 0 \\
&= \nabla u \times \nabla v
\end{aligned}
$$

which can be inserted to the LHS, yielding the desired result.