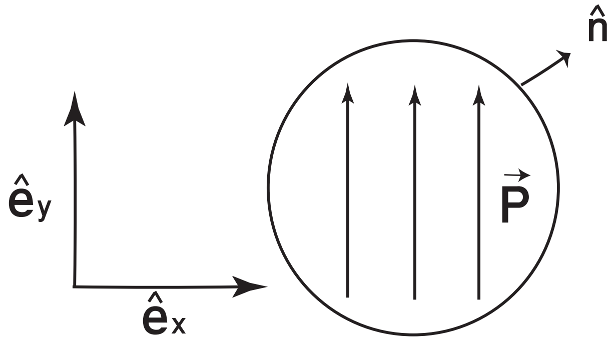
Solution HW 6: Dielectrics and Polarization

1. **Answer:**



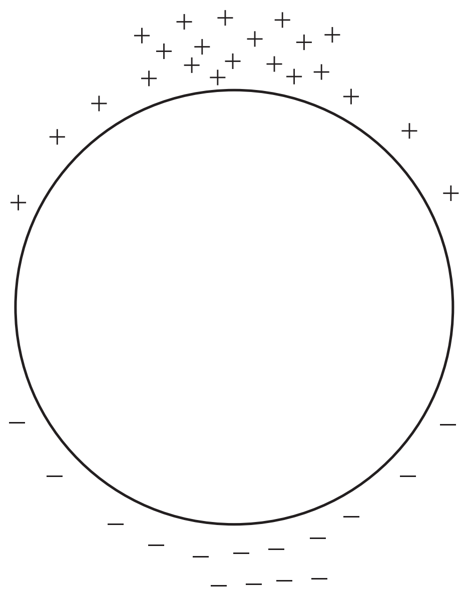
Inside the sphere, $\rho_p = -\nabla \cdot \mathbf{P} = 0$. Since $\mathbf{P}$ = constant. On the sphere surface $\sigma_p(\mathbf{r}_s) = \mathbf{P} \cdot \hat{\mathbf{n}}$, where

$$\hat{\mathbf{n}} = \sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_z,$$

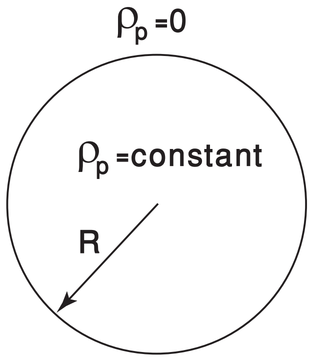
and θ is the angle with the vertical axis. Since

$$\begin{aligned} \mathbf{P} &= P\ \hat{\mathbf{e}}_z \\ \sigma_p(\mathbf{r}_s) &= P\ \hat{\mathbf{e}}_z \cdot (\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_z) \\ &= P\ \cos\theta. \end{aligned}$$

The varying charge on the surface of the sphere can be depicted as below.



2. **Answer:**

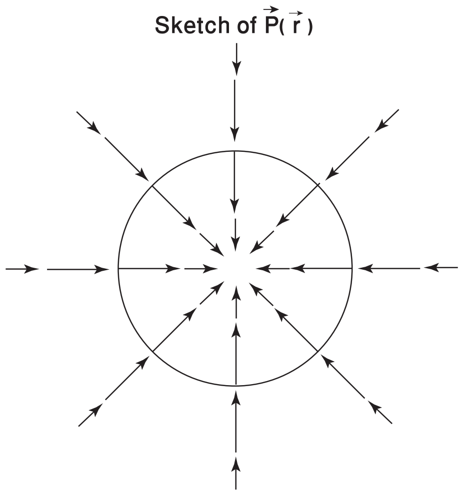


Gauss's law

$$\nabla \cdot \mathbf{E} = +\frac{\rho}{\varepsilon_0} \quad (\rho \text{ inside a sphere})$$

has a solution:

$$\mathbf{E}(\mathbf{r}) = \begin{Bmatrix} \dfrac{\rho}{3\varepsilon_0}\,\mathbf{r}, & r < R \\ \dfrac{\rho\ R^3}{3\varepsilon_0\ r^3}\,\mathbf{r}, & r > R \end{Bmatrix}$$



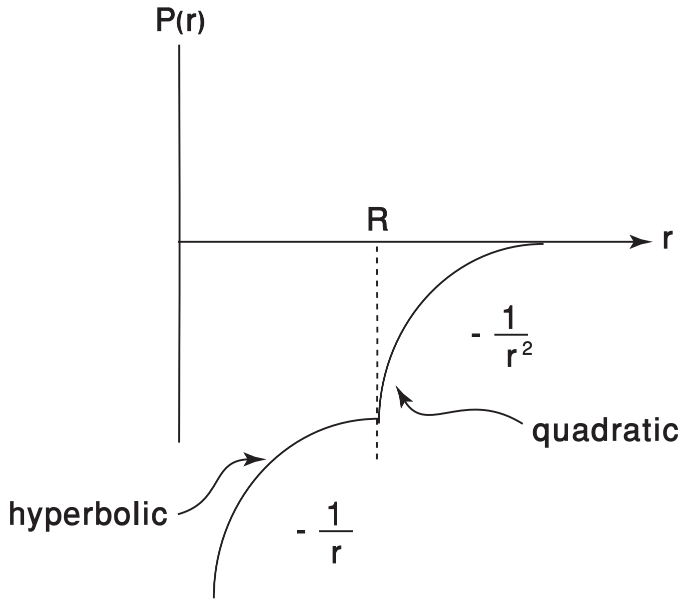
Similarly, here we need to solve,

$$\nabla \cdot \mathbf{P} = -\rho_p$$

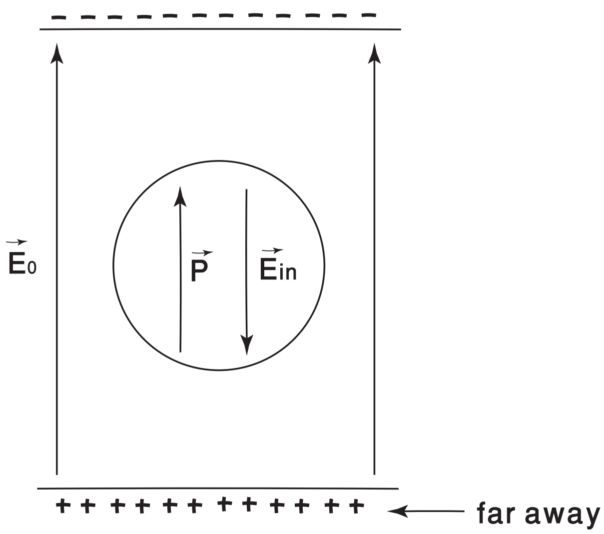
leading to (by analogy)

$$\mathbf{P}(\mathbf{r}) = \begin{Bmatrix} -\dfrac{\rho_p}{3}\,\mathbf{r}, & r < R \\ -\dfrac{\rho_p\ R^3}{3\ r^3}\,\mathbf{r}, & r > R \end{Bmatrix}$$

$\mathbf{P}$ is non-uniform and is non-zero outside the sphere too. A rough sketch is shown in the accompanying figure.



3. **Answer:**



$$
\begin{aligned}
\text{Now }\ \mathbf{E}_{\text{in}} &= -\frac{\mathbf{P}}{3\varepsilon_0} \quad \text{(derived in class, } \mathbf{P} \text{ is uniform inside the dielectric)} \\
\mathbf{E} \text{ (total inside the dielectric)} &= \mathbf{E}_p + \mathbf{E}_0 \\
\text{Now }\ \mathbf{P} &= -3\varepsilon_0 \mathbf{E}_p \\
&= -3\varepsilon_0 (\mathbf{E} - \mathbf{E}_0)
\end{aligned}
$$

We also have

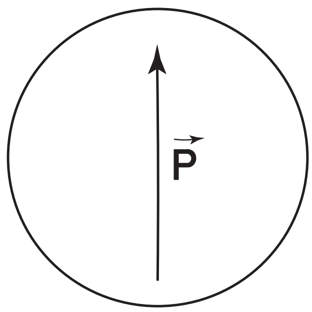
$$\begin{aligned}
\mathbf{P} &= \varepsilon_0\chi_e\mathbf{E} \\
\therefore\ \varepsilon_0\chi_e E &= -3\varepsilon_0(E-E_0) \\
\varepsilon_0\chi_e E + 3\varepsilon_0 E &= 3\varepsilon_0\mathbf{E}_0 \\
E(\chi_e+3) &= 3E_0 \\
E_0 &= \frac{E(\chi_e+3)}{3} \\
\text{Since}\quad \chi_e &= \kappa-1 \\
E &= \left(\frac{3}{\kappa+2}\right)E_0 \\
\text{Hence}\quad \mathbf{E} &= \mathbf{E}_0\left(\frac{3}{\kappa+2}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Similarly}\quad E_p &= E-E_0 \\
&= E_0\left(\frac{3}{\kappa+2}-1\right) = E_0\left(\frac{3-\kappa-2}{\kappa+2}\right) = E_0\left(\frac{1-\kappa}{\kappa+2}\right) \\
\text{Hence}\quad \mathbf{E}_p &= \mathbf{E}_0\left(\frac{1-\kappa}{\kappa+2}\right)
\end{aligned}$$

$$\text{Finally}\quad \mathbf{P} = -3\varepsilon_0\left(\frac{1-\kappa}{\kappa+2}\right)\mathbf{E}_0.$$

4. **Answer:**

(a)



$$
\begin{aligned}
\mathbf{E}_p(\text{in}) &= \mathbf{E}_{\text{in}} = -\frac{\mathbf{P}}{3\varepsilon_0}, \qquad r < R \\
\mathbf{E}_p(\text{out}) &= \mathbf{E}_{\text{out}} = -\frac{V}{4\pi\varepsilon_0}\left(\frac{3(\mathbf{P}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{P}}{r^3}\right), \qquad r > R
\end{aligned}
$$

Without loss of generality,

$$\mathbf{P} = P\hat{\mathbf{e}}_z.$$

θ is the polar angle from the vertical.

$$\mathbf{P}\cdot\hat{\mathbf{r}} = P\cos\theta$$

Inside the sphere

$$
\begin{aligned}
\mathbf{D}^{(\text{in})} &= \varepsilon_0\mathbf{E}_{\text{in}} + \mathbf{P} = \varepsilon_0\left(-\frac{\mathbf{P}}{3\varepsilon_0}\right) + \mathbf{P} = \frac{2}{3}\mathbf{P} \\
D_\perp^{(\text{in})} &= D_r^{(\text{in})} = \frac{2}{3}P\cos\theta.
\end{aligned}
$$

The normal component is the radial component.

Outside the sphere

$$
\begin{aligned}
\mathbf{P} &= 0 \\
\mathbf{D}^{(\text{out})} &= \varepsilon_0\mathbf{E}_{\text{out}} = \frac{V}{4\pi}\left(\frac{3(\mathbf{P}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{P}}{r^3}\right) \\
D_\perp^{(\text{out})} &= D_r^{(\text{out})} = \frac{V}{4\pi r^3}\big(3P\cos\theta - P\cos\theta\big)
\end{aligned}
$$

If $r = R$

$$D_\perp^{(\text{out})} = D_r^{(\text{out})} = \frac{4}{3}\pi R^3\ \frac{P}{4\pi R^3}(2\cos\theta) = \frac{2P\cos\theta}{3}.$$

Hence $D_\perp$ is continuous across the surface of the dielectric.

(b) We now compute the normal components of the electric field.

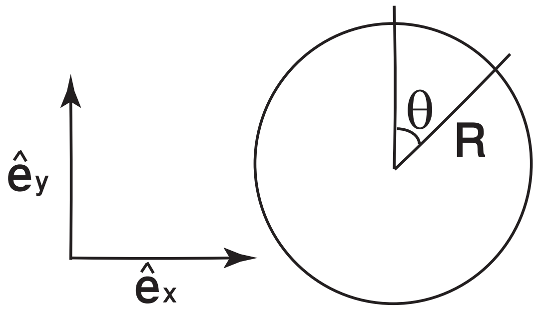
$$
\begin{aligned}
E_\perp^{(\text{in})} &= -\frac{P}{3\varepsilon_0}\cos\theta \\
E_\perp^{(\text{out})} &= \frac{2P}{3\varepsilon_0}\cos\theta
\end{aligned}
$$

Discontinuity is

$$E_{\perp}^{(\text{out})} - E_{\perp}^{(\text{in})} = \frac{P}{3\varepsilon_0}\cos\theta(2+1) = \frac{P}{\varepsilon_0}\cos\theta.$$

The electric field's $\perp$ component is discontinuous, but the D's $\perp$ component is continuous.

(c) Find field at the center of a uniformly polarized sphere by explicit integration.



There is only charge on the surface (see Q1.).

$$\begin{aligned}
\text{Since} \quad \sigma &= P\cos\theta \\
\mathbf{E}(\mathbf{r}=0) &= \frac{1}{4\pi\varepsilon_0}\int_S \frac{d^2r' \ \sigma(\mathbf{r}')}{r'^3}(-\mathbf{r}) \\
&= -\frac{P}{4\pi\varepsilon_0 \ R^3}\int_S d^2r' \ \cos\theta \ \mathbf{R}
\end{aligned}$$

Let's find the x-component:

$$\begin{aligned}
E_x &= -\frac{P}{4\pi\varepsilon_0 \ R^3}\int_s d^2r' \ \cos\theta \ x \\
x &= R\sin\theta\cos\phi \\
E_x &= -\frac{P}{4\pi\varepsilon_0 \ R^3}\int d\theta \ d\phi \ R^2 \ \sin\theta \ \cos\theta \ (R\sin\theta\cos\phi) \\
&= -\frac{P}{4\pi\varepsilon_0}\left(\int_0^{2\pi} d\phi \ \cos\phi\right)\left(\int_0^{\pi} d\theta \ \sin^2\theta \ \cos\theta\right) = 0
\end{aligned}$$

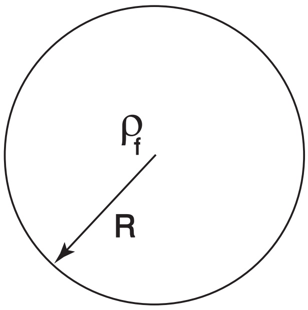
Similarly, by symmetry $E_y = 0$. whereas, for the z-component

$$\begin{aligned}
E_z &= -\frac{P}{4\pi\varepsilon_0\ R^3}\int_s d^2r'\ \cos\theta\ z \\
&= -\frac{P}{4\pi\varepsilon_0\ R^3}\int\limits_{\phi=0}^{2\pi}\int\limits_{\theta=0}^{\pi} d\theta\ d\phi\ R^2\ \sin\theta\ \cos\theta\ (R\cos\theta) \\
&= -\frac{P}{4\pi\varepsilon_0}(2\pi)\int\limits_0^{\pi} d\theta\ \sin\theta\ \cos^2\theta \\
&= -\frac{P}{2\varepsilon_0}\cdot\left(-\frac{\cos^3\theta}{3}\right)\Bigg|_0^{\pi} = \frac{P}{6\varepsilon_0}(-2) = -\frac{P}{3\varepsilon_0}.
\end{aligned}$$

Hence $\mathbf{E}$ at center is $-\dfrac{P}{3\varepsilon_0}$ as desired.

5. **Answer:**

(a)



The density ρ_f is uniform. Let's find $\mathbf{D}$ using Gauss's law:

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho_f \\
\int_S d\mathbf{S}\cdot\mathbf{D} &= D(4\pi r^2) = \frac{4}{3}\pi r^3\rho_f \\
\mathbf{D} &= \frac{r\rho_f}{3}\hat{\mathbf{r}} \\
\text{Now}\quad \mathbf{D} &= \varepsilon_0\mathbf{E} + \mathbf{P} \\
\mathbf{P} &= \mathbf{D} - \varepsilon_0\mathbf{E} = \mathbf{D} - \frac{\varepsilon_0}{\epsilon_0}\mathbf{D} = \mathbf{D}\left(1 - \frac{1}{\kappa}\right) \\
\text{Hence}\quad \mathbf{P} &= \frac{r\rho_f}{3}\hat{\mathbf{r}}\left(\frac{\kappa-1}{\kappa}\right) = \mathbf{r}\frac{\rho_f}{3}\left(\frac{\kappa-1}{\kappa}\right).
\end{aligned}$$

(b)

$$\begin{aligned}
\rho_p &= -\nabla \cdot \mathbf{P} = -\frac{\rho_f}{3}\left(\frac{\kappa-1}{\kappa}\right)\nabla \cdot \mathbf{r} = -\rho_f\left(\frac{\kappa-1}{\kappa}\right). \\
\sigma_p &= \mathbf{P} \cdot \hat{\mathbf{n}} = \frac{\rho_f}{3}\left(\frac{\kappa-1}{\kappa}\right)\mathbf{r} \cdot \hat{\mathbf{n}} = \frac{\rho_f}{3}\left(\frac{\kappa-1}{\kappa}\right)R.
\end{aligned}$$

(c)

$$\text{Total polarization charge on surface} = \frac{\rho_f}{3}\left(\frac{\kappa-1}{\kappa}\right)R(4\pi R^2) = Q_f\left(\frac{\kappa-1}{\kappa}\right).$$