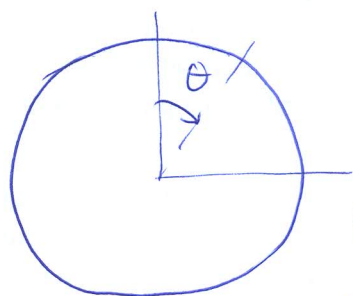# Spherical multipole examples

Ex 1

$\sigma(\theta) = \sigma_0 \cos\theta$.

We want to find the **potential** due to this surface charge distribution.

For the **exterior** multipole expansion, we have:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$r > R$

where

$$A_{lm} = \frac{4\pi}{2l+1} \int d^3r' \, \rho(\vec{r}) \, r'^l \, Y_{lm}^*(\theta',\phi')$$

The charge distribution is azimuthally symmetric, so let's first find A_{lm} for this special case $\left(\rho(\vec{r}') \right) = \rho(r', \theta', \phi')$
$$= \rho(r', \theta').$$

$$A_{lm} = \frac{4\pi}{2l+1} \int d\phi' \, d\theta' \, dr' \, r'^2 \sin\theta' \, \rho(r', \theta') \, r'^l \, Y_{lm}^*(\theta', \phi') \quad \rule[-2mm]{0.4pt}{5mm} \text{①}$$

Now

$$\int_0^{2\pi} d\phi' \, Y_{lm}(\theta', \phi') = 2\pi \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta') \, \delta_{mo}$$

∴ Since the only ϕ' dependent term in ① $Y_{lm}^*(\theta', \phi')$

$$A_{lm} = \frac{4\pi}{2l+1} \cdot 2\pi \sqrt{\frac{2l+1}{4\pi}} \, \delta_{m0} \int d\theta' d\alpha' \, \alpha'^2 \, \underline{\sin\theta'} \, \rho(\alpha', \theta') \, \alpha'^l \, P_\ell(\cos\theta')$$

Now

$$\Rightarrow \boxed{A_{lm} = \sqrt{\frac{4\pi}{2l+1}} \, \delta_{m0} \int d^3\alpha' \, \rho(\alpha', \theta') \, \alpha'^l \, P_\ell(\cos\theta')} \quad \text{AZIMUTHAL}$$

In the problem at hand.

$$d^3\alpha' \, \rho(\alpha', \theta') \quad \longrightarrow \quad \oint dS \, \sigma_0 \, \cos\theta' = dS \, \sigma_0 \, P_1(\cos\theta') \, \delta(\alpha'-R)$$
$$\times \, \delta(\alpha'-R)$$

$$\therefore \quad A_{lm} = \sqrt{\frac{4\pi}{2l+1}} \, \delta_{m0} \int dS \, \sigma_0 \, P_1(\cos\theta') \, \alpha'^l \, P_\ell(\cos\theta')$$

$$= \sqrt{\frac{4\pi}{2l+1}} \, \delta_{m0} \int_{\theta'=0}^{\pi} d\theta' \, R^2 \, \sin\theta' \, \sigma_0 \, R^l \, P_1(\cos\theta') \, P_\ell(\cos\theta') \underbrace{\int_{\phi=0}^{2\pi} d\phi'}_{\textcircled{2}}$$

Now $\displaystyle\int_{-1}^{1} dx \, P_\ell(x) \, P_{\ell'}(x) = \frac{2}{2l+1} \, \delta_{\ell\ell'}$ (an identity)

if $x = \cos\theta'$, $\qquad dx = -\sin\theta' \, d\theta'$

$x = -1$, corresponds to $\theta' = \pi$,

$x = +1$, corresponds to $\theta' = 0$

$$\int_{-1}^{1} dx \, P_L(x) \, P_{L'}(x)$$

$$= - \int_{\pi}^{0} d\theta' \sin\theta' \, P_\ell(\cos\theta') \, P_2(\cos\theta')$$

$$= \int_{0}^{\pi} d\theta' \sin\theta' \, P_1(\cos\theta') \, P_\ell(\cos\theta') = \left(\frac{2}{2\ell+1}\right) \delta_{\ell\ell'}$$

one can identify this integral ② , leading to:

$$A_{\ell m} = \sqrt{\frac{4\pi}{2\ell+1}} \, \delta_{mo} \, R^{2+\ell} \, \sigma_0 \left(\frac{2}{2\ell+1}\right) \delta_{\ell 1} \cdot 2\pi$$

$$\boxed{A_{\ell m} = \left(\frac{4\pi}{2\lambda+1}\right)^{3/2} R^{2+\ell} \, \sigma_0 \, \delta_{mo} \delta_{\ell 1}}$$

$$\phi(\vec{n}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{4\pi}{2\ell+1}\right)^{3/2} R^{2+\ell} \sigma_0 \, \delta_{mo}\delta_{\ell 1} \frac{Y_{\ell m}(\theta,\phi)}{n^{\ell+1}}$$

$\imath > R$

$$\phi(\vec{n}) = \frac{1}{4\pi\varepsilon_0} \frac{R^3 \sigma_0 \, Y_{10}(\theta,\phi)}{n^2} \cdot \left(\frac{4\pi}{3}\right)^{3/2}$$

Now $\quad Y_{10}(\theta,\phi) = \sqrt{\dfrac{3}{4\pi}} \cos\theta \quad$ (Table provided)

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \, \sigma_0 \frac{R^3}{r^2} \sqrt{\frac{3}{4\pi}} \cos\theta \cdot \left(\frac{4\pi}{3}\right)^{3/2}$$

$$\frac{3}{2} - \frac{1}{2} = 1.$$

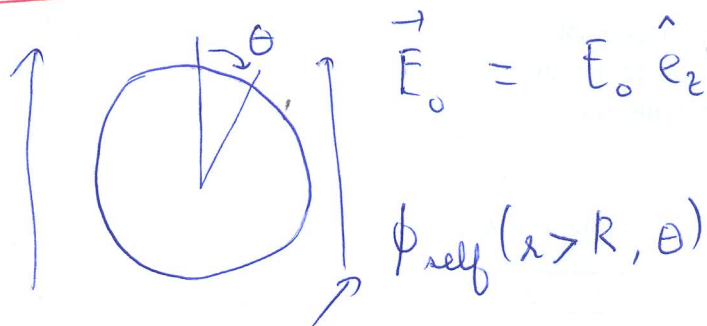$$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{4\pi}{3}\right) \sigma_0 \frac{R^3}{r^2} \cos\theta \,, \quad r > R}$$

Similarly we can show that for $r < R$ (interior

multipole expansion)

$$\boxed{\phi(\vec{r}) = \frac{\sigma}{3\varepsilon_0} \, r \cos\theta}.$$

Q: Find the electric fields due to these potentials.

Now a conducting sphere is placed inside a
uniform electric field

$$\vec{E}_0 = E_0 \, \hat{e}_z$$

$$\phi_{self}(r > R, \theta)$$

no ϕ dependence

$$A_{lm} = \sqrt{\frac{4\pi}{2l+1}} \, \delta_{mo} \int d^3 r' \, \rho(r', \theta') \cdot r'^l \, P_l(\cos\theta')$$

$$\phi(r > R)_{self} = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sqrt{\frac{4\pi}{2l+1}} \left[\left[\int d^3x' \, \rho(x',\theta') \frac{4\pi}{2l+1} r'^l P_l(\cos\theta') \right] \underbrace{\frac{Y_{l0}(\theta,\phi)}{r^{l+1}}}_{} \right]$$

$$Y_{l0}(\theta,\phi) \propto P_l(\cos\theta)$$

$$\therefore \quad \phi_{self}(r > R) = \frac{A}{r} + \frac{B}{r^2} P_1(\cos\theta) + \frac{C}{r^3} P_2(\cos\theta) + \cdots$$

$$= \frac{A}{r} + \frac{B}{r^2} \cos\theta + \frac{C}{r^3} P_2(\cos\theta) + \cdots$$

Inside

$$\vec{E}_{self} = -E_0 \hat{e}_z \longleftarrow E_0$$

$$\phi_{self}(r < R) = E_0 z = E_0 \, r\cos\theta$$

$$\phi_{self}(r = R) = E_0 R \cos\theta$$

$$\Rightarrow \quad E_0 R = B/r^2 \Rightarrow \quad B = E_0 R \cdot R^2 = E_0 R^3 \quad \boxed{\text{compare}} \checkmark$$

$$\phi_{self}(r > R) = \frac{E_0 R^3}{r^2} \cos\theta \equiv \frac{1}{4\pi\varepsilon_0} \frac{P\cos\theta}{r^2}$$

$$\boxed{\vec{P} = (4\pi\varepsilon_0 R^3)\, \vec{E}}$$